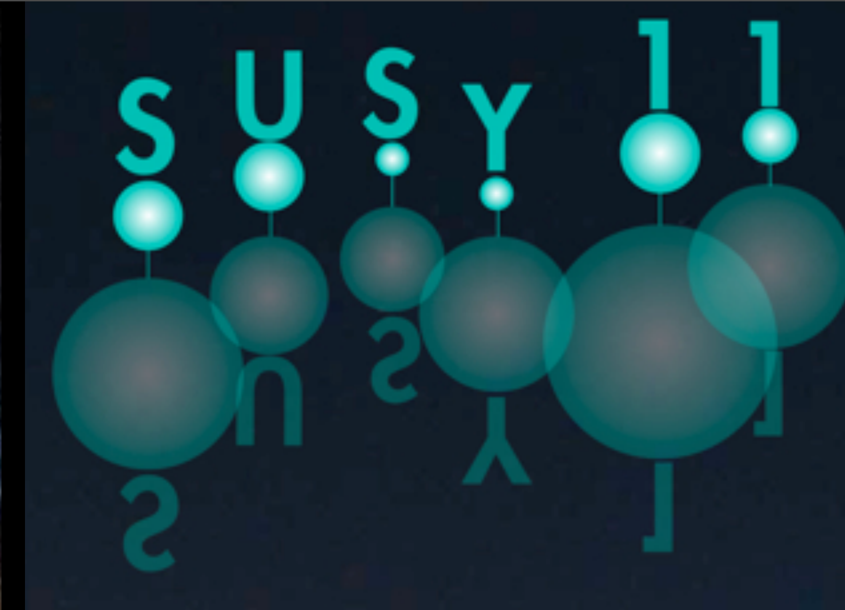

PARTICLE PHYSICS 2011

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Perturbation Theory

- Free-particle states have been eigenstates of Hamiltonian we have seen no interactions and no scattering

- There is no known method -other than perturbation theory- that could be used to include nonlinear terms in Hamiltonian
  -- or Lagrangian --
  that will couple different Fourier modes to one another
  --and the particles that occupy them--

- In order to obtain closer description of real world we are forced to resort to approximation methods

- In perturbation theory we divide Hamiltonian into two parts

H_0 ☛ is a Hamiltonian for which we know how to solve equations of motion

$$H_0|\phi_n\rangle = E_n|\phi_n\rangle \quad \text{with} \quad \langle\phi_m|\phi_n\rangle = \int_V \phi_m^* \; \phi_n \, d^3x = \delta_{mn}$$ ☆

$V(\vec{x}, t)$ ☛ is a perturbing interaction

Nonrelativistic perturbation theory

- normalized solution to one particle in a box of volume V
- Since only soluble field theory is free-field theory take for H_0 sum of all free particle Hamiltonians -with physical masses appearing in them-
- for sake of simplicity consider theory with one scalar field
- objective is to solve Schrödinger equation

$$[H_0 + V(\vec{x}, t)]\psi = i\partial_t \psi$$

in presence of an interaction potential $V(\vec{x}, t)$

- Any solution can be expressed as

$$|\psi\rangle = \sum_n c_n(t) |n\rangle e^{-iE_n t} = \sum_n c_n(t)\, \phi_n(\vec{x})\, e^{-iE_n t}$$

Nonrelativistic Perturbation Theory

When ✦ is substituted in Schrödinger equation

we get an equation for coefficients $c_n(t)$

$$\sum_n c_n(t) V(\vec{x},t) |n\rangle e^{-iE_n t} = i \sum_n \dot{c}_n(t) |n\rangle e^{-iE_n t}$$

or equivalently

$$\sum_n c_n(t) V(\vec{x},t) \phi_n(\vec{x}) e^{-iE_n t} = i \sum_n \dot{c}_n(t) \phi_n(\vec{x}) e^{-iE_n t}$$

Multiplying by ϕ_f^*

integrating over volume

and using orthogonality relation ☆

obtain coupled linear differential equations for coefficients

$$\dot{c}_f = -i \sum_n c_n(t) \int \phi_f^* V \phi_n \, d^3x \, e^{i(E_f - E_n)t}$$

Hints for the calculation

$$\sum_n E_n c_n(t)\phi_n(\vec{x})e^{-iE_nt} + \sum_n c_n(t)\; V(\vec{x},t)\; \phi_n(\vec{x})e^{-iE_nt} \quad = \quad i\sum_n \dot{c}_n(t)\,\phi_n(\vec{x})\,e^{-iE_nt}$$
$$+ \quad i(-iE_n)c_n(t)\phi_n(\vec{x})e^{-iE_nt}$$

Nonrelativistic perturbation theory

- Assume that before potential V acts particle is in an eigenstate i of unperturbed Hamiltonian

- We therefore set at time $t = -T/2$

  every $c_n(-T/2) = 0$ for $n \neq i$ and $c_i(-T/2) = 1$

- The relation ☛

$$|\psi\rangle = \sum_n c_n(t)|n\rangle$$

shows that system state $|\psi\rangle = |i\rangle$ as desired

- Replacing the initial condition into ♛ we get

$$\dot{c}_f = -i \int d^3x \; \phi_f^* V \phi_i \, e^{i(E_f - E_i)t}$$

Nonrelativistic perturbation theory

- Provided that potential is small and transient
  --as a first approximation--
  Assume that these initial conditions remain true at all times
- To find amplitude for system to be in state $|f\rangle$ at t
  project out eigenstate $|f\rangle$ from $|\psi\rangle$ by calculating

$$c_f(t) = -i \int_{-T/2}^{t} dt' \int d^3x \; \phi_f^* V \phi_i \, e^{i(E_f - E_i)t'}$$ ✖

at time $T/2$ after interaction has ceased we have

$$T_{fi} \equiv c_f(T/2) = -i \int_{-T/2}^{T/2} dt \int d^3x \, [\phi_f(\vec{x}) e^{-iE_f t}]^* \, V(\vec{x}, t) [\phi_i(\vec{x}) e^{-iE_i t}]$$ ✪

which can be rewritten in covariant form

$$T_{fi} = -i \int d^4x \, \phi_f^*(x) \, V(x) \, \phi_i(x)$$

- expression for $c_f(t)$ is only a good approximation if $c_f(t) \ll 1$
  as this has been assumed in obtaining result

Transition probability per unit time

It is tempting to identify $|T_{fi}|^2$ with probability that particle is scattered from an initial state $|i\rangle$ to a final state $|f\rangle$

To see whether this identification is possible

consider case in which $V(\vec{x},t) = V(\vec{x})$ is time independent

☛ then using

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} dq\, e^{iqp} = \delta(p)$$

☛ ✪ becomes

$$\begin{aligned} T_{fi} &= -iV_{fi}\int_{-\infty}^{\infty} dt\, e^{i(E_f-E_i)t} \\ &= -2\pi i\, V_{fi}\,\delta(E_f - E_i) \end{aligned}$$

♻

with

$$V_{fi} \equiv \int d^3x\; \phi_f^*(\vec{x})\, V(\vec{x})\, \phi_i(x)$$

Transition probability per unit time

δ -funtion in ♻ expresses that energy of particle is conserved in transition $i \to f$

By uncertainty principle

this means that infinite time separates states i and f

and $|T_{fi}|^2$ is therefore not a meaningful quantity

Define instead a transition probability per unit time

$$W = \lim_{T\to\infty} \frac{|T_{fi}|^2}{T}$$

Squaring ♻

$$\begin{aligned} W &= \lim_{T\to\infty} 2\pi \frac{|V_{fi}|^2}{T} \delta(E_f - E_i) \int_{-T/2}^{+T/2} dt\, e^{i(E_f - E_i)t} \\ &= \lim_{T\to\infty} 2\pi \frac{|V_{fi}|^2}{T} \delta(E_f - E_i) \int_{-T/2}^{+T/2} dt \\ &= 2\pi |V_{fi}|^2 \delta(E_f - E_i) \end{aligned}$$

This equation can only be given physical meaning after integrationover a set of initial and final states

Hints for the calculation

$$[\delta(E_f - E_i)]^2 = \delta(E_f - E_i) \lim_{T\to\infty} \frac{1}{2\pi} \int_{-T/2}^{+T/2} dt\ e^{i(E_f - E_i)t}$$

$$= \delta(E_f - E_i) \lim_{T\to\infty} \frac{T}{2\pi}$$

Transition Rate

- In particle physics we usually deal with situations where we begin with a specified initial state and end up in one set of final states
- We denote with $\rho(E_f)$ the density of final states

i.e. $\rho(E_f)dE_f$ is number of states in energy interval $(E_f, E_f + dE_f)$

- Integration over this density -- imposing energy conservation -- leads to the transition rate

$$W_{fi} = 2\pi \int dE_f \; \rho(E_f) \; |V_{fi}|^2 \; \delta(E_f - E_i)$$
$$= 2\pi |V_{fi}|^2 \; \rho(E_i)$$

This is famous Fermi's golden rule

Iterative Process

Clearly we can improve on above approximation

by inserting result for $c_n(t)$ in right-hand side of ♛

$$\dot{c}_f(t) = \cdots + (-i)^2 \left[\sum_n V_{ni} \int_{-T/2}^{t} dt' \, e^{i(E_n - E_i)t'} \right] V_{fn} \, e^{i(E_f - E_n)t}$$

where dots represent first order result

Correction to T_{fi} is

$$T_{fi} = \cdots - \sum_n V_{fn} \, V_{ni} \int_{-\infty}^{\infty} dt \, e^{i(E_f - E_n)t} \int_{-\infty}^{t} dt' \, e^{i(E_n - E_i)t'}$$

To make integral over dt' meaningful

must include a term in exponent involving a small quantity $\epsilon > 0$

which we let go to zero after integration

$$\int_{-\infty}^{t} dt' \, e^{i(E_n - E_i - i\epsilon)t'} = i \frac{e^{i(E_n - E_i - i\epsilon)t}}{E_i - E_n + i\epsilon}$$

Higher order corrections

- Second order correction to T_{fi} is given by

$$T_{fi} = \cdots - 2\pi i \sum_n \frac{V_{fn} V_{ni}}{E_i - E_n + i\epsilon} \delta(E_f - E_i)$$

rate for $i \to f$ transition is given by Fermi's Golden rule

with replacement

$$V_{fi} \to V_{fi} + \sum_n V_{fn} \frac{1}{E_i - E_n + i\epsilon} V_{ni} + \ldots$$

- this equation is the perturbation series for the amplitude with terms to first, second, . . . order in V

Symmetries and Invariants

Remarkable connection between symmetries and conservation laws are summarized in Noether's theorem:

Any differentiable symmetry of the action of a physical system

has a corresponding conservation law

Theorem grants observed selection rules in nature to be expressed directly in terms of symmetry requirements in Lagrangian density

e.g. translational symmetry

Under infinitesimal displacement $x'_\mu = x_\mu + \epsilon_\mu$

☛ Lagrangian changes by the amount $\delta\mathcal{L} = \mathcal{L}' - \mathcal{L} = \epsilon_\mu \, \partial^\mu \mathcal{L}$

if $\mathcal{L}$ is translationally invariant

☛ it has no explicit coordinate dependence $\mathcal{L}(\phi, \partial^\mu \phi)$

Translational Symmetry

Recalling $$\delta\phi = \phi(x+\epsilon) - \phi(x) = \epsilon_\nu \, \partial^\nu \phi(x)$$

we have
$$\begin{aligned} \delta\mathcal{L} &= \partial_\phi \mathcal{L}\, \delta\phi + \partial_{\partial^\mu \phi}\mathcal{L}\ \delta(\partial^\mu \phi) \\ &= \partial_\phi \mathcal{L}\, \delta\phi + \partial_{\partial^\mu \phi}\mathcal{L}\ \partial^\mu(\delta\phi) \\ &= \partial_\phi \mathcal{L}\, \delta\phi + \partial_{\partial^\mu \phi}\mathcal{L}\ \partial^\mu(\epsilon_\nu \partial^\nu \phi) \end{aligned}$$

integration by parts leads to
$$\delta\mathcal{L} = \partial^\mu(\partial_{\partial^\mu \phi}\mathcal{L}\ \epsilon_\nu \partial^\nu \phi) - \delta\phi\, \partial^\mu(\partial_{\partial^\mu \phi}\mathcal{L}) + \partial_\phi \mathcal{L}\ \delta\phi$$

using Euler-Lagrange equation
$$\delta\mathcal{L} = \partial^\mu(\partial_{\partial^\mu \phi}\mathcal{L}\ \epsilon_\nu \partial^\nu \phi) = \epsilon_\mu \partial^\mu \mathcal{L}$$

therefore ☛ $$\partial^\mu(\partial_{\partial^\mu \phi}\mathcal{L}\ \epsilon_\nu \partial^\nu \phi) - \epsilon_\mu \partial^\mu \mathcal{L} = 0$$

because this holds for arbitrary displacements ϵ_μ
$$\partial^\mu[\partial_{\partial^\mu \phi}\mathcal{L}\ \partial_\nu \phi - g_{\mu\nu}\ \mathcal{L}] = 0$$

Energy momentum 4-vector

define ☛ energy-momentum stress tensor

$$\mathfrak{I}_{\mu\nu} = -g_{\mu\nu}\ \mathcal{L} + \partial_{\partial^\mu \phi}\mathcal{L}\ \partial_\nu \phi$$

From this differential conservation law one finds

$$P_\nu = \int d^3x\, \mathfrak{I}_{0\nu} = \int d^3x\, \pi\, \partial_\nu \phi - g_{0\nu}\ \mathcal{L}$$

and so $\partial^t P_\nu = 0$

we have already seen that $\mathfrak{I}_{00}$ is Hamiltonian density

$$\mathfrak{I}_{00} = \pi\, \dot{\phi} - \mathcal{L} = \mathcal{H}$$

and

$$\int d^3x\, \mathfrak{I}_{00} = H$$

Therefore ☛ we can identify operator P_ν

as conserved energy-momentum 4-vector

Gauge Invariance

- ✔ Importance of connection between symmetry properties and invariance of physical quantities can hardly be overemphasized
- ✔ Homogeneity and Isotropy of spacetime imply Lagrangian is invariant under time displacements is unaffected by translation of entire system and does not change if system is rotated an infinitesimal angle
- ✔ these particular measurable properties of spacetime lead to conservation of energy, momentum, and angular momentum
- ✔ However these are only 3 of various invariant symmetries in nature which are regarded as cornerstones of particle physics
- ✔ we will focus attention on conservation laws associated with "internal" symmetry transformations that do not mix fields with internal spacetime properties i.e. ☛ transformations that commute with spacetime components of wave function

Charge Conservation

free fermion of mass m is described by a complex field $\psi(x)$

Inspection of Dirac's lagrangian $\mathcal{L}_{\rm Dirac} = \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi$

shows that $\psi(x)$ is invariant under global phase transformation

$$\psi(x) \to \exp(i\alpha)\ \psi(x)$$ ♔

where single parameter α could run continuously over real numbers

Noether's theorem implies the existence of a conserved current

To see ☛ this we need to study invariance of $\mathcal{L}$

under infinitesimal $U(1)$ transformations $\psi \to (1+i\alpha)\psi$

Invariance requires the lagrangian to be unchanged ☛ that is

$$\begin{aligned} \delta\mathcal{L} &= \partial_\psi\mathcal{L}\ \delta\psi + \partial_{\partial_\mu\psi}\mathcal{L}\ \delta(\partial_\mu\psi) + \delta\bar{\psi}\ \partial_{\bar{\psi}}\mathcal{L} + \delta(\partial_\mu\bar{\psi})\ \partial_{\partial_\mu\bar{\psi}}\mathcal{L} \\ &= \partial_\psi\mathcal{L}\ (i\alpha\psi) + \partial_{\partial_\mu\psi}\mathcal{L}\ (i\alpha\partial_\mu\psi) + \dots \\ &= i\alpha\ \left[\partial_\psi\mathcal{L} - \partial_\mu(\partial_{\partial_\mu\psi}\mathcal{L})\right]\psi + i\alpha\partial_\mu(\partial_{\partial_\mu\psi}\mathcal{L}\ \ \psi) + \dots \end{aligned}$$ ♘

Charge Conservation

term in square brackets vanishes by virtue of Euler-Lagrange Eq.

-- for ψ and similarly for $\bar{\psi}$ --

♞ reduces to eq. for a conserved current $\partial_\mu j^\mu = 0$

$$j^\mu = -\frac{i}{2}\left(\partial_{\partial_\mu \psi}\mathcal{L}\, \psi - \bar{\psi}\, \partial_{\partial_\mu \bar{\psi}}\mathcal{L}\right) = \bar{\psi}\gamma^\mu\psi$$

using ☛ $\mathcal{L}_{\text{Dirac}} = \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi$

It follows that the charge $Q = \int d^3x\, j^0$

must be a conserved quantity

because of $U(1)$ phase invariance

Maxwell-Dirac Lagrangian

A global phase transformation is surely not most general invariance
more convenient to have independent phase changes at each point

We thus generalize ♔ to include local phase transformations

$$\psi \to \psi' \equiv \exp[i\alpha(x)]\ \psi$$

derivative $\partial_\mu \alpha(x)$ breaks invariance of Dirac Lagrangian
which acquires an additional phase change at each point

$$\delta\mathcal{L}_{\rm Dirac} = \bar\psi\ i\gamma^\mu\ [i\partial_\mu\alpha(x)]\ \psi$$

Dirac Lagrangian $\mathcal{L}_{\rm Dirac} = \bar\psi(i\gamma^\mu\partial_\mu - m)\psi$

is not invariant under local gauge transformations
but if we seek a modified derivative $\partial_\mu \to D_\mu \equiv \partial_\mu + ieA_\mu$
which tranforms covariantly under phase transformations

$D_\mu\psi \to e^{i\alpha(x)} D_\mu\psi$ ☛ then local gauge invariance can be restored

$$\begin{aligned}\mathcal{L} &= \bar\psi\ (i\not{D} - m)\ \psi \\ &= \bar\psi\ (i\not{\partial} - m)\ \psi - e\ \bar\psi\ \not{A}(x)\ \psi\end{aligned}$$

Maxwell-Dirac Lagrangian

Namely ☛ if $\psi \to \psi'$ and $A \to A'$ we have

$$\begin{aligned} \mathcal{L}' &= \bar{\psi}' \, (i\partial\!\!\!/ - m) \, \psi' - e \, \bar{\psi}' \, A\!\!\!/' \, \psi' \\ &= \bar{\psi} \, (i\partial\!\!\!/ - m) \, \psi - \bar{\psi} \, [\partial\!\!\!/ \alpha(x)] \, \psi - e \, \bar{\psi} A\!\!\!/' \, \psi \end{aligned}$$

we can ensure $\mathcal{L} = \mathcal{L}'$ if we demand

vector potential A_μ to change by a total divergence

$$A'_\mu(x) = A_\mu(x) - \frac{1}{e}\partial_\mu \alpha(x)$$ ♪

which does not change the electromagnetic field strength $F_{\mu\nu}$

In other words ☛ by demanding local phase invariance in ψ

we must introduce a gauge field A_μ

that couples to fermions of charge e

in exactly same way as photon field

Wilson line

Alternative approach to visualize consequences of local gauge invariance

as particle of charge e moves in spacetime from point A to point B

wave function undergoes a phase change

$$\Phi_{AB} = \exp\left(-ie\int_A^B A_\mu(x)dx^\mu\right) \quad ♣$$

$-eA_\mu(x)$ parametrizes infinitesimal phase change in $(x^\mu, x^\mu + dx^\mu)$

integral in ♣ for points at finite separation is known as a Wilson line

A crucial property of Wilson line is that it depends on path taken

and therefore Φ_{AB} is not uniquely defined

Wilson Loop

- ✔ If C is a closed path that returns to A -- i.e. a Wilson loop --

$$\Phi_C = \exp\left(-ie\oint A_\mu(x)dx^\mu\right)$$ ☀

phase becomes a nontrivial function of A_μ
that is by construction locally gauge invariant

- ✔ Note that for a Wilson loop
any change in contribution to Φ_C from integral up to given point x_μ^0
will be compensated by and equal and opposite contribution
from integral departing from x_μ^0
- ✔ To verify this claim ☛ we express closed path integral ☀
as a surface integral via Stokes' theorem

$$\oint A_\mu(x)dx^\mu = \int F_{\mu\nu}(x)d\sigma^{\mu\nu}$$

$d\sigma^{\mu\nu}$ ☛ element of surface area

- ✔ One can now check by inspection that Wilson loop
is invariant under changes of $A_\mu(x)$ by a total divergence

Abelian Gauge Theory

To obtain QED Lagrangian we need to include kinetic term ☛

$$\mathcal{L}_{\text{Maxwell}} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + e A_\mu j^\mu$$

that accounts for energy and momentum of free electromagnetic fields

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \bar{\psi}(i\not{\partial} - m)\psi - e\bar{\psi}\not{A}\psi$$

If electromagnetic current is defined as ☛ $e j_\mu \equiv e\bar{\psi}\gamma_\mu\psi$

this Lagrangian leads to Maxwell's equations

$$\epsilon^{\mu\nu\rho\sigma}\partial_\nu F_{\rho\sigma} = 0, \qquad \partial_\mu F^{\mu\nu} = e\, j^\nu$$

local phase changes ❦ form a $U(1)$ group of transformations

Since such transformations commute with one another

group is said to be Abelian

Electrodynamics is thus an Abelian gauge theory

Yang-Mills Theories

If by imposing local phase invariance on Dirac's Lagrangian we are we are lead to interacting theory of QED ☛ then in an analogous way one can hope to infer structure of other interesting theories by starting from more general fundamental symmetries

Pioneer work by Yang and Mills considered that a charged particle moving along in spacetime could undergo not only phase changes but also changes of identity --a quark can change its color from red to blue or change its flavor from u to d --

Such a kind of transformation requires a generalization of local phase rotation invariance to invariance under any continuous symmetry group

Coefficient eA_μ of infinitesimal displacement dx_μ should be replaced should be replaced by a $n \times n$ matrix $-g(\mathbf{x}) \equiv -\mathbf{g}\mathbf{A}_\mu^\mathbf{a}(\mathbf{x})\mathbf{t}_\mathbf{a}$

acting in n-dimensional space of particle's degrees of freedom

☛ g is coupling constant

☛ $\mathbf{t}_\mathbf{a}$ define a linearly independent basis set of matrices

☛ A_μ^a are their coefficients

Non-abelian Gauge theories

Wilson lines can be generalized to Yang-Mills transformations

careful must be taken as some subtleties arise
because integral in exponent now contains matrices $\mathbf{A}_\mu(x)$
which do not necessarily commute with one another
at different points of spacetime
consequently a path-ordering $(P\{\})$ is needed
we introduce a parameter s of path P
which runs from zero at $x = A$ to one at $x = B$

Wilson line is then defined as power series expansion of exponential
with matrices in each term ordered so that
higher values of s stand to the left

$$\mathbf{\Phi_{AB}} = \mathcal{P}\left\{\exp\left(\mathbf{ig}\int_{\mathbf{0}}^{\mathbf{1}}\mathbf{ds}\,\frac{\mathbf{dx}^{\mu}}{\mathbf{ds}}\,\mathbf{A}_{\mu}(\mathbf{x})\right)\right\}$$

If basis matrices t_a do not commute with one another
theory is said to be non-Abelian

Requirements

to ensure that changes in phase or identity conserve probability

we demand $\mathbf{\Phi}_{AB}$ be a unitary matrix ☛ $\mathbf{\Phi}^\dagger_{AB}\mathbf{\Phi}_{AB} = \mathbb{I}$

to separate out pure phase changes from remaining transformations

-- in which $\mathbf{A}_\mu(x)$ is a multiple of unit matrix --

consider only transformations such that det $(\mathbf{\Phi}_{AB}) = 1$

This becomes evident if we note that near identity any unitary matrix

can be expanded in terms of Hermitian generators of $SU(N)$

hence for infinitesimal separation between A and B

we can write ☛ $\mathbf{\Phi}_{AB} = \mathbb{I} + i\epsilon(gA^a_\mu \mathbf{t}_a) + \mathcal{O}(\epsilon^2)$

or equivalently

$$\begin{aligned} \mathbb{I} &= \mathbf{\Phi}^\dagger_{AB}\mathbf{\Phi}_{AB} \\ &= \mathbb{I} + ig\epsilon[\mathbf{A}_\mu(x)^\dagger - \mathbf{A}_\mu(x)] + \mathcal{O}(\epsilon^2) \end{aligned}$$

This shows that we must consider only transformations such that

$$\begin{aligned} \det\left(e^{ig\,A^a_\mu\,\mathbf{t}_a}\right) &= e^{ig\,A^a_\mu \mathrm{Tr}(\mathbf{t}_a)} \\ &= 1 \end{aligned}$$

corresponding to traceless $\mathbf{A}_\mu(x)$

Cheat Sheet Allowed

$$U(N) = SU(N) \times U(1)$$

$SU(N)$ is a compact group

e.g. $SU(2)$ gives rotations on sphere of radius 1

$U(1)$ is non-compact

leads to changes on the amplitude of a vector

to separate pure phase changes from the remaining transformations we must take $SU(N)$ groups

SU(N)

- The $n \times n$ basis matrices $\mathbf{t}_a$ must be Hermitian and traceless

  There are $n^2 - 1$ of them

  corresponding to number of independent $SU(N)$ generators

  Basis matrices satisfy commutation relations

$$[\mathbf{t}_i, \mathbf{t}_j] = i c_{ijk} \mathbf{t}_k$$

  where the c_{ijk} are structure constants characterizing group

- In fundamental representation of $SU(2)$

  generators are proportional to Pauli matrices $(\mathbf{t}_i = \sigma_i/2)$

  structure constants are defined by Levi-Civita symbol $(c_{ijk} = \epsilon_{ijk})$

- In fundamental representation of $SU(3)$

  Generators of are Gell-Mann matrices $\mathbf{t}_i = \lambda_i/2$

  normalized such that Tr $(\lambda_i \lambda_j) = 2\delta_{ij}$

  structure constants $c_{ijk} = f_{ijk}$

  are fully antisymmetric under interchange of any pair of indices

$$f_{123} = 1, \quad f_{458} = f_{678} = \frac{\sqrt{3}}{2}, \quad f_{147} = f_{165} = f_{246} = f_{257} = f_{345} = f_{376} = \frac{1}{2}$$

Non-abelian field strength

By considering an infinitesimal closed-path transformation

analogous to ☀ but for matrices $\mathbf{A}_\mu(x)$ that do not commute

we write field-strength tensor $\mathbf{F}_{\mu\nu} = F^a_{\mu\nu}\mathbf{t}_a$

for a non-abelian transformation:

$$\mathbf{F}_{\mu\nu} = \partial_\mu \mathbf{A}_\nu - \partial_\nu \mathbf{A}_\nu - ig[\mathbf{A}_\mu, \mathbf{A}_\nu]$$

or equivalently

$$F^i_{\mu\nu} = \partial_\mu A^i_\nu - \partial_\nu A^i_\mu + g c_{ijk} A^j_\mu A^k_\nu$$

Yang-Mills Lagrangian

alternative way to introduce non-Abelian gauge fields

by analogy with ♕ demand that theory involving fermions

be invariant under local transformations, ψ

$$\psi(x) \to \psi'(x) = V(x)\psi(x) \equiv \exp\left[i\alpha_a(x)\mathbf{t}^a\right]\,\psi(x)$$

where V is an arbitrary unitary matrix $(V^\dagger V = \mathbb{I})$

which we show parametrized by its general form

A summation over repeated suffix a is implied

Yang-Mills Lagrangian

Duplicating preceding discussion for $U(1)$ gauge group

we demand

$$\mathcal{L} \to \mathcal{L}'$$

where

$$\begin{aligned} \mathcal{L}' &\equiv \overline{\psi}'(i\partial\!\!\!/ - m)\psi' \\ &= \overline{\psi}V^\dagger(i\partial\!\!\!/ - m)V\psi \\ &= \overline{\psi}(i\partial\!\!\!/ - m)\psi + i\psi V^\dagger\gamma^\mu(\partial_\mu V)\psi \end{aligned}$$

Last term --as in abelian case-- spoils invariance of $\mathcal{L}$

As before it can be compensated if we replace

$$\partial_\mu \to \mathbf{D}_\mu \equiv \partial_\mu - ig\mathbf{A}_\mu(x)$$

under transformation Lagrangian $\mathcal{L} = \overline{\psi}(i\mathbf{D}\!\!\!\!/ - m)\psi$

becomes

$$\begin{aligned} \mathcal{L}' &\equiv \overline{\psi}'(i\mathbf{D}\!\!\!\!/' - m)\psi' \\ &= \overline{\psi}V^\dagger(i\partial\!\!\!/ + g\mathbf{A}\!\!\!/' - m)V\psi \\ &= \mathcal{L} + \overline{\psi}[g(V^\dagger\mathbf{A}\!\!\!/'V - \mathbf{A}\!\!\!/) + iV^\dagger(\partial\!\!\!/V)]\psi \end{aligned}$$

which is equal to $\mathcal{L}$ if we take ☛ $\mathbf{A}'_\mu = V\mathbf{A}_\mu V^\dagger - \frac{i}{g}(\partial_\mu V)V^\dagger$

Full Yang-Mills Lagrangian

- ✔ Covariant derivative acting on ψ transforms in same way as ψ itself
- ✔ under a gauge transformation ☛ $\mathbf{D}_\mu\psi \to \mathbf{D}_\mu{}'\psi' = V\mathbf{D}_\mu\psi$
- ✔ Field strength $\mathbf{F}_{\mu\nu}$ transforms as $\mathbf{F}_{\mu\nu} \to \mathbf{F}'_{\mu\nu} = V\mathbf{F}_{\mu\nu}V^\dagger$

As in abelian case it can be computed via $[\mathbf{D}_\mu, \mathbf{D}_\nu] = -\mathbf{igF}_{\mu\nu}$

Both sides transform as $V(\quad)V^\dagger$ under a local gauge transformation

To obtain propagating gauge fields we follow steps of QED

add kinetic term $-(1/4)F^i_{\mu\nu}F^{i\mu\nu}$ to Lagrangian

Full Lagrangian for gauge fields interacting with matter fields

$$\mathcal{L} = -\tfrac{1}{2}\mathrm{Tr}(\mathbf{F}_{\mu\nu}\,\mathbf{F}^{\mu\nu}) + \overline{\psi}(i\,\not{\mathbf{D}} - m)\psi$$

Recall ☛ $\mathbf{F}_{\mu\nu} = F^i_{\mu\nu}$

written for gauge group generators normalized such that

$$\mathrm{Tr}\,(\mathbf{t}_i\mathbf{t}_j) = \delta_{ij}/2$$

Non-abelian self interactions

Interaction of a gauge field with fermions

corresponds to interaction Lagrangian $\Delta\mathcal{L} = g\bar{\psi}(x)\gamma^{\mu}\mathbf{A}_{\mu}(x)\psi(x)$

$\mathbf{A}_{\mu}, \mathbf{A}_{\nu}$ term in $\mathbf{F}_{\mu\nu}$ leads to self-interactions of non-Abelian fields

arising solely from the kinetic term

They have no analogue in QED

arise on account of non-abelian character of gauge group

yielding three-and four-field vertices

of form

$$\Delta\mathcal{L}_{K}^{(3)} = (\partial_{\mu}A_{\nu}^{i})gc_{ijk}A^{\mu j}A^{\nu k}$$

and

$$\Delta\mathcal{L}_{K}^{(4)} = -\frac{g^{2}}{4}c_{ijk}c_{imn}A^{\mu j}A^{\nu k}A_{\mu}^{m}A_{\nu}^{n}$$

respectively

These self-interactions

are a paramount property of non-Abelian gauge theories

drive remarkable asymptotic freedom of QCD,

which leads to its becoming weaker at short distances

allowing application of perturbation theory

Isospin

Isospin arises because nucleon may be view as having internal degree of freedom with two allowed states ☛ proton and neutron which nuclear interaction does not distinguish

Consider description of two-nucleon system

Each nucleon has spin $\frac{1}{2}$ -- with spin states $\uparrow$ and $\downarrow$ -- following rules for addition of angular momenta

composite system may have total spin $S = 1$ or $S = 0$

Composition of these spin triplet and spin singlet states is

$$\begin{cases} |S=1, M_s=1\rangle = \uparrow\uparrow \\ |S=1, M_s=0\rangle = \sqrt{\tfrac{1}{2}}(\uparrow\downarrow + \downarrow\uparrow) \\ |S=1, M_s=-1\rangle = \downarrow\downarrow \end{cases}$$

$$|S=0, M_S=0\rangle = \sqrt{\tfrac{1}{2}}(\uparrow\downarrow - \downarrow\uparrow)$$

Isospin

Each nucleon is similarly postulated to have isospin $T = \frac{1}{2}$

with $T_3 = \pm\frac{1}{2}$ for protons and neutrons respectively

$T = 1$ and $T = 0$ states of nucleon-nucleon system

can be constructed in exact analogy to spin

$$\begin{cases} |T=1, T_3=1\rangle = \psi_p^{(1)}\psi_p^{(2)} \\ |T=1, T_3=0\rangle = \sqrt{\tfrac{1}{2}}(\psi_p^{(1)}\psi_n^{(2)} + \psi_n^{(1)}\psi_p^{(2)}) \\ |T=1, T_3=-1\rangle = \psi_n^{(1)}\psi_n^{(2)} \end{cases}$$

$$|T=0, T_3=0\rangle = \sqrt{\tfrac{1}{2}}(\psi_p^{(1)}\psi_n^{(2)} - \psi_n^{(1)}\psi_p^{(2)})$$

Isospin

most positively charged particle is chosen to have maximum value of T_3

nucleon field operators will transform according to

$$U \begin{pmatrix} \psi_p(x) \\ \psi_n(x) \end{pmatrix} U^{-1} = \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix} \begin{pmatrix} \psi_p(x) \\ \psi_n(x) \end{pmatrix} \equiv \mathcal{U} \begin{pmatrix} \psi_p(x) \\ \psi_n(x) \end{pmatrix}$$

Preservation of commutation relations requires that $\mathcal{U}$ be unitary

Such 2×2 unitary matrix is characterized by four parameters

when common phase factor is taken out -- we have 3 parameters --

conventional way of writing general form for U is

-- ommiting phase factor --

$$\mathcal{U} = e^{(i/2)\alpha\,.\,\tau}$$

where three traceless hermitian canonical 2×2 matrices

$$\tau_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \tau_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \tau_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

☛ are just Pauli spin matrices

Isospin

Close similarity between ✽ and way we express rotational invariance suggests a way of characterizing invariance

We will speak of an invariance under rotations in an **internal** space

Isospin T is analog of angular momentum

$$U = e^{i\alpha\,.\,\mathbf{T}}$$

Rotational invariance implies that isospin is conserved

For an infinitesimal rotation ✽ reads

$$\psi(x) + i\alpha_i[T_i, \psi(x)] = \psi(x) + \tfrac{1}{2} i\alpha_i\tau_i\,\psi(x)$$

i.e.

$$[T_i, \psi(x)] = \frac{1}{2}\tau_i\psi(x)$$

where we represent $\begin{matrix}\psi_p(x)\\ \psi_n(x)\end{matrix}$ by $\psi(x)$

It is easily seen that these relations are satisfied by

$$\mathbf{T} = \frac{1}{2}\int \mathbf{d^3x}\,\psi^\dagger(\mathbf{x})\,\tau\,\psi(\mathbf{x})$$

Isospin

Note that $$T_3 = \frac{1}{2}\int d^3x\,[\psi_p^\dagger(x)\,\psi_p(x) - \psi_n^\dagger(x)\,\psi_n(x)]$$

Hence, charge operator for nucleons Q may be written as

$$Q = \int d^3x\;\psi_p^\dagger(x)\,\psi_p(x) = \int d^3x\;\psi^\dagger(x)\,\tfrac{1+\tau_3}{2}\,\psi(x)$$

We may introduce baryon-number operator N_B by definition

$$N_B = \int d^3x\;[\psi_p^\dagger(x)\psi_p(x) + \psi_n^\dagger(x)\psi_n(x) + \ldots]$$

Extra terms -- not written down -- are similar contributions from other fields carrying baryon number

If we consider only protons and neutrons ☛ $$Q = \tfrac{1}{2}N_B + T_3$$

It follows from easily derived commutation relations

$$[T_i,\;T_j] = i\epsilon_{ijk}T_k$$

that

$$[Q,\;T_i] \neq 0 \quad i = 1, 2$$

so that charge violates isospin conservation

Antiparticle isospin multiplets

Construction of antiparticle isospin multiplets requires care

It is well illustrated by a simple example

Consider a particular isospin transformation of nucleon doublet

a rotation through π about the 2-axis leads to

$$\begin{pmatrix} \psi'_p \\ \psi'_n \end{pmatrix} = e^{-i\pi(\tau_2/2)} \begin{pmatrix} \psi_p \\ \psi_n \end{pmatrix} = -i\tau_2 \begin{pmatrix} \psi_p \\ \psi_n \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \psi_p \\ \psi_n \end{pmatrix} \quad ✹$$

we define antinucleon states using particle-antiparticle

conjugation operator $C, \; C\psi_p = \psi_{\bar{p}}, \; C\psi_n = \psi_{\bar{n}}$

Applying C to ✹

$$\begin{pmatrix} \psi'_{\bar{p}} \\ \psi'_{\bar{n}} \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \psi_{\bar{p}} \\ \psi_{\bar{n}} \end{pmatrix} \quad ✣$$

ANTIPARTICLE ISOSPIN MULTIPLETS

WE WANT ANTIPARTICLE DOUBLET

TO TRANSFORM IN EXACTLY SAME WAY AS PARTICLE DOUBLET

WE MUST THEREFORE MAKE TWO CHANGES

- REORDER DOUBLET SO THAT MOST POSITIVELY CHARGED PARTICLE HAS $T_3 = +\frac{1}{2}$
- INTRODUCE MINUS SIGN TO KEEP MATRIX TRANSFORMATION IDENTICAL TO ✣

WE OBTAIN

$$\begin{pmatrix} -\psi'_{\bar{n}} \\ \psi'_{\bar{p}} \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} -\psi_{\bar{n}} \\ \psi_{\bar{p}} \end{pmatrix}$$

THAT IS ANTIPARTICLE DOUBLET $(-\psi_{\bar{n}}, \psi_{\bar{p}})$

TRANSFORMS EXACTLY AS PARTICLE DOUBLET (ψ_p, ψ_n)

THIS IS A SPECIAL PROPERTY OF $SU(2)$

Isospin of nucleon antinucleon pair

A composite system of a nucleon-antinucleon pair has isospin states

$$\begin{cases} |T=1,T_3=1\rangle = -\psi_p\psi_{\bar{n}} \\ |T=1,T_3=0\rangle = \sqrt{\frac{1}{2}}(\psi_p\psi_{\bar{p}} - \psi_n\psi_{\bar{n}}) \\ |T=1,T_3=-1\rangle = \psi_n\psi_{\bar{p}} \end{cases}$$

$$|T=0,T_3=0\rangle = \sqrt{\frac{1}{2}}(\psi_p\psi_{\bar{p}} + \psi_n\psi_{\bar{n}})$$


"That's all Folks!"