Classical Mechanics

Dynamics of a System of Particles

- ☞ Center of Mass ✓
- ☞ Liner Momentum of the System ✓
- ☞ Angular Momentum of the System ✓
- ☞ Energy of the System ✓
- ☞ Elastic and Inelastic Collisions ✓
- ☞ Reduced Mass ✓
- ☞ Cross Sections

Warm-up: Reduced Mass

- **Consider 2 objects**
  - $1^{\rm st}$ object is of mass m_1 and is located at position vector $\vec{r}_1$
  - $2^{\rm nd}$ object is of mass m_2 and is located at position vector $\vec{r}_2$

Let the first object exert a force $\vec{f}_{21}$ on the second

$\Downarrow$

Newton's Third Law ☛ $2^{\rm nd}$ object exerts an equal and opposite force on the $1^{\rm st}$

$$\vec{f}_{12} = -\vec{f}_{21}$$

Suppose that there are no other forces in the problem

$\Downarrow$

The equations of motion of our two objects are

$$m_1 \frac{d^2\vec{r}_1}{dt^2} = -\vec{f}$$

$$m_2 \frac{d^2\vec{r}_2}{dt^2} = \vec{f}$$

☞ $\vec{f} = \vec{f}_{21}$

Warm-up: Reduced Mass (cont'd)

The center of mass of our system is located at

$$\vec{r}_{\rm cm} = \frac{m_1\,\vec{r}_1 + m_2\,\vec{r}_2}{m_1 + m_2}$$

$$\Downarrow$$

$$\vec{r}_1 = \vec{r}_{\rm cm} - \frac{m_2}{m_1+m_2}\,\vec{r} \qquad\qquad \vec{r}_2 = \vec{r}_{\rm cm} + \frac{m_1}{m_1+m_2}\,\vec{r}$$

☞ $\vec{r} = \vec{r}_2 - \vec{r}_1$

Substituting into the equations of motion
(making use of the fact that the c.m. of an isolated system does not accelerate)
we find that both equations yield

$$\mu\,\frac{d^2\vec{r}}{dt^2} = \vec{f}$$

$\mu = \frac{m_1\,m_2}{m_1+m_2}$ ☞ is called the reduced mass

$$\Downarrow$$

We have effectively converted a 2-body problem into an equivalent 1-body problem

In the equivalent problem

- $\vec{f}$ is the same as that acting on both objects (modulo a minus sign)
- the mass μ is different and is less than either of m_1 or m_2

Cross Sections

Let us now consider scattering due to the collision of two particles

- $1^{\rm st}$ particle is of mass m_1 and is located at position vector $\vec{r}_1$
- $2^{\rm nd}$ particle is of mass m_2 and is located at position vector $\vec{r}_2$

We restrict our discussion to particles which interact via conservative central forces

By definition ☞ there is zero net linear momentum in the c.m. frame at all times

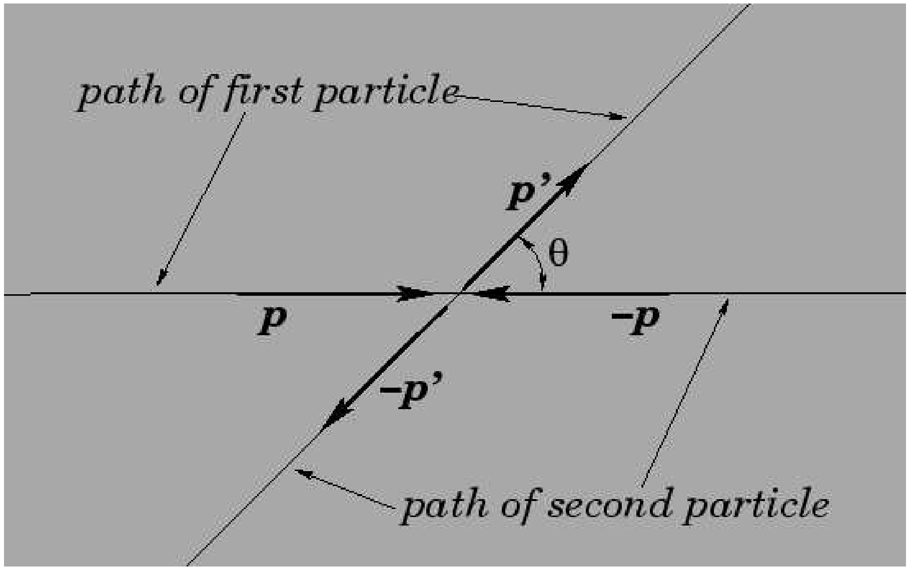
if the $1^{\rm st}$ particle approaches the collision point with momentum $\vec{p}$ $\Downarrow$

the $2^{\rm nd}$ must approach with momentum $-\vec{p}$

Likewise ☞

if the $1^{\rm st}$ particle recedes from the collision point with momentum $\vec{p'}$ $\Downarrow$

then the $2^{\rm nd}$ must recede with momentum $-\vec{p'}$



Cross Sections (cont'd)

The interaction force is conservative

$\Downarrow$

the total kinetic energy before and after the collision must be the same

It follows that

☞ **magnitude of the final momentum vector $\vec{p}\,'$**
is equal to the magnitude of the initial momentum vector $\vec{p}$

$\Downarrow$

the collision event is completely specified
once the angle θ through which the first particle is scattered is given

Recall that
in the c.m. frame the $2^{\rm nd}$ particle is scattered through the same angle

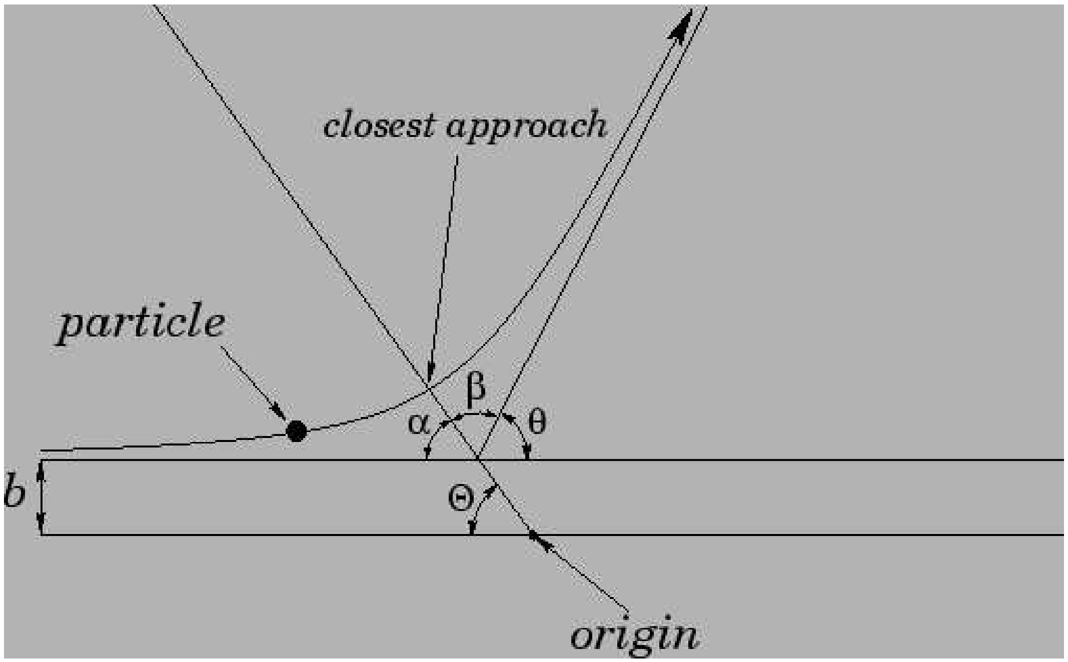
Cross Sections (cont'd)

Suppose that the two particles interact via a potential $U(r)$
(where r is the distance separating the particles)

⇓

the 2-body problem can be converted into the equivalent 1-body problem

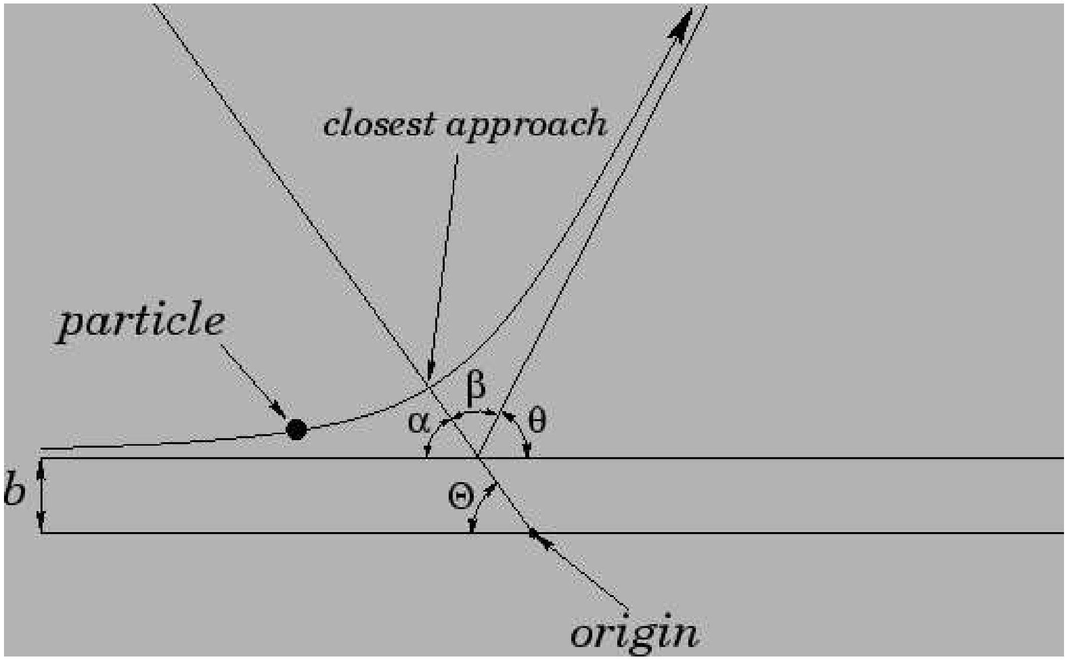
In this equivalent problem
a particle of mass $\mu = m_1\, m_2/(m_1 + m_2)$ is scattered in the fixed potential $U(r)$
(where r is now the distance from the origin)



Cross Sections (cont'd)

- The $\vec{r}$ of the particle in the equivalent problem
  - ☞ relative position vector $\vec{r}_2 - \vec{r}_1$ in the original problem
- The angle θ through which the particle is scattered is the same in both set ups

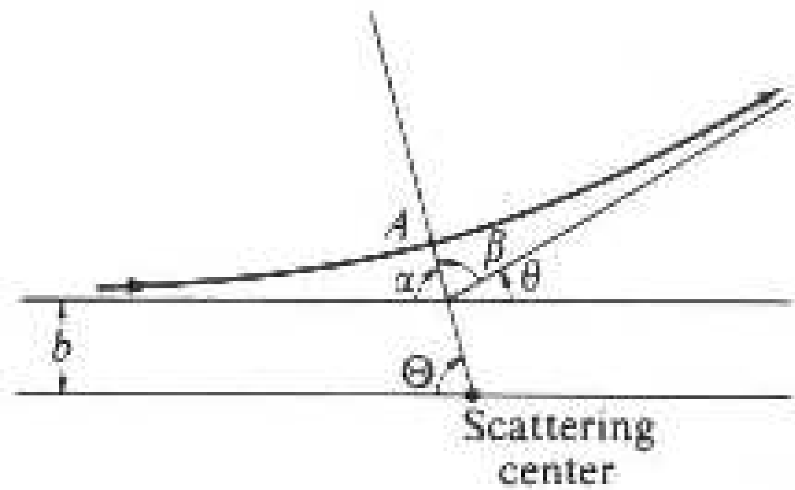
The scattering angle θ is largely determined by the so-called impact parameter b which is the distance of closest approach of the two particles in the absence of an interaction potential



In the equivalent problem b is the distance of closest approach to the origin in the absence of an interaction potential

Cross Sections (cont'd)

If $b = 0$ ☞ head-on collision ⇔ particles reverse direction after colliding ☚ $\theta = \pi$
If b is large ☞ we expect the two particles to miss one another entirely ☚ $\theta = 0$
The scattering angle θ is a decreasing function of the impact parameter b



Take plane polar coordinates (r, ϑ) for the particle in the equivalent problem

let particle approach the origin from the direction $\vartheta = 0$
and attain its closest distance to the origin when $\vartheta = \Theta$

from symmetry ☞ the angle $\alpha = \beta$ ⇓ from simple geometry ☞ $\alpha = \Theta$

$$\theta = \pi - 2\,\Theta \qquad (\dagger)$$

Recall that

The total energy of our planet is a conserved quantity

$$\mathcal{E} = \frac{v^2}{2} - \frac{G\,M}{r}$$

and that

$$\vec{v} = \dot{r}\ \boldsymbol{e}_r + r\,\dot{\theta}\ \boldsymbol{e}_\theta$$

$$\Downarrow$$

the total energy per unit mass of an object in orbit around the Sun is

$$\mathcal{E} = \frac{\dot{r}^{\,2} + r^2\,\dot{\theta}^{\,2}}{2} - \frac{G\,M}{r}$$

using

- $h = r^2\,\dot{\theta}$
- $u = r^{-1}$
- $\dot{r} = -h\,\frac{du}{d\theta}$
- r_c ☞ latus rectum of the orbit
- $r_c = \frac{h^2}{G\,M}$
- $u_c = r_c^{-1}$

the orbital energy per unit mass is

$$\mathcal{E} = \frac{h^2}{2}\left[\left(\frac{du}{d\theta}\right)^2 + u^2 - 2\,u\,u_c\right]$$

Cross Sections (cont'd)

By analogy with the orbital energy per unit mass
the conserved total energy E in the equivalent problem
(which can be shown to be the same as the total energy in the original problem)
is given by

$$E = \frac{\mu\, h^2}{2} \left[\left(\frac{du}{d\vartheta} \right)^2 + u^2 \right] + U(u),$$

- $u = r^{-1}$
- h is the angular momentum per unit mass in the equivalent problem

The impact parameter specifies the angular momentum per unit mass

$$\begin{aligned} h &= b\, v_\infty \\ &= b \left(\frac{2\, E}{\mu} \right)^{1/2} \equiv b \left(\frac{2\, T_0^*}{\mu} \right)^{1/2} \end{aligned}$$

v_∞ ☞ the approach velocity in the equivalent problem at large r
the total energy E must equal the kinetic energy T_0^* at $r = \infty$ where $U = 0$

Cross Sections (cont'd)

It follows that

$$T_0^* = T_0^*\, b^2 \left[\left(\frac{du}{d\vartheta} \right)^2 + u^2 \right] + U(u)$$

The above equation can be rearranged to give

$$\frac{d\vartheta}{du} = \frac{b}{\sqrt{1 - b^2\, u^2 - U(u)/T_0^*}}$$

Integration yields

$$\Theta = \int_0^{u_{max}} \frac{b\, du}{\sqrt{1 - b^2\, u^2 - U(u)/T_0^*}} \qquad (\ddagger)$$

$u_{max} = 1/r_{min}$ ☞ r_{min} is the distance of closest approach
by symmetry ☞ $(du/d\vartheta)_{u_{max}} = 0 \Rightarrow 1 - b^2\, u_{max}^2 - U(u_{max})/T_0^* = 0$

Equations (†) and (‡) enable us to calculate the function $b(\theta)$
for given interaction potential $U(r)$ and energy T_0^* of the two particles in the c.m.

$b(\theta)$ tells us which impact parameter corresponds to which scattering angle and vice versa

Cross Sections (cont'd)

Instead of two particles

$\Downarrow$

consider two counter-propagating beams of identical particles
(with the same properties as the two particles described above)
which scatter one another via binary collisions

What is the angular distribution of the scattered particles?

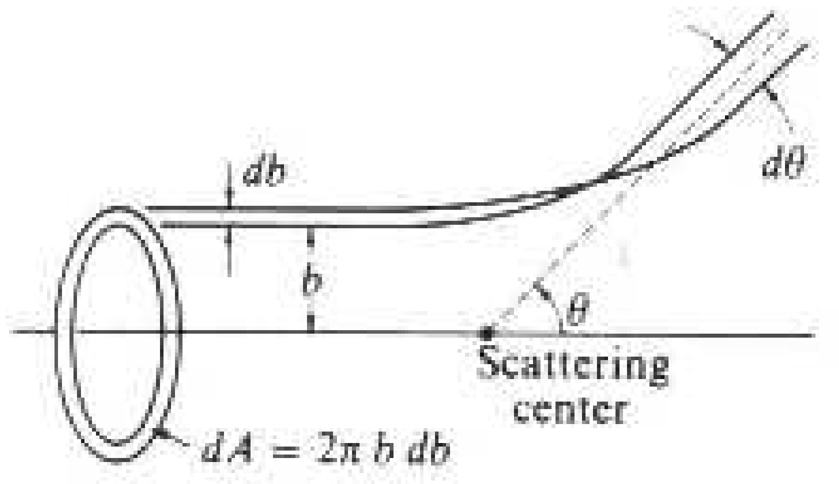
- Consider pairs of particles whose impact parameters lie in the range b to $b + db$
- The particles scattering angles lie in the range θ to $\theta + d\theta$
  θ ☞ is determined from inverting the function $b(\theta)$ and $d\theta = \frac{db}{|db(\theta)/d\theta|}$
  [the modulus of $db(\theta)/d\theta$ is because $b(\theta)$ is a decreasing function of θ]

Assuming (as seems reasonable) that the scattering is azimuthally symmetric
the range of solid angle into which the particles are scattered is

$$d\Omega = 2\pi \,\sin\theta \,d\theta = \frac{2\pi\,\sin\theta\,db}{|db/d\theta|}$$

Cross Sections (cont'd)

The annulus cross-sectional area through which incoming particles must pass if they are to have impact parameters in the range b to $b + db$ is



$$d\sigma = 2\pi\, b\, db$$

The differential scattering cross-section is then

$$\frac{d\sigma}{d\Omega} = \frac{b}{\sin\theta}\left|\frac{db}{d\theta}\right|$$

Cross Sections (cont'd)

The differential scattering cross-section has units of area per steradian
It specifies the effective target area for scattering into a given range of solid angle
For two uniform beams scattering off one another
the differential scattering cross-section
effectively specifies the probability of scattering into a given range of solid angle

The total scattering cross-section is

(the integral of the differential cross-section over all solid angles)

$$\sigma_{\rm tot} = \int \frac{d\sigma}{d\Omega}\, d\Omega$$

$\sigma_{\rm tot}$ measures the effective target area for scattering in any direction

If the flux of particles per unit area per unit time of the two beams is $\mathcal{F}$
(otherwise known as the intensity)

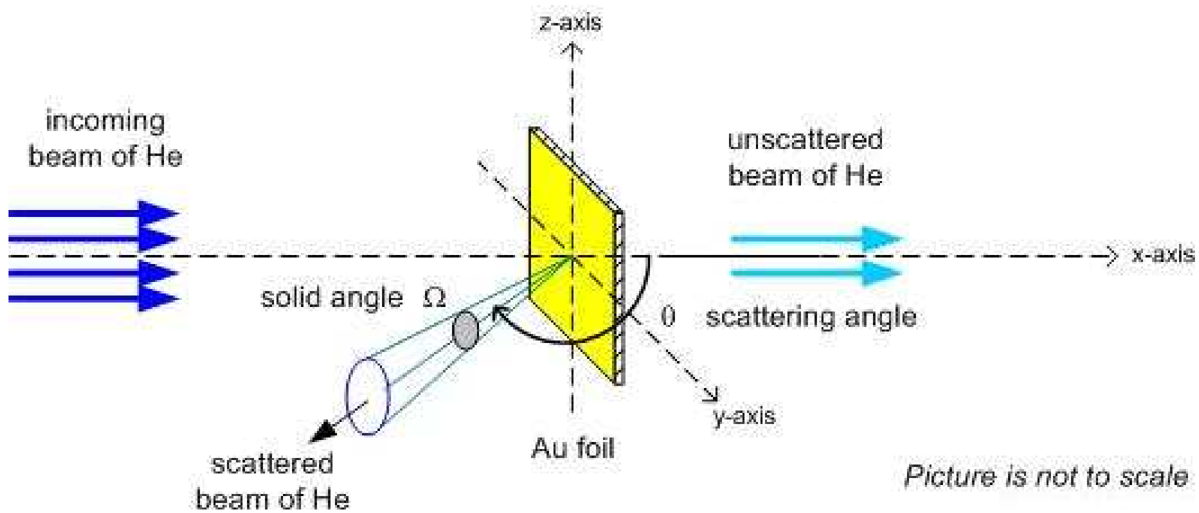
$$\Downarrow$$

the number of particles of a given type scattered per unit time is simply

$$\mathcal{N} = \mathcal{F}\, \sigma$$

Rutherford Scattering Formula

- In 1909 Ernest Rutherford was motivated to study the scattering of alpha particles by thin metal fouls
- He constructed a genious experiment that becomes a standard method to probe the subatomic world: the scattering process



- It is rather remarkable that quantum mechanical treatment of Coulomb scattering leads to same result as does the classical derivation
- This is indeed a fortunate circumstance because, if it were otherwise, the disagreement at this early stage between classical theory and experiment might have seriously delayed the progress of nuclear physics

Rutherford Scattering Formula

Consider the scattering of particles in a Coulomb field

$$U(r) = \frac{1}{4\pi\epsilon_0}\,\frac{q_1\,q_2}{r} = \frac{k}{r}$$

Equation (‡) becomes

$$\Theta = \int_{r_{\min}}^{\infty} \frac{(b/r)\,dr}{\sqrt{r^2 - (k/T_0^*)r - b^2}}$$

Integration leads to

$$\cos\Theta = \frac{(\kappa/b)}{\sqrt{1+(\kappa/b)^2}} \qquad \kappa \equiv \frac{k}{2T_0^*}$$

After a bit of algebra this equation can be re-written as

$$b^2 = \kappa^2\,\tan^2\Theta$$

Using $\Theta = \pi/2 - \theta/2 \Rightarrow b = \kappa\,\cot(\theta/2)$

Rutherford Scattering Formula (cont'd)

$$\frac{db}{d\theta} = -\frac{\kappa}{2}\,\frac{1}{\sin^2(\theta/2)}$$

$$\sigma(\theta) = \frac{b}{\sin\theta}\left|\frac{db}{d\theta}\right| = \frac{\kappa^2}{2}\,\frac{\cot(\theta/2)}{\sin\theta\,\sin^2(\theta/2)}$$

Using the trigonometric identity

$$\sin\theta = 2\sin(\theta/2)\,\cos(\theta/2)$$

we obtain Rutherford scattering formula

$$\sigma(\theta) = \frac{\kappa^2}{4}\,\frac{1}{\sin^4(\theta/2)} = \frac{k^2}{16T_0^{*2}}\,\frac{1}{\sin^4(\theta/2)}$$