Classical Mechanics

Dynamics of a System of Particles

- ☞ Center of Mass
- ☞ Liner Momentum of the System
- ☞ Angular Momentum of the System
- ☞ Energy of the System
- ☞ Reduced Mass
- ☞ Elastic and Inelastic Collisions

Internal Forces Between the Particles in a System

☺ Thus far we have treated our dynamical problems in terms of single particles
even though we have considered extended objects
(such as projectiles, rockets, and planets)
we have not had deal with the internal interactions
between the many particles that make up the extended body

Newton's Third Law plays a prominent role in the dynamic of a system

Assumptions:

✍ The forces exerted by two particles α and β on each other are

equal in magnitude and oposite in direction

If $\vec{f}_{\alpha\beta}$ represents the force on the αth particle due to the βth particle

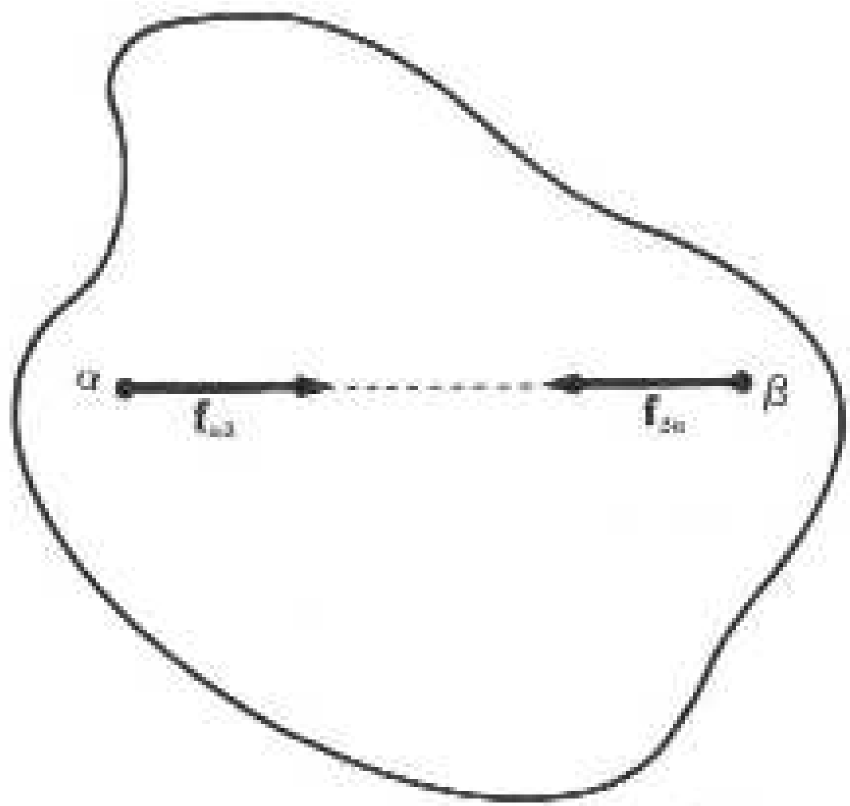
$\Downarrow$

the so called "weak" form of Newton's Third Law is

$$\vec{f}_{\alpha\beta} = -\vec{f}_{\beta\alpha}$$

✍ The forces exerted by two particles α and β on each other
in addition to being equal and opposite
must lie on a stright line joining the two particles
This more restricted form of Newton's Third Law is often called strong form

Strong Form of Newton's Third Law



Center of Mass

Consider a system composed of n particles with masses m_α ☞ $\alpha = 1 \dots n$
the total mass of the system is

$$M = \sum_\alpha m_\alpha$$

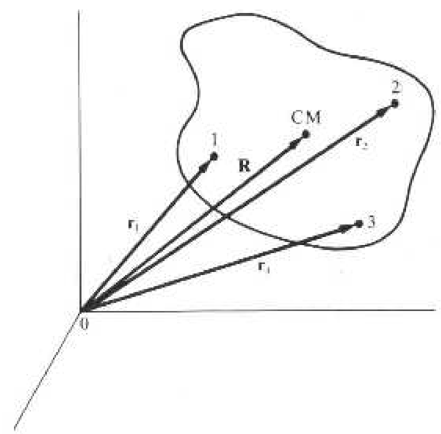
If the vector connecting the origin with the αth particle is $\vec{r}_\alpha$

$$\Downarrow$$

the vector defining the position of the system's center of mass is

$$\vec{R} = \frac{1}{M} \sum_\alpha m_\alpha \vec{r}_\alpha$$

For a continuos distribution of mass ☞ $\vec{R} = \frac{1}{M} \int \vec{r} \ dm$



Linear Momentum of a System

If a certain group of particles constitute a system

⇓

the resultant force acting on a particle within the system is composed of two parts

- external force $\vec{F}_\alpha^{(\mathrm{e})}$
  - ☞ the resultant of all forces whose origin lies outside the system
- internal force $\vec{f}_\alpha$
  - ☞ the resultant of the forces arising from the interaction of all other $(n-1)$ particles with the αth particle

$$\vec{f}_\alpha = \sum_\beta \vec{f}_{\alpha\beta} \Rightarrow \vec{f}_{\alpha\beta} \equiv \text{force on the } \alpha\text{th particle due to } \beta\text{th particle}$$

The total force acting on the αth particle is then

$$\vec{F}_\alpha = \vec{F}_\alpha^{(\mathrm{e})} + \vec{f}_\alpha$$

According to Newton's Third Law

$$\vec{f}_{\alpha\beta} = -\vec{f}_{\beta\alpha}$$

Linear Momentum of a System (cont'd)

$$\dot{\vec{p}}_\alpha = m_\alpha \ddot{\vec{r}} = \vec{F}^{(\mathrm{e})}_\alpha + \vec{f}_\alpha$$

$$\frac{d^2}{dt^2}(m_\alpha \vec{r}_\alpha) = \vec{F}^{(\mathrm{e})}_\alpha + \sum_\beta \vec{f}_{\alpha\beta}$$

summing over α

$$\frac{d^2}{dt^2}\sum_\alpha m_\alpha \vec{r}_\alpha = \sum_\alpha \vec{F}^{(\mathrm{e})}_\alpha + \sum_\alpha \sum_\beta \vec{f}_{\alpha\beta} \quad \alpha\neq\beta$$

- the summation in the left hand side ☞ $M\vec{R}$
- $\alpha = \beta$ do not enter in the second sum of the right hand side ☞ $\vec{f}_{\alpha\alpha} = 0$
- sum of all external forces ☞ $\sum_\alpha \vec{F}^{(\mathrm{e})}_\alpha = \vec{F}$

$$\sum_{\alpha,\beta\neq\alpha} \vec{f}_{\alpha\beta} = \sum_{\alpha<\beta}(\vec{f}_{\alpha\beta} + \vec{f}_{\beta\alpha})$$

which vanishes from Newton's Thrid Law ⇓

$$M\ddot{\vec{R}} = \vec{F}$$

Linear Momentum of a System (cont'd)

The total linear momentum of the system is

$$\vec{P} = \sum_\alpha m_\alpha \dot{\vec{r}}_\alpha = \frac{d}{dt}\sum_\alpha m_\alpha \vec{r}_\alpha = \frac{d}{dt}(M\vec{R}) = M\dot{\vec{R}}$$

$$\Downarrow$$

$$\dot{\vec{P}} = M\ddot{\vec{R}} = \vec{F}$$

Summing up

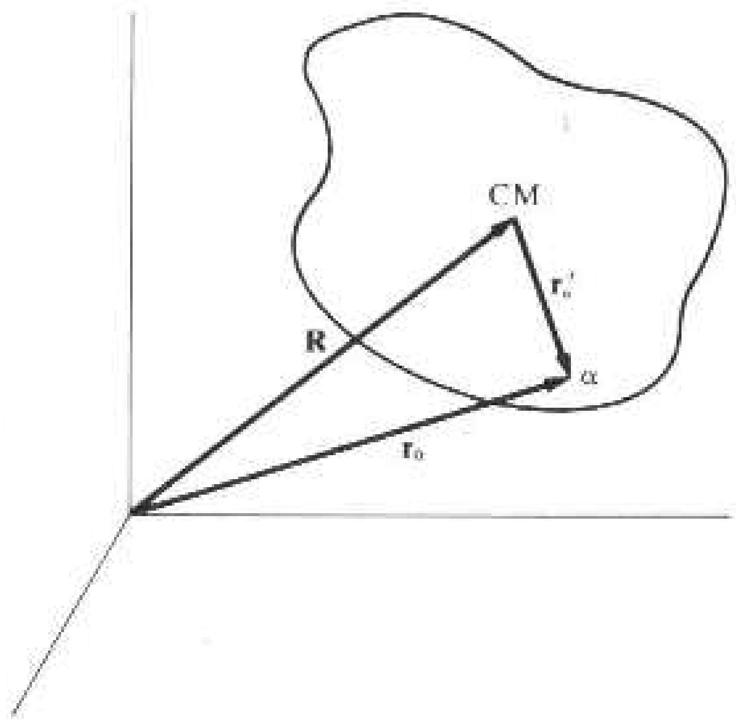
- *The center of mass of a system moves as if it were a single particle (of mass equal to the total mass of the system) acted on by the total external force and independent of the nature of the internal forces (as long as they follow $\vec{f}_{\alpha\beta} = -\vec{f}_{\beta\alpha}$)*
- *The linear momentum of a system is the same as if a single particle of mass M were located at the position of the center of mass and moving in the manner the center of mass moves*
- *The total linear momentum for a system free of external forces is a constant and equal to the momentum of the center of mass*

Angular Momentum of a System (cont'd)

It is convinient to describe a system by a position vector with respect to the c.m
The position vector r_α in the inertial reference frame system becomes

$$\vec{r}_\alpha = \vec{R} + \vec{r'}_\alpha$$

$\vec{r'}_\alpha$ ☞ position of the vector particle α wrt c.m.



Angular Momentum of a System (cont'd)

The angular momentum of the αth particle about the origin is

$$\vec{L}_\alpha = \vec{r}_\alpha \times \vec{p}_\alpha$$

summing over α

$$\begin{aligned}
\vec{L} &= \sum_\alpha \vec{L}_\alpha \\
&= \sum_\alpha \vec{r}_\alpha \times \vec{p}_\alpha \\
&= \sum_\alpha \vec{r}_\alpha \times m_\alpha \dot{\vec{r}}_\alpha \\
&= \sum_\alpha (\vec{r\,}'_\alpha + \vec{R}) \times m_\alpha (\dot{\vec{r}}'_\alpha \times \dot{\vec{R}}) \\
&= \sum_\alpha m_\alpha [(\vec{r\,}'_\alpha \times \dot{\vec{r}}'_\alpha) + (\vec{r\,}'_\alpha \times \dot{\vec{R}}) + (\vec{R} \times \dot{\vec{r}}'_\alpha) + (\vec{R} \times \dot{\vec{R}})]
\end{aligned}$$

Angular Momentum of a System (cont'd)

The middle two terms can be written as

$$\left(\sum_\alpha m_\alpha \vec{r'}_\alpha\right) \times \dot{\vec{R}} + \vec{R} \times \frac{d}{dt}\left(\sum_\alpha m_\alpha \vec{r'}_\alpha\right)$$

which vanishes because

$$\sum_\alpha m_\alpha \vec{r'}_\alpha = \sum_\alpha m_\alpha (\vec{r}_\alpha - \vec{R}) = \sum_\alpha m_\alpha \vec{r}_\alpha - \vec{R}\sum_\alpha m_\alpha$$

$$\sum_\alpha m_\alpha \vec{r'}_\alpha = M\vec{R} - M\vec{R} \equiv 0$$

- This indicates that $\sum_\alpha m_\alpha \vec{r'}_\alpha$ specifies the position of the center of mass in the center of mass coordinate system ☞ it is a null vector

$$\vec{L} = M\vec{R} \times \dot{\vec{R}} + \sum_\alpha \vec{r'}_\alpha \times \vec{p'}_\alpha = \vec{R} \times \vec{P} + \sum_\alpha \vec{r'}_\alpha \times \vec{p'}_\alpha$$

➠ *The total angular momentum about an origin is the sum of the angular momentum of the center of mass about that origin + the angular momentum of the system about the position of the center of mass*

Angular Momentum of a System (cont'd)

The time derivative of the angular momentum of the αth particle is

$$\dot{\vec{L}}_\alpha = \vec{r}_\alpha \times \dot{\vec{p}}_\alpha$$

$$\dot{\vec{L}}_\alpha = \vec{r}_\alpha \times \left(\vec{F}_\alpha^{(\mathrm{e})} + \sum_\beta \vec{f}_{\alpha\beta} \right)$$

summing over alpha

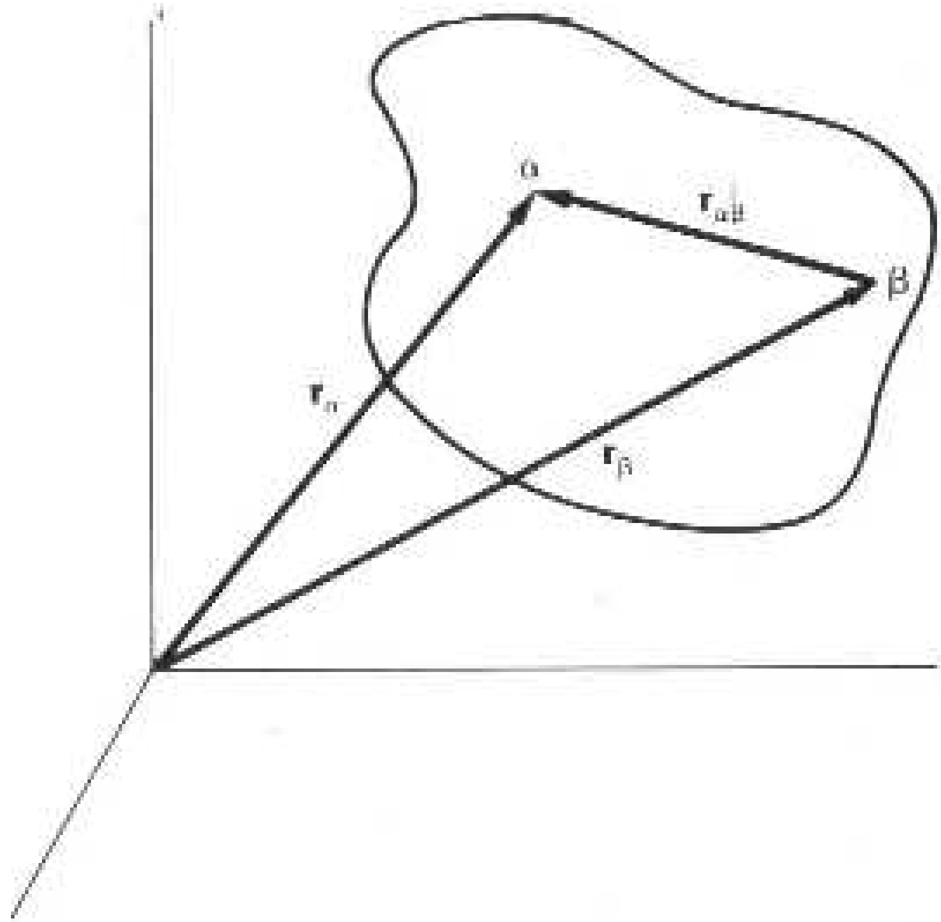
$$\dot{\vec{L}} = \sum_\alpha \dot{\vec{L}}_\alpha = \sum_\alpha (\vec{r}_\alpha \times \vec{F}_\alpha^{(\mathrm{e})}) + \sum_{\alpha,\beta\neq\alpha} (\vec{r}_\alpha \times \vec{f}_{\alpha\beta})$$

the last term may be written as

$$\sum_{\alpha,\beta\neq\alpha} (\vec{r}_\alpha \times \vec{f}_{\alpha\beta}) = \sum_{\alpha<\beta} [(\vec{r}_\alpha \times \vec{f}_{\alpha\beta}) + (\vec{r}_\beta \times \vec{f}_{\beta\alpha})]$$

Angular Momentum of a System (cont'd)

The vector connecting the αth particle and the βth particle is



$$\vec{r}_{\alpha\beta} \equiv \vec{r}_\alpha - \vec{r}_\beta$$

Angular Momentum of a System (cont'd)

since $\vec{f}_{\alpha\beta} = -\vec{f}_{\beta\alpha}$

$$\Downarrow$$

$$\sum_{\alpha,\beta\neq\alpha} (\vec{r}_\alpha \times \vec{f}_{\alpha\beta}) = \sum_{\alpha<\beta} (\vec{r}_\alpha - \vec{r}_\beta) \times \vec{f}_{\alpha\beta}$$
$$= \sum_{\alpha<\beta} (\vec{r}_{\alpha\beta} \times \vec{f}_{\alpha\beta})$$

Applying the strong version of Newton's Third Law ☞ $\vec{r}_{\alpha\beta} \times \vec{f}_{\alpha\beta} \equiv 0$

$$\Downarrow$$

$$\dot{\vec{L}} = \sum_\alpha \vec{r}_\alpha \times \vec{F}^{(\mathrm{e})}_\alpha = \sum_\alpha \tau^{(\mathrm{e})}_\alpha = \tau^{(\mathrm{e})}$$

If the resultant of external torques about a given axis vanish
☞ *angular momentum of the system about that axis remains constant in time*

Recall that:

If some work W_{12} is done on a particle by a force $\vec{F}$ in transforming the particle from condition 1 to condition 2 $\Rightarrow W_{12} \equiv \int_1^2 \vec{F} \,.\, d\vec{r}$

$$\begin{aligned}
\vec{F} \,.\, d\vec{r} &= m\frac{d\vec{v}}{dt} \,.\, \frac{d\vec{r}}{dt}dt \\
&= m\frac{d\vec{v}}{dt} \,.\, vdt \\
&= \frac{m}{2}\frac{d}{dt}(\vec{v} \,.\, \vec{v})dt \\
&= \frac{m}{2}\frac{d}{dt}(v^2)dt \\
&= d\left(\frac{1}{2}mv^2\right)
\end{aligned}$$

$\vec{F} \,.\, d\vec{r}$ is an exact differential $\Rightarrow$ work done by the total force $\vec{F}$ acting on a particle is equal to its change in kinetic energy ΔT

$$W_{12} = \left(\frac{1}{2}mv^2\right)\Big|_1^2 = \frac{1}{2}m(v_2^2 - v_1^2) = T_2 - T_1$$

If $T_1 > T_2 \Rightarrow W_{12} < 0$

☛ the particle has done work with a resulting decrease in its kinetic energy

Energy of the System

The work done on the system

in moving it from a configuration 1 (in which all coordinates $\vec{r}_\alpha$ are specified) to a configuration 2 (in which the coordinates $\vec{r}_\alpha$ have some different specification) **is**

$$
\begin{aligned}
W_{12} &= \sum_\alpha \int_1^2 \vec{F}_\alpha \, . \, d\vec{r}_\alpha \\
&= \sum_\alpha \int_1^2 d(m_\alpha \, v_\alpha^2/2) \\
&= T_2 - T_1
\end{aligned}
$$

$\vec{F}_\alpha$ ☞ net resultant force acting on particle α

$T = \sum_\alpha T_\alpha = \sum_\alpha m_\alpha \, v_\alpha^2/2$

☞ **Note that the individual particles may be rearranged in such a process but the position of the c.m. could remain stationary**

Energy of the System (cont'd)

using the relation ☞ $\dot{\vec{r}}_\alpha = \dot{\vec{r}'}_\alpha + \dot{\vec{R}}$

$$\begin{aligned}\dot{\vec{r}}_\alpha \, . \, \dot{\vec{r}}_\alpha &= v_\alpha^2 = (\dot{\vec{r}'}_\alpha + \dot{\vec{R}}) \, . \, (\dot{\vec{r}'}_\alpha + \dot{\vec{R}}) \\ &= (\dot{\vec{r}'}_\alpha . \dot{\vec{r}'}_\alpha) + 2\,(\dot{\vec{r}'}_\alpha \, . \, \dot{\vec{R}}) + (\dot{\vec{R}} \, . \, \dot{\vec{R}}) \\ &= v_\alpha'^2 + 2\,(\dot{\vec{r}'}_\alpha \, . \, \dot{\vec{R}}) + V^2\end{aligned}$$

$\vec{v}' \equiv \dot{\vec{r}'}$ and $\vec{V}$ ☚ velocity of the c.m.

$$\begin{aligned}T &= \sum_\alpha m_\alpha \, v_\alpha^2/2 \\ &= \sum_\alpha m_\alpha \, v_\alpha'^2/2 + \sum_\alpha m_\alpha V^2/2 + \dot{\vec{R}} \, . \, \frac{d}{dt} \sum_\alpha m_\alpha \vec{r'}_\alpha\end{aligned}$$

☞ $\sum_\alpha m_\alpha \vec{r'}_\alpha \equiv 0$

$$T = \frac{1}{2} \sum_\alpha m_\alpha v_\alpha'^2 + \frac{1}{2} M V^2$$

Energy of the System (cont'd)

- *The total kinetic energy of the system is equal to the sum of the kinetic energy of a particle of mass M moving with velocity of the c.m. and the kinetic energy of the motion of the indiuvidual particles relative to the c.m.*

HOMEWORK

Use a procedure similar to that in obtaining the conservation of energy of a particle in a conservative system to show that

- *The total energy for a conservative system is constant*

Reduced Mass

- **Consider 2 objects**
  - 1^{st} object is of mass m_1 and is located at position vector $\vec{r}_1$
  - 2^{nd} object is of mass m_2 and is located at position vector $\vec{r}_2$

Let the first object exert a force $\vec{f}_{21}$ on the second

$$\Downarrow$$

Newton's Third Law ☛ 2^{nd} object exerts an equal and opposite force on the 1^{st}

$$\vec{f}_{12} = -\vec{f}_{21}$$

Suppose that there are no other forces in the problem

$$\Downarrow$$

The equations of motion of our two objects are

$$m_1 \frac{d^2\vec{r}_1}{dt^2} = -\vec{f}$$

$$m_2 \frac{d^2\vec{r}_2}{dt^2} = \vec{f}$$

☞ $\vec{f} = \vec{f}_{21}$

Reduced Mass (cont'd)

The center of mass of our system is located at

$$\vec{r}_{\rm cm} = \frac{m_1\,\vec{r}_1 + m_2\,\vec{r}_2}{m_1 + m_2}$$

$$\Downarrow$$

$$\vec{r}_1 = \vec{r}_{\rm cm} - \frac{m_2}{m_1+m_2}\,\vec{r} \qquad\qquad \vec{r}_2 = \vec{r}_{\rm cm} + \frac{m_1}{m_1+m_2}\,\vec{r}$$

☞ $\vec{r} = \vec{r}_2 - \vec{r}_1$

Substituting into the equations of motion
(making use of the fact that the c.m. of an isolated system does not accelerate)
we find that both equations yield

$$\mu\,\frac{d^2\vec{r}}{dt^2} = \vec{f}$$

$\mu = \frac{m_1\,m_2}{m_1+m_2}$ ☞ is called the reduced mass

$$\Downarrow$$

We have effectively converted a 2-body problem into an equivalent 1-body problem
In the equivalent problem

- $\vec{f}$ is the same as that acting on both objects (modulo a minus sign)
- the mass μ is different and is less than either of m_1 or m_2

Elastic and Inelastic Collisions

Next ☞ apply conservation laws to the interaction of two particles

Take advantage of simplifications by describing collisions on the c.m. system

$$\begin{array}{l} m_1 = \\ m_2 = \end{array} \text{ mass of the } \left\{ \begin{array}{c} \text{moving} \\ \text{struck} \end{array} \right\} \text{ particle}$$

In general * quantities refer to the c.m. system

$$\left.\begin{array}{lcc} \vec{u}_1 & = & \text{initial} \\ \vec{v}_1 & = & \text{final} \end{array}\right\} \text{velocity of } m_1 \text{in the lab system}$$

$$\left.\begin{array}{lcc} \vec{u}_1^* & = & \text{initial} \\ \vec{v}_1^* & = & \text{final} \end{array}\right\} \text{velocity of } m_1 \text{in the c.m. system}$$

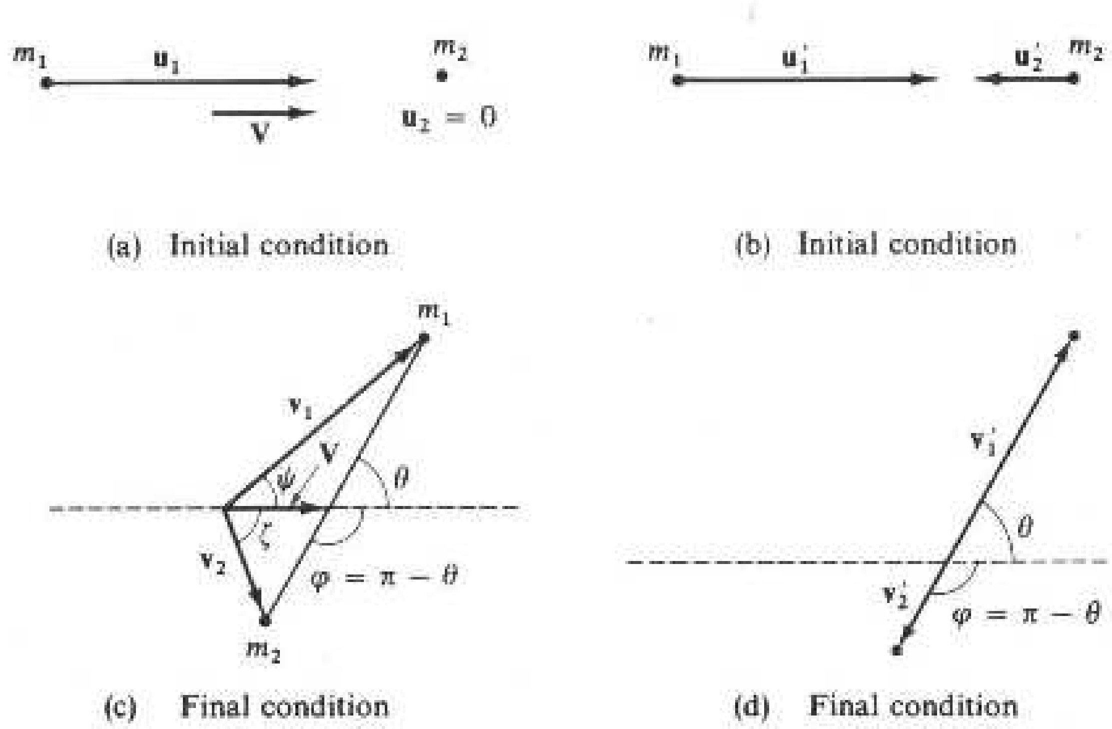
similarly for $\vec{u}_2 = 0, \; \vec{v}_2, \; \vec{u}_2^*, \; \vec{v}_2^*$

$$\begin{array}{l} T_0 = \\ T_0^* = \end{array} \text{ total initial kinetic energy in } \left\{ \begin{array}{c} \text{lab} \\ \text{c.m} \end{array} \right\} \text{ system}$$

$$\begin{array}{l} T_1 = \\ T_1^* = \end{array} \text{ total final kinetic energy of } m_1 \text{ in } \left\{ \begin{array}{c} \text{lab} \\ \text{c.m} \end{array} \right\} \text{ system}$$

similarly for T_2 and T_2^*

Elastic and Inelastic Collisions (cont'd)



$\vec{V}$ ☞ velocity of the c.m. in the lab system
ψ ☞ angle through which m_1 is deflected in the lab system
ζ ☞ angle through which m_2 is deflected in the lab system
θ ☞ angle through which m_1 and m_2 are deflected in the c.m. system

Energy Conservation in Particle's Collisions

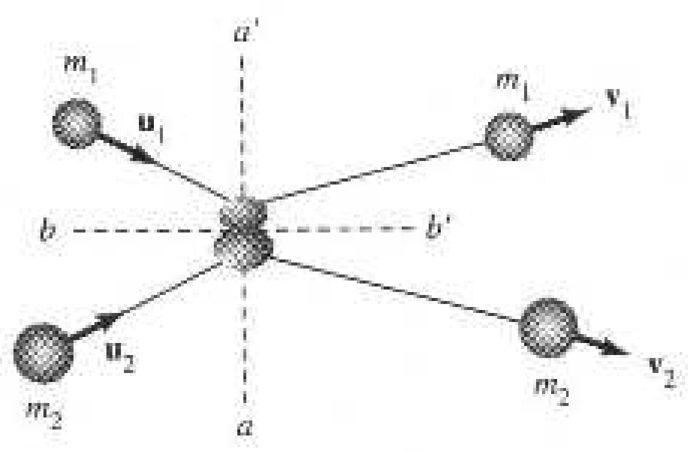
$$Q + \frac{1}{2}m_1u_1^2 + \frac{1}{2}m_2u_2^2 = \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2$$

Q ☞ energy loss or win in the collision

- ↗ $Q > 0$ ☞ exoergic collision ☚ kinetic energy is gained
- ➡ $Q = 0$ ☞ elastic collision ☚ kinetic energy is conserved
- ↘ $Q < 0$ ☞ endoergic collision ☚ kinetic energy is lost

Coefficient of restitution ☞ Newton's rule

$$\epsilon = \frac{|v_2 - v_1|}{|u_2 - u_1|}$$



only applies to $\vec{v}$ components along the normal (aa') to the plane of contact (bb')