# Classical Mechanics

## Hamiltonian Mechanics

- Generalized Momenta
- Hamilton's Equations
- Phase Space
- Liouville's Theorem
- The Virial Theorem

# Generalized Momenta

**Consider the motion of a single particle moving in one dimension**

**the kinetic energy is**

$$T = \frac{1}{2} m \dot{x}^2$$

- $m$ ☞ the mass of the particle
- $x$ ☞ its displacement

**the particle's linear momentum is**

$$p = m \dot{x}$$

**This can also be written as**

$$p = \frac{\partial T}{\partial \dot{x}} = \frac{\partial \mathcal{L}}{\partial \dot{x}}$$

because $\mathcal{L} = T - U$ and the potential energy $U$ is independent of $\dot{x}$

# Generalized Momenta (cont'd)

Consider a dynamical system described by $\mathcal{F}$ generalized coordinates $q_i$ (for $i = 1, \ldots, \mathcal{F}$)

By analogy with our previous discussion

$$\Downarrow$$

we can define generalized momenta of the form

$$p_i = \frac{\partial \mathcal{L}}{\partial \dot{q}_i}$$

$p_i$ ☞ is sometimes called the momentum conjugate to the coordinate $q_i$

Lagrange's equation

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) - \frac{\partial \mathcal{L}}{\partial q_i} = 0$$

can be re-written as

$$\frac{dp_i}{dt} = \frac{\partial \mathcal{L}}{\partial q_i}$$

Note that a generalized momentum
does not necessarily have the dimensions of linear momentum

# Generalized Momenta (cont'd)

Suppose that the Lagrangian $\mathcal{L}$ does not depend explicitly on some coordinate $q_k$

$$\Downarrow$$

$$\frac{dp_k}{dt} = \frac{\partial \mathcal{L}}{\partial q_k} = 0$$

$$\Downarrow$$

$$p_k = \text{const.}$$

The coordinate $q_k$ is said to be ignorable in this case

CONCLUSION

the generalized momentum associated with an ignorable coordinate
is a constant of the motion

EXAMPLE

The Lagrangian for a particle moving in a central potential
is independent of the angular coordinate $\theta$

$$\Downarrow$$

$\theta$ ☞ is an ignorable coordinate

$p_\theta = \frac{\partial \mathcal{L}}{\partial \dot{\theta}} = m\, r^2\, \dot{\theta}$ is a constant of the motion

Of course ☞ $p_\theta$ is the angular momentum about the origin
which is conserved because a central force exerts no torque about the origin

# Hamilton's Equations

Consider a dynamical system with $\mathcal{F}$ degrees of freedom
described by the generalized coordinates $q_i$ ☞ $i = 1, \ldots, \mathcal{F}$
Suppose that neither $T$ nor $U$ depend explicitly on the time $t$
In conventional dynamical systems

- the potential energy is generally independent of the $\dot{q}_i$
- the kinetic energy takes the form of a homogeneous quadratic function of the $\dot{q}_i$

$$T = \sum_{i,j=1,\mathcal{F}} m_{ij}\, \dot{q}_i\, \dot{q}_j$$

- $m_{ij}$ depends on the $q_i$ but not on the $\dot{q}_i$

$$\Downarrow$$

$$\sum_{i=1,\mathcal{F}} \dot{q}_i \frac{\partial T}{\partial \dot{q}_i} = 2\,T$$

The generalized momentum conjugate to the $i$th generalized coordinate is

$$p_i = \frac{\partial \mathcal{L}}{\partial \dot{q}_i} = \frac{\partial T}{\partial \dot{q}_i}$$

- $\mathcal{L} = T - U$ is the Lagrangian of the system
- $U$ is independent of the $\dot{q}_i$

# Hamilton's Equations (cont'd)

Consider the function

$$H = \sum_{i=1,\mathcal{F}} \dot{q}_i \, p_i - \mathcal{L} = \sum_{i=1,\mathcal{F}} \dot{q}_i \, p_i - T + U$$

If all of the conditions discussed above are satisfied

$$\Downarrow$$

$$H = T + U$$

In other words ☞ the function $H$ is equal to the total energy of the system

Consider the variation of the function $H$

$$\delta H = \sum_{i=1,\mathcal{F}} \left( \delta\dot{q}_i \, p_i + \dot{q}_i \, \delta p_i - \frac{\partial \mathcal{L}}{\partial \dot{q}_i} \, \delta\dot{q}_i - \frac{\partial \mathcal{L}}{\partial q_k} \, \delta q_k \right)$$

The first and third terms in the bracket cancel because ☞ $p_i = \partial \mathcal{L}/\partial \dot{q}_i$

Now ☞ Lagrange's equation can be written as $\dot{p}_i = \partial \mathcal{L}/\partial q_i$

$$\Downarrow$$

$$\delta H = \sum_{i=1,\mathcal{F}} (\dot{q}_i \, \delta p_i - \dot{p}_i \, \delta q_i) \qquad (\clubsuit)$$

# Hamilton's Equations (cont'd)

Suppose now ☞ that we can express the total energy of the system $H$ solely as a function of the $q_i$ and the $p_i$ ☞ with no explicit dependence on the $\dot{q}_i$

In other words ☞ suppose that we can write $H = H(q_i, p_i)$

When the energy is written in this fashion

$$\Downarrow$$

it is generally termed the Hamiltonian of the system

The variation of the Hamiltonian function takes the form

$$\delta H = \sum_{i=1,\mathcal{F}} \left( \frac{\partial H}{\partial p_i}\, \delta p_i + \frac{\partial H}{\partial q_i}\, \delta q_i \right) \qquad (\spadesuit)$$

A comparison of Eqs. (♣) and (♠) yields

$$\dot{q}_i = \frac{\partial H}{\partial p_i} \qquad \text{and} \qquad \dot{p}_i = -\frac{\partial H}{\partial q_i}$$

for $i = 1, \ldots, \mathcal{F}$

These $2\mathcal{F}$ first-order differential equations are known as Hamilton's equations

Hamilton's equations are often a useful alternative to Lagrange's equations which take the form of $\mathcal{F}$ second-order differential equations

# 1D Harmonic Oscillator

Consider a one-dimensional harmonic oscillator

The kinetic and potential energies of the system are written

$$T = (1/2)\, m\, \dot{x}^{\,2} \qquad \text{and} \qquad U = (1/2)\, k\, x^2$$

- $x$ ☞ is the displacement
- $m$ ☞ is the mass
- $k > 0$

The generalized momentum conjugate to $x$ is

$$p = \frac{\partial T}{\partial \dot{x}} = m\, \dot{x} \Rightarrow T = \frac{1}{2}\, \frac{p^2}{m}$$

The Hamiltonian of the system takes the form

$$H = T + U = \frac{1}{2}\, \frac{p^2}{m} + \frac{1}{2}\, k\, x^2$$

Hamilton's equations yield

$$\dot{x} = \frac{\partial H}{\partial p} = \frac{p}{m} \qquad \text{(which is just a restatement of the kinetic energy)}$$

$$\dot{p} = -\frac{\partial H}{\partial x} = -k\, x \qquad \text{(Newton's second law of motion for the system)}$$

# Motion in a Central Potential

Consider a particle of mass $m$ moving in the central potential $U(r)$

The kinetic energy is

$$T = \frac{1}{2}\, m\, (\dot{r}^{\,2} + r^2\, \dot{\theta}^{\,2})$$

$\Rightarrow$ $(r, \theta)$ ☞ are plane polar coordinates

The generalized momenta conjugate to $r$ and $\theta$ are

$$p_r = \frac{\partial T}{\partial \dot{r}} = m\, \dot{r}$$

$$p_\theta = \frac{\partial T}{\partial \dot{\theta}} = m\, r^2\, \dot{\theta}$$

$$\Downarrow$$

$$T = \frac{1}{2\, m} \left( p_r^2 + \frac{p_\theta^2}{r^2} \right)$$

The Hamiltonian of the system takes the form

$$H = \frac{1}{2\, m} \left( p_r^2 + \frac{p_\theta^2}{r^2} \right) + U(r)$$

# Motion in a Central Potential (cont'd)

Hamilton's equations yield

$$\dot{r} = \frac{\partial H}{\partial p_r} = \frac{p_r}{m} \qquad \text{and} \qquad \dot{\theta} = \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{m\, r^2}$$

(which are just restatements of the generalized momenta)

$$\dot{p}_r = -\frac{\partial H}{\partial r} = \frac{p_\theta^2}{m\, r^3} - \frac{\partial U}{\partial r} \qquad (\digamma)$$

$$\dot{p}_\theta = -\frac{\partial H}{\partial \theta} = 0$$

The last equation implies that

$$\frac{p_\theta}{m} = r^2\, \dot{\theta} = h \qquad (\kappa)$$

$\Rightarrow$ $h$ ☞ is a constant

This can be combined with Eq. $(\digamma)$ to give

$$\frac{\dot{p}_r}{m} = \ddot{r} = \frac{h^2}{r^3} - \frac{\partial V}{\partial r} \qquad (\varkappa)$$

$\Rightarrow$ $V = U/m$

Of course ☛ Eqs. $(\kappa)$ and $(\varkappa)$ are the conventional equations of motion for a particle moving in a central potential

# Motion in a Central Potential

**Consider a particle of mass $m$ moving in 2D in the central potential $U(r)$**
**this is clearly a two degree of freedom dynamical system**
**The particle's instantaneous position is most conveniently specified**

$$\Downarrow$$

**plane polar coordinates $r$ and $\theta$**

**these are our two generalized coordinates**
**The square of the particle's velocity can be written as**

$$v^2 = \dot{r}^{\,2} + (r\,\dot{\theta})^2$$

**the Lagrangian of the system takes the form**

$$\mathfrak{L} = \frac{1}{2}\,m\,(\dot{r}^{\,2} + r^2\,\dot{\theta}^{\,2}) - U(r)$$

$$\frac{\partial \mathfrak{L}}{\partial \dot{r}} = m\,\dot{r} \qquad \frac{\partial \mathfrak{L}}{\partial r} = m\,r\,\dot{\theta}^{\,2} - dU/dr$$

$$\frac{\partial \mathfrak{L}}{\partial \dot{\theta}} = m\,r^2\,\dot{\theta} \qquad \frac{\partial \mathfrak{L}}{\partial \theta} = 0$$

# Motion in a Central Potential (cont'd)

**Equations of motion ☞ Lagrange equations**

$$\frac{d}{dt}\left(\frac{\partial \mathfrak{L}}{\partial \dot{r}}\right) - \frac{\partial \mathfrak{L}}{\partial r} = 0 \qquad \frac{d}{dt}\left(\frac{\partial \mathfrak{L}}{\partial \dot{\theta}}\right) - \frac{\partial \mathfrak{L}}{\partial \theta} = 0$$

$$\frac{d}{dt}(m\,\dot{r}) - m\,r\,\dot{\theta}^{\,2} + \frac{dU}{dr} = 0$$

$$\frac{d}{dt}\left(m\,r^2\,\dot{\theta}\right) = 0$$

$$\ddot{r} - r\,\dot{\theta}^{\,2} = -\frac{dV}{dr}$$

$$r^2\,\dot{\theta} = h$$

☞ $V = U/m$

☞ $h =$ constant

# Phase Space

Recall that

$\Downarrow$

The generalized coordinates $q_j$ define an $s$ dimensional configuration space
with every point representing a state of the system
Likewise

$\Downarrow$

The generalized momenta $p_j$ define an $s$-dimensional momentum space
representing a certain condition of motion of the system
**Hamilton phase space** ☞ $2s$ dimensional space consisting of $q_j$ and $p_j$
which allows us to represent both the positions and the momenta of all particles
If the position and momenta of all the particles in a system are known
(at a given time)

$\Downarrow$

these quantities can be used as initial conditions ☞ $(q_j(0), p_j(0))$

$\Downarrow$

- the subsequent motion of the system is completely determined
- the representative point describing the system moves along a unique path phase

# Phase Diagrams

State of motion of 1-dimensional oscillator is completely specified by 2 quantities
$x(t)$ and $\dot{x}(t)$
(two quantities needed because differential equation of motion is second order)

$x(t)$ and $\dot{x}(t)$ define coordinates of points in a 2-dimensional space

$\Downarrow$

**The Phase Space**

In two dimensions the phase space is a phase plane

for general oscillator with $n$ degrees of freedom ☞ 2n-dimensional phase space

- ☛ As the time varies the point $P(x, \dot{x})$ describing the state of the oscillating particle will move along a certain phase path in the phase plane
- ☛ For different initial conditions the motion will be described by different paths
- ☛ The totality of all phase paths constitutes the phase diagram of the oscillator

# Phase Diagrams (cont'd)

## simple harmonic oscillator

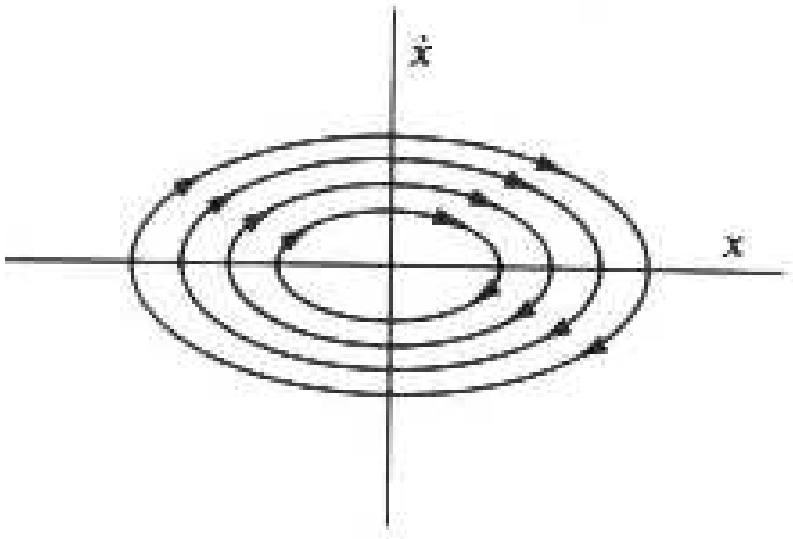
$$x(t) = A\sin(\omega_0 t - \delta)$$

$$\dot{x}(t) = A\omega_0 \cos(\omega_o t - \delta)$$

Eliminating $t$

$$\frac{x^2}{A^2} + \frac{\dot{x}^2}{A^2\omega_0^2} = 1$$

This equation represents a family of elipses



# Phase Diagrams (cont'd)

Total Energy $E = kA^2/2$ Angular Frequency $\omega_0^2 = k/m$

$$\Downarrow$$

$$\frac{x^2}{2E/k} + \frac{\dot{x}^2}{2E/m} = 1$$

**Each phase path corresponds to a definite total energy of the oscillator**

**No two phase paths of the oscillator can cross!!!**

If they could cross ☞ this would imply that for a given set of initial conditions the motion could proceed along different paths

$$\Downarrow$$

This is impossible since the solution of the differential equation is unique

# Statistical Mechanics

For complex systems with a large collection of particles ☞ gas molecules

⇓

we are unable to identify the particular phase space point representing the system

⇓

we must devise some alternative approach to study the dynamics of such sytems
One possibility is to fill the phase space with a collection of points
each representing a *possible* condition of the system

⇓

☞ any of which could be the actual system

We imagine a large number of systems

☞ each consistent with the known constraints

We are unable to discuss the details of the particles' motion in the actual system

⇓

we substitute a discussion of an ensamble of equivalent systems
Each representative point corresponds to a single system of the ensamble
the motion of a particular point represents the independent motion of that system
No two of the phase path may ever intersect!!!

# Liouville's Theorem

THEOREM

– Joseph Liouville (1809-1882) –

The density of representative points in phase space corresponding to the motion of a system of particles remains constant during the motion

$\Downarrow$

Define a density in phase $\rho$

- must be sufficiently large to contain a large number of representative points

The volume elements of the phase space defining the density

- must be also sufficiently small so that the density varies continuously

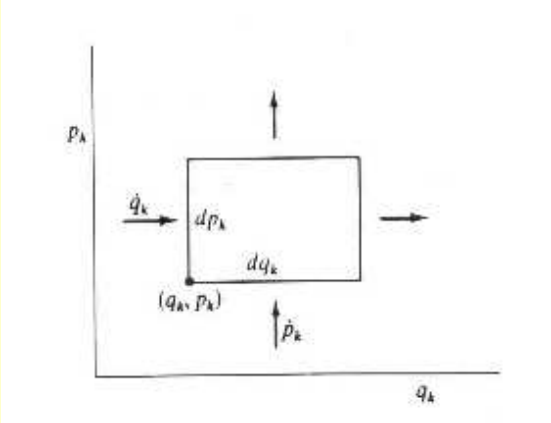
The number $N$ of systems whose representative points lie within a volume $dv$ is

$$N = \rho\, dv$$

- $dv = dq_1\, dq_2 \ldots dq_s\, dp_1\, dp_2 \ldots dp_s$
- $s$ ☞ number of degrees of freedom in the ensamble

# Liouville's Theorem (cont'd)

Consider an element of area in the $q_k - p_k$ plane in phase space




the number of representative points
moving across the left-hand edge into the area per unit time is

$$\rho \frac{dq_k}{dt}\, dp_k = \rho \dot{q}_k\, dp_k$$

and the number moving across the lower edge into the area per unit time is

$$\rho \frac{dp_k}{dt}\, dq_k = \rho \dot{p}_k\, dq_k$$

the total number of representative points
moving into the area $dq_k\ dp_k$ per unit time is

$$\rho(\dot{q}_k\, dp_k + \dot{p}_k\, dq_k)$$

# Liouville's Theorem (cont'd)

By Taylor series expansion
the number of representative points moving out of the area per unit time is

$$\left[\rho\dot{q}_k + \frac{\partial}{\partial q_k}(\rho\dot{q}_k)\,dq_k\right] dp_k + \left[\rho\dot{p}_k + \frac{\partial}{\partial p_k}(\rho\dot{p}_k)\,dp_k\right] dq_k$$

the total increase in density in $dq_k\,dp_k$ per unit time is

$$\frac{\partial\rho}{\partial t}\,dq_k\,dp_k = -\left[\frac{\partial}{\partial q_k}(\rho\dot{q}_k) + \frac{\partial}{\partial p_k}(\rho\dot{p}_k)\right] dq_k\,dp_k$$

After dividing by $dq_k\,dp_k$ and summing this expression over all possible values of $k$

$$\frac{\partial\rho}{\partial t} + \sum_{k=1}^{s}\left(\frac{\partial\rho}{\partial q_k}\dot{q}_k + \rho\frac{\partial\dot{q}_k}{\partial q_k} + \frac{\partial\rho}{\partial p_k}\dot{p}_k + \rho\frac{\partial\dot{p}_k}{\partial p_k}\right) = 0$$

If the second partial derivatives of $H$ are continuos ☞ Hamilton's equations yield

$$\frac{\partial\dot{q}_k}{\partial q_k} + \frac{\partial\dot{p}_k}{\partial p_k} = 0$$

$$\Downarrow$$

$$\frac{\partial\rho}{\partial t} + \sum_{k=1}^{s}\left(\frac{\partial\rho}{\partial q_k}\frac{dq_k}{dt} + \frac{\partial\rho}{\partial p_k}\frac{dp_k}{dt}\right) = 0 \Rightarrow \frac{d\rho}{dt} = 0$$

# The Virial Theorem

Consider a collection of particles
whose position vectors $\vec{r}_\alpha$ and momenta $\vec{p}_\alpha$ are bounded
☞ i.e., remain finite at all times

Define a quantity

$$S \equiv \sum_\alpha \vec{p}_\alpha \ . \ \vec{r}_\alpha$$

$$\Downarrow$$

the time derivative of $S$ is

$$\frac{dS}{dt} = \sum_\alpha (\vec{p}_\alpha \ . \ \dot{\vec{r}}_\alpha + \dot{\vec{p}}_\alpha \ . \ \vec{r}_\alpha)$$

$$\Downarrow$$

The average value of $dS/dt$ over a time interval $\tau$ is

$$\begin{aligned} \left\langle \frac{dS}{dt} \right\rangle &= \frac{1}{\tau} \int_0^\tau \frac{dS}{dt} \, dt \\ &= \frac{S(\tau) - S(0)}{\tau} \end{aligned}$$

# The Virial Theorem (cont'd)

$$\left\langle \frac{dS}{dt} \right\rangle = \frac{S(\tau) - S(0)}{\tau}$$

If the system motion is periodic $\Rightarrow S(\tau) = S(0)$ and $\langle \dot{S} \rangle$ ☞ vanishes
If the system does not exhibit any periodicity

$$\Downarrow$$

we can make $\langle \dot{S} \rangle$ as small as desired by allowing $\tau \to \infty$

$$\Downarrow$$

in this limit

$$\left\langle \sum_\alpha \vec{p}_\alpha \cdot \dot{\vec{r}}_\alpha \right\rangle = -\left\langle \sum_\alpha \dot{\vec{p}}_\alpha \cdot \vec{r}_\alpha \right\rangle \Rightarrow \left\langle 2\sum_\alpha T_\alpha \right\rangle = -\left\langle \sum_\alpha \vec{F}_\alpha \cdot \vec{r}_\alpha \right\rangle$$

The sum over $T_\alpha$ is the total kinetic energy of the system

$$\langle T \rangle = -\frac{1}{2}\left\langle \sum_\alpha \vec{F}_\alpha \cdot \vec{r}_\alpha \right\rangle \qquad (\mathfrak{V})$$

The rhs of Eq. $(\mathfrak{V})$ was called by Clausius (1822-1888) the virial of the system
*The average kinetic energy of a system of particles is equal to its virial*

# The Virial Theorem (cont'd)

If the forces $\vec{F}_\alpha$ can be derived from potentials $U_\alpha$

$$\Downarrow$$

$$\langle T \rangle = \frac{1}{2} \left\langle \sum_\alpha \vec{r}_\alpha \,.\, \vec{\nabla} U_\alpha \right\rangle$$

If two particles interact according to a central power law force

$$F \propto r^n \text{ ☞ } U = k r^{n+1}$$

$$\Downarrow$$

$$\vec{r} \,.\, \vec{\nabla} U = r \frac{dU}{dr} = k(n+1) r^{n+1} = (n+1) U$$

and the Virial Theorem becomes

$$\langle T \rangle = \frac{n+1}{2} \langle U \rangle$$

If the particles have gravitational interaction ☞ $n = -2$

$$\Downarrow$$

$$\langle T \rangle = -\frac{1}{2} \langle U \rangle$$

# Homework

A particle moves in a circular orbit in a force field given by $F(r) = -k/r^2$
Show that ☞ if $k$ suddenly decreases to half its original value

$$\Downarrow$$

the particle's orbit is parabolic

CLUES

☞ the potential energy decreases to half its former value

When $k \longrightarrow k/2$

☞ the kinetic energy remains the same

Since the orbital is circular ☞ $T = \langle T \rangle$ and $U = \langle U \rangle$
For $F \propto 1/r^2$ ☞ the Virial Theorem states

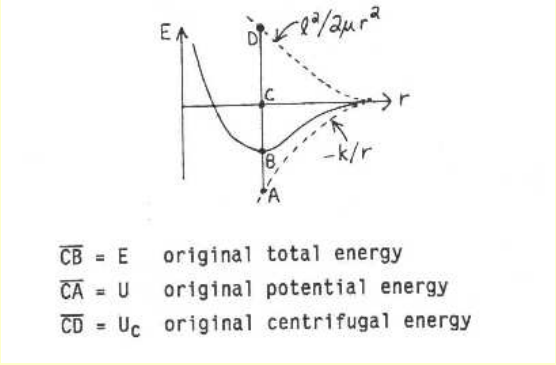
$$\langle T \rangle = -\frac{1}{2} \langle U \rangle$$

$$\Downarrow$$

$$E = T + U = -\frac{1}{2}U + U = \frac{1}{2}U$$

# Homework (cont'd)

Consider the energy diagram




The point $B$ is obtained from $\overline{CB} = \overline{CA} - \overline{CD}$

According to the Virial Theorem ☞ $E = U/2 \rightarrow \overline{CB} = \overline{CA}/2$

$$\Downarrow$$

$$\overline{CD} = \overline{CB} = \overline{BA}$$

$$\Downarrow$$

If $U$ suddenly is halved ☞ the total energy is raised by an amount equal to $\overline{CB}$

$$\Downarrow$$

the total energy is raised from $B$ to $C$ ☞ $E_{\rm final} = 0$ and the orbit is parabolic