Classical Mechanics

Fundamental aspects of Newton's theory of motion

- ☞ Newton's Laws
- ☞ Inclined Plane
- ☞ Projectiles
- ☞ Conservation Theorems

Newton's Laws

I. *A body remains at rest or in uniform motion unless acted upon by a force*

II. *A body acted upon by a force moves in such a manner that the time rate of change of momentum equals the force*

III. *If two bodies exert forces on each other, these forces are equal in magnitude and opposite in direction*

I is also known as Law of Inertia

It provides us only with a qualitative notion ragarding a force

$\Downarrow$

In the absence of forces a particle moves with constant velocity $\vec{v}$

II relates force to the time of rate of change of momentum

Newton defined momentum as $\vec{p} = m\vec{v}$ ☚ $m =$ particle's mass

Newton's Second Law then reads

$$\vec{F} = \frac{d\vec{p}}{dt} = \dot{\vec{p}} = \frac{d}{dt}(m\vec{v})$$

III states that for two isolated bodies $\vec{F}_1 = -\vec{F}_2$

Frames of Reference

For the laws of motion to have a meaning ⇓

motion of bodies must be measured relative to some reference frame

Inertial Frame: a frame where Newton's laws are valid

Galilean Invariance or Principle of Newtonian Relativity

If Newton's laws are valid in one reference frame ☚ they are also valid in any reference frame in uniform motion with respect to the first system

⇓

$$\vec{F} = m\ddot{\vec{r}}$$

⇓

A change of coordinates involving constant velocity

does not influence the equation of motion

Theory of Relativity has shown that

the concepts of absolute rest and absoulte inertial frame are meaningless

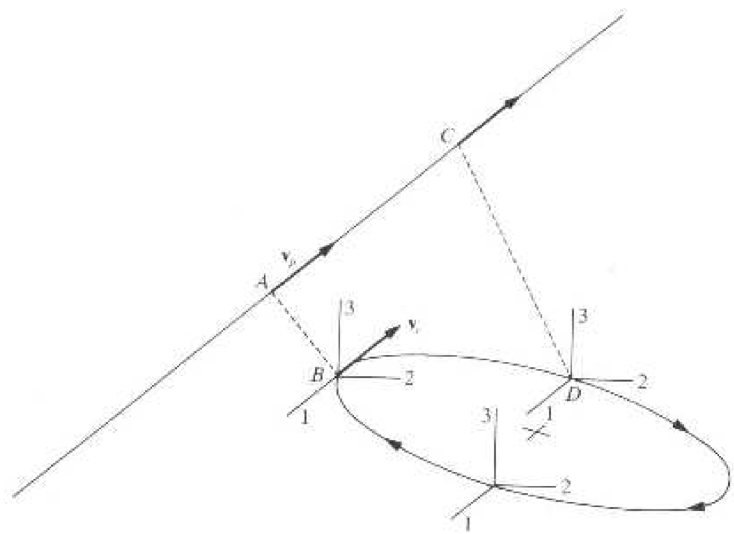
In reference frames described with respect to "fixed" stars

Newtonian dynamics is valid to high degree of accuracy

Equation of Motion for a Particle

Particle's vector equation of motion

- **independent of the position of the origin of the coordinate system**
- **independent of its orientation in space**



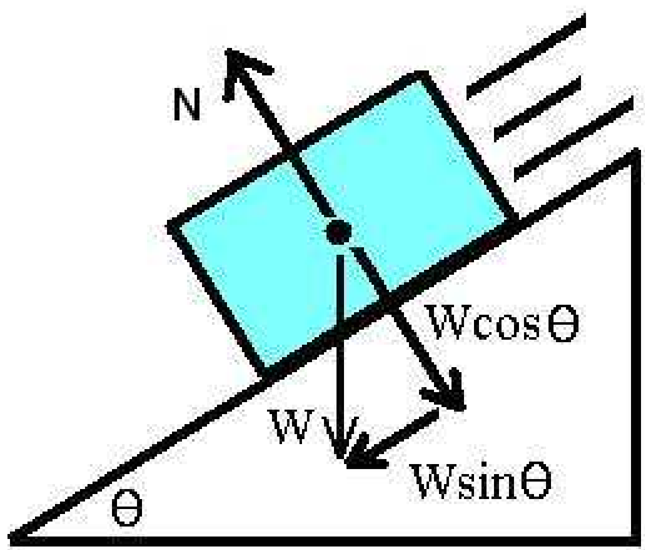
Rotating coordinate systems do not qualify as inertial reference frames

Equation of motion assuming m does not vary with time

$$\vec{F} = \frac{d}{dt}(m\vec{v}) = m\frac{d\vec{v}}{dt} = m\ddot{\vec{r}}$$

Inclined Plane

Find acceleration of block sliding without friction inclined plane ☛ $\theta = 30^\circ$



$$\sum_i \vec{F}_i = \vec{W} + \vec{N} = m\ddot{\vec{r}}$$

y-direction ☛ $-|\vec{W}|\cos\theta + |\vec{N}| = 0$ x-direction ☛ $|\vec{W}|\sin\theta = m\ddot{x}$

$$\ddot{x} = \frac{|\vec{W}|}{m}\sin\theta = \frac{mg\,\sin\theta}{m} = g\sin\theta$$

$$\ddot{x} = g\,\sin 30^\circ = \frac{g}{2} = 4.9\ \mathrm{m/s^2}$$

Inclined Plane (cont'd)

Find blocks velocity after it moves from rest a distance x_d

$$[\ddot{x} = g\,\sin\theta]$$

$$\Downarrow$$

$$2\dot{x}\ddot{x} = 2\dot{x}\,g\,\sin\theta$$

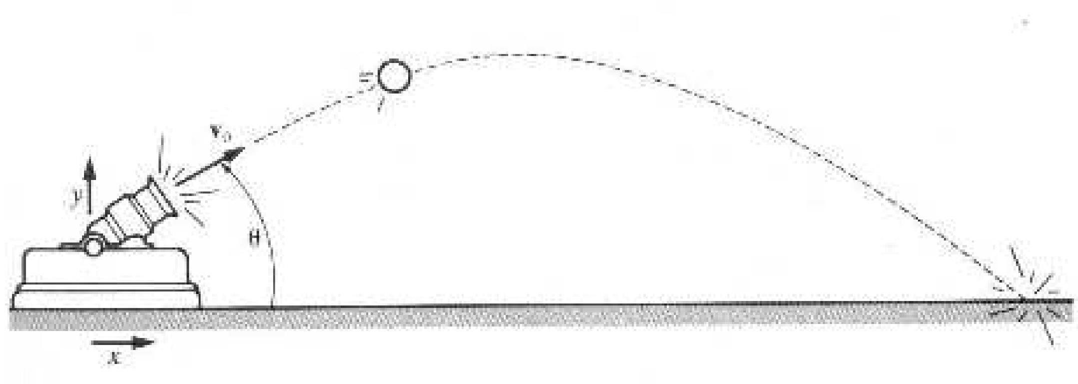
$$\frac{d}{dt}(\dot{x}^2) = 2g\,\sin\theta\,\frac{dx}{dt}$$

$$\int_0^{v_d^2} d(\dot{x}^2) = 2g\,\sin\theta\int_0^{x_d} dx$$

$$v_d^2 = 2\,g\sin\theta\;x_d$$

$$v_d = \sqrt{2\,g\,\sin\theta\;x_d}$$

Projectile Motion in 2-Dimensions



x-direction ☚ $m\ddot{x} = 0$ y-direction ☚ $-mg = m\ddot{y}$

Neglect the height of the gun and assume $x = y = 0$ @ $t = 0$

$$\ddot{x} = 0 \qquad \ddot{y} = -g$$
$$\dot{x} = v_0 \cos\theta \qquad \dot{y} = -g\,t + v_0\,\sin\theta$$
$$x = v_0\,t\,\cos\theta \qquad y = v_0\,t\,\sin\theta - g\,t^2/2$$

speed ☚ $v(t) = \sqrt{\dot{x}^2 + \dot{y}^2} = (v_0^2 + g^2\,t^2 - 2v_0\,g\,t\,\sin\theta)^{1/2}$

displacement ☚ $r(t) = \sqrt{x^2 + y^2} = (v_0^2\,t^2 + g^2\,t^4/4 - v_0\,g\,t^3\,\sin\theta)^{1/2}$

Projectile Range

Range: **Value of x when the projectile falls back to ground**

$$\Downarrow$$

$$\boldsymbol{y = 0}$$

$$y = t\left(v_0 \sin\theta - \frac{g\,t}{2}\right) = 0$$

$$y = 0 \quad \begin{cases} ☛\ t = 0 \\ ☛\ t = T \end{cases}$$

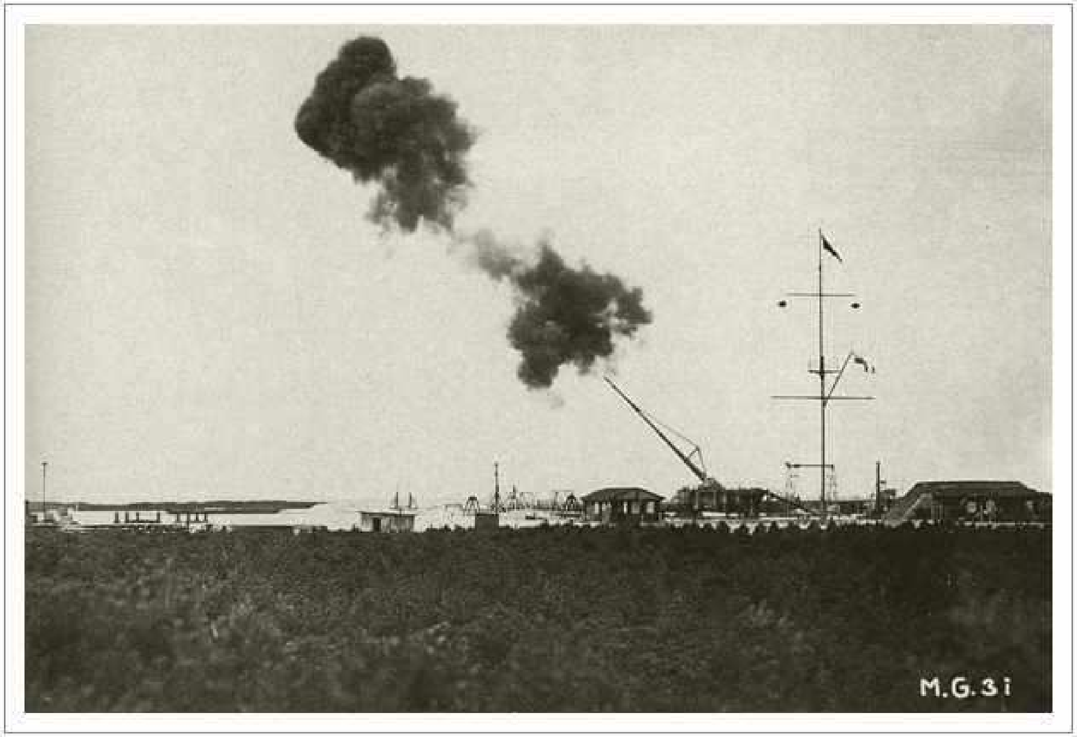
$$\Updownarrow$$

$$T = \frac{2\,v_0\,\sin\theta}{g}$$

$$\text{range} \equiv R = x(t = T) = \frac{2v_0^2}{g}\sin\theta\,\cos\theta = \frac{v_0^2}{g}\sin(2\theta)$$

Big Bertha

On March 28th 1918 during World War I
the Germans used a long-range gun named Big Bertha to bombard Paris



Its muzzle velocity was 1,450 m/s and the angle of elevation $\theta = 55^\circ$

Big Bertha (cont'd)

According to our recent estimate

$$R = \frac{v_0^2}{g}\sin(2\theta) = \left(\frac{1450\ \mathrm{m/s}}{9.8\ m/s^2}\right)^2 [\sin(110^\circ)] = 202\ \mathrm{km}$$

Could a shell fired by a Big Bertha from 202 km caused this damage?

Big Bertha (cont'd)

NO $\rightarrow$ because of the air resistance

☛ Big Bertha's actual range was 120 km

present day

Americans invented a much better weapon ☛ no resistance at all!!!

Effects of Retarding Forces

$\vec{F}$'s acting on a particle are not necessary constant

If a particle falls in a constant gravitational field ☛ $\vec{F}_g = m\vec{g}$

NEVERTHELESS

Retarding forces F_r that are function of the instantaneous speed may exist

To a good approximation $F_r \propto v^n$

Total $\vec{F}$ acting on a particle

$$\vec{F} = m\,\vec{g} - m\,k\,v^n\,\frac{\vec{v}}{v}$$

$\vec{v}/v$ ☛ unit vector in the direction of $\vec{v}$

k ☛ positive constant that specifies the strength of the retarding force

Experimentally ☛ for relatively small objects moving in air

$n \simeq 1$ for $v < 24$ m/s

$n \sim 2$ is usually taken for speeds up to the speed of sound

In the power-law approximation the equation of motion can be integrated

Effects of Air Resistance to the Motion of Projectiles

INITIAL CONDITIONS

$$x(t=0) = 0 = y(t=0)$$

$$\dot{x}(t=0) = v_0 \cos\theta \equiv U$$

$$\dot{y}(t=0) = v_0 \sin\theta \equiv V$$

EQUATION OF MOTION

$$m\ddot{x} = -km\dot{x}$$

$$m\ddot{y} = -km\dot{y} - mg$$

$$\Downarrow$$

$$m\frac{dv}{dt} = -kmv \Rightarrow \ln v = -kt + \mathcal{C}$$

FROM INITIAL CONDITION $\mathcal{C} = \ln U$ ☚ $v = Ue^{-kt}$

$$x = U\int e^{-kt}dt = -\frac{U}{k}e^{-kt} + \mathcal{K}$$

FROM INITIAL CONDITION $\mathcal{K} = U/k$ ☚ $x = \frac{U}{k}(1 - e^{-kt})$

Effects of Air Resistance to the Motion of Projectiles (cont'd)

HOMEWORK:

show that a similar procedure using the equation in y-direction leads to

$$y = -\frac{gt}{k} + \frac{kV + g}{k^2}(1 - e^{-kt})$$

t when the projectile falls back to ground

$$T = \frac{kV + g}{gk}\ (1 - e^{-kt})$$

Transcendental Equation

Two approaches

- ☛ Perturbation Method to find an approximate solution
- ☛ Numerical method that can be as accurate as desired

Effects of Air Resistance to the Motion of Projectiles (cont'd)

Perturbation Method

☞ Consider an expansion parameter or coupling constant that is normally small

EXAMPLE

retarding force constant k

We solved the case $k = 0$

We turn on retarding force but keep it small

Expansion of the exponential term

$$T = \frac{kV + g}{gk} \left\{ kT - \frac{1}{2!}\, k^2 T^2 + \frac{1}{3!}\, k^3 T^3 - \dots \right\}$$

Keeping terms in ↑ up to k^3 ☞ $T = \frac{2V/g}{1+kV/g} + \frac{1}{3} k T^2$

Expansion of denominator in power series up to second order

$$\frac{1}{1 + kV/g} = 1 - (kV/g) + (kV/g)^2 - \dots$$

Keeping terms up to first order ☞ $T = \frac{2V}{g} + \left(\frac{T^2}{3} - \frac{2V^2}{g^2} \right) k + \mathcal{O}(k^2)$

In the limit $k \to 0$ ☞ we recover solution without air resistance

$$T(k = 0) = T_0 = 2\ V/g = 2\, v_0\ \sin\theta/g$$

Effects of Air Resistance to the Motion of Projectiles (cont'd)

Approximate expression for flight time

$$T \approx \frac{2V}{g}\left(1 - \frac{kV}{3g}\right)$$

Writing equation of x-direction in expanded form

$$x = \frac{U}{k}\left(kt - \frac{1}{2}k^2t^2 + \frac{1}{6}k^3t^3 \ldots\right)$$

keeping terms only through first order of k

$$R' \approx U\left(T - \frac{1}{2}kT^2\right)$$

Replacing by T ☛ $R' \approx \frac{2UV}{g}\left(1 - \frac{4kV}{3g}\right)$ ☺ but $2UV/g = v_0^2 \sin(2\theta)/g = R$

$$\Downarrow$$

$$R' \approx R\left(1 - \frac{4kV}{3g}\right)$$

valid when $k \ll g/V = g/(v_0 \sin\theta)$

Effects of Air Resistance to the Motion of Projectiles (cont'd)

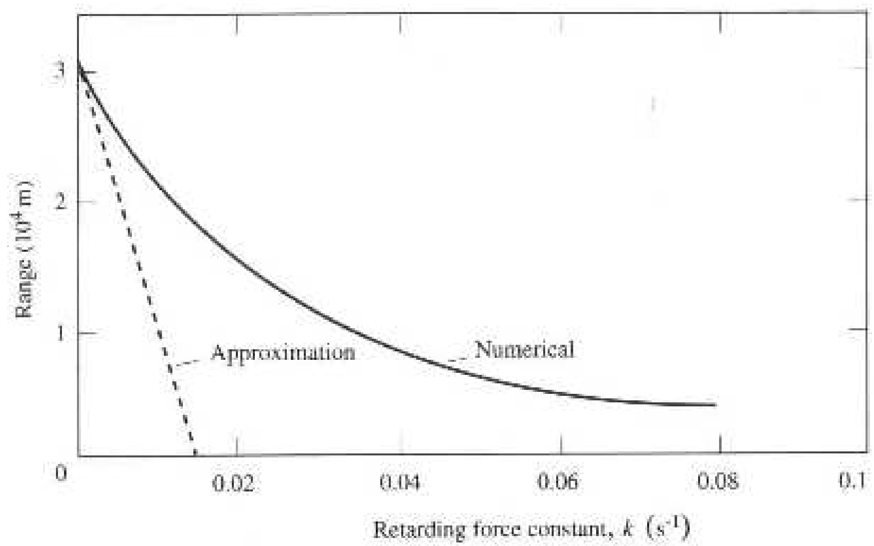
HOMEWORK

Solve numerically

$$T = \frac{kV + g}{gk}\left(1 - e^{-kt}\right)$$

show that the linear approximation is inaccurate for k values as low as 0.01 s^{-1}
show that it incorrectly indicates the range is zero for $k > 0.014\ \text{s}^{-1}$

HOMEWORK'S HINT



Conservation Theorems

Newtonian mechanics of a single particle

- We derive important theorems regarding conserved quantities
- We DO NOT prove the conservation of the various quantities
- We derive consequences of Newton's laws of dynamics

$\Downarrow$

Implications must be put to the test of experiment

$\Downarrow$

Verification supplies a measure of confirmation of the dynamical laws

- Conservation theorems are found to be valid in many circumstances

$\Downarrow$

Important part of the proof for the correctness of Newton's laws

$\Downarrow$

(at least in classical physics)

Conservation Theorems (cont'd)

I. **The total liner momentum $\vec{p}$ of a particle is conserved when the total force on it is zero**

II. **The angular momentum $\vec{L}$ of a particle subject to no torque $\vec{\tau}$ is conserved**

III. **The total energy E of a particle in a conservative force field is a constant in time**

$I \to$ is derived from vector equation ☚ $\dot{\vec{p}} = 0$

$\Downarrow$

applies for each component of the linear momentum

Let $\vec{s}$ be some constant vector such $\vec{F} \cdot \vec{s} = 0 \; \forall$ time

$\Downarrow$

$$\dot{\vec{p}} \cdot \vec{s} = 0$$

integration leads to

$\Downarrow$

$$\vec{p} \cdot \vec{s} = \text{constant}$$

Conservation Theorems (cont'd)

☞ **The angular momentum of a particle**
(with respect to an origin from which the position vector $\vec{r}$ is measured) **is**

$$\vec{\boldsymbol{L}} = \vec{\boldsymbol{r}} \times \vec{\boldsymbol{p}}$$

☞ **The torque (or moment of force) with respect to the same origin is**

$$\vec{\boldsymbol{\tau}} = \vec{\boldsymbol{r}} \times \vec{\boldsymbol{F}}$$

$$\downarrow$$

position vector from the origin to the point where the force is applied

$$\vec{\tau} = \vec{r} \times \dot{\vec{p}}$$

$$\Downarrow$$

$$\dot{\vec{L}} = \frac{d}{dt}(\vec{r} \times \vec{p}) = (\dot{\vec{r}} \times \vec{p}) + (\vec{r} \times \dot{\vec{p}})$$

But of course ☚ $\dot{\vec{r}} \times p = \dot{\vec{r}} \times mv = m(\dot{\vec{r}} \times \dot{\vec{r}}) = 0$

$$\dot{\vec{L}} = \vec{r} \times \dot{\vec{p}} = \vec{\tau}$$

If $\vec{\tau} = 0 \Rightarrow \dot{\vec{L}} = 0$ ☚ $\vec{L}$ is a vector constant in time

Conservation Theorems (cont'd)

If some work W_{12} is done on a particle by a force $\vec{F}$ in transforming the particle from condition 1 to condition 2 $\Rightarrow W_{12} \equiv \int_1^2 \vec{F} \, . \, d\vec{r}$

$$\begin{aligned} \vec{F} \, . \, d\vec{r} &= m \frac{d\vec{v}}{dt} \, . \, \frac{d\vec{r}}{dt} dt \\ &= m \frac{d\vec{v}}{dt} \, . \, v dt \\ &= \frac{m}{2} \frac{d}{dt} (\vec{v} \, . \, \vec{v}) dt \\ &= \frac{m}{2} \frac{d}{dt} (v^2) dt \\ &= d \left(\frac{1}{2} m v^2 \right) \end{aligned}$$

$\vec{F} \, . \, d\vec{r}$ is an exact differential $\Rightarrow$ work done by the total force $\vec{F}$ acting on a particle is equal to its change in kinetic energy ΔT

$$W_{12} = \left(\frac{1}{2} m v^2 \right) \Big|_1^2 = \frac{1}{2} m (v_2^2 - v_1^2) = T_2 - T_1$$

If $T_1 > T_2 \Rightarrow W_{12} < 0$

☛ the particle has done work with a resulting decrease in its kinetic energy

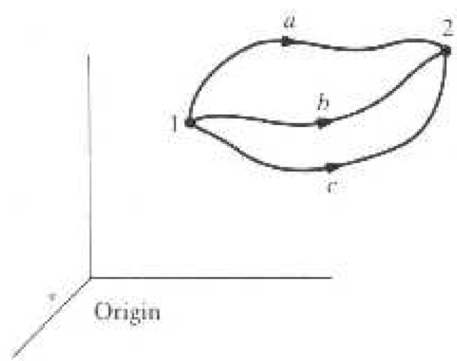
Conservation Theorems (cont'd)

A force is said to be conservative

$\Updownarrow$

the work required to move a particle from one position to another without any change in kinetic energy depends only on the original and final positions and not on the exact path taken by the particle

Example: A constant gravitational field



- If a particle of mass m is raised through a height h (by any path) $\Downarrow$ an amount of work mgh has been done on the particle
- The particle can do equal amount of work in returning to its original position
- The capacity to do work is known as potential energy of the particle

Conservation Theorems (cont'd)

Potential Energy

⇓

required work to transport the particle from point 1 to a point 2
(with not net change in kinetic energy)

$$\int_1^2 \vec{F} \, . \, d\vec{r} = U_1 - U_2$$

If $\vec{\nabla} \times \vec{F} = 0 \Leftrightarrow \vec{F} = -\vec{\nabla} U$

$$\int_1^2 \vec{F} \, . \, d\vec{r} = -\int_1^2 (\vec{\nabla} U) \, . \, d\vec{r} = -\int_1^2 dU = U_1 - U_2$$

- The potential energy is defined only to within an additive constant
- The force defined by $\vec{\nabla} U$ is no different from that defined by $\vec{\nabla}(U+$ constant$)$
- Only differences of potential energy are physically meaningful
- The $\vec{v}$ of a particle is in general different in different inertial reference frames
- It is impossible to ascribe an absolute kinetic energy to a particle

Conservation Theorems (cont'd)

Total (mechanical) energy of a particle ☛ $E \equiv T + U$

$$\frac{dE}{dt} = \frac{dT}{dt} + \frac{dU}{dt}$$

Recall that $\vec{F} \cdot d\vec{r} = dT$ ☛ $\dot{T} = \vec{F} \cdot \dot{\vec{r}}$

$$\begin{aligned} \frac{dU}{dt} &= \sum_i \frac{\partial U}{\partial x_i}\frac{dx_i}{dt} + \frac{\partial U}{\partial t} \\ &= (\vec{\nabla} U) \cdot \dot{\vec{r}} + \frac{\partial U}{\partial t} \end{aligned}$$

If $\vec{F}$ is a conservative force ☛ $\vec{F} = -\vec{\nabla} U$

$$\begin{aligned} \frac{dE}{dt} &= \vec{F} \cdot \dot{\vec{r}} + (\vec{\nabla} U) \cdot \dot{\vec{r}} + \frac{\partial U}{\partial t} \\ &= (\vec{F} + \vec{\nabla} U) \cdot \dot{\vec{r}} + \frac{\partial U}{\partial t} \\ &= \frac{\partial U}{\partial t} \end{aligned}$$

If $\partial U/\partial t = 0$ the total (mechanical) energy of a particle is a constant in time

Energy

- The concept of energy was not nearly as popular in Newton's time as it is today
- Early in the XIX century ☚ became clear that heat was another form of energy (and not a form of fluid called caloric that flowed between hot and cold bodies)
- Count Rumford is given credit for realizing that the amount of heat generated during the boring of a cannon was caused by friction and not the caloric

If frictional energy is just energy ☚ a total conservation of energy occurs

- Hermann von Helmholtz (1821-1894) formulated the general law of energy conservation in 1847