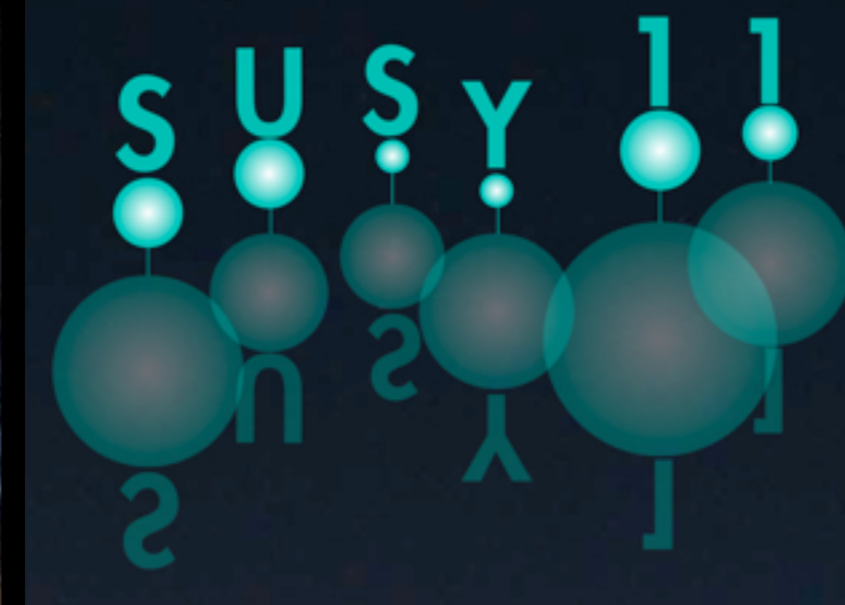

Particle Physics 2011

Luis Anchordoqui

Dirac Equation

today ☛

we will construct wave equation for spin-1/2 relativistic particles

Following Dirac we proceed by analogy with non-relativistic QM

write equation which -- unlike Klein-Gordon equation-- is linear in ∂t

In order to be covariant ☛ it must also be linear in $\vec{\nabla}$

Hamiltonian has general form

$$H \;\psi(x) = (\vec{\alpha}\,.\,\vec{p} + \beta\, m)\;\psi(x)$$

4 coefficients $\beta, \alpha_1, \alpha_2$ and α_3

are determined by requirement that a free particle -- of mass m --

must satisfy relativistic energy momentum relation

$$\begin{aligned} H^2\psi &= (\alpha_i p_i + \beta m)(\alpha_j p_j + \beta m)\psi \\ &= (\underbrace{\alpha_i^2}_{1}\; p_i^2 + \underbrace{(\alpha_i\alpha_j + \alpha_j\alpha_i)}_{0}\; p_i p_j + \underbrace{(\alpha_i\beta + \beta\alpha_i)}_{0}\; p_i\, m + \underbrace{\beta^2}_{1}\; m^2)\;\psi \end{aligned}$$ ❉

Anticommutation Relations

- From ❇ we see that
  all the coefficients α_i and β anticommute with each other
  and hence they cannot simply be numbers
- We are lead to consider matrices $\alpha^k (k = 1, 2, 3)$ and β
  which are required to satisfy the condition

$$\alpha^k \alpha^l + \alpha^l \alpha^k \equiv \{\alpha^k, \alpha^l\} = 2\delta^{kl}, \quad \{\alpha^k, \beta\} = 0, \text{ and } \beta^2 = 1$$

is the unit matrix

- It turns out that the lowest dimensionality matrices which guarantee relativistic energy momentum relation also holds true ☛ 4×4
- A four-component quantity $\psi_\alpha(x)$ which satisfies the Dirac equation

$$i\,\partial_t\,\psi_\rho(x) = -i\,[\alpha_{\rho\sigma}]^k\;\partial_{x^k}\,\psi_\sigma(x) + m\;\beta_{\rho\sigma}\;\psi_\sigma(x)$$

☛ called a Spinor

- Its transformation properties are different from that of a 4-vector

Dirac-Pauli & Weyl representations

Specific representation of matrices α^k and β

- ✓ Dirac Pauli representation ☛

$$\vec{\alpha} = \begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix}$$

and

$$\beta = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix}$$

- ✓ Weyl or chiral representation ☛
  2 x 2 block form

$$\vec{\alpha} = \begin{pmatrix} -\vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix}$$

and

$$\beta = \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix}$$

Covariant form of Dirac Equation

On multiplying Dirac's equation by β from the left we obtain

$$i\,\beta\,\partial_t\psi = -i\,\beta\,\vec{\alpha}\,.\,\vec{\nabla}\,\psi + m\psi$$

which can be rewritten as

$$i\gamma^0\partial_t\psi + i\gamma^k\partial_{x^k}\psi - m\psi = 0 \quad ♠$$

or equivalently

$$(i\gamma^\mu\partial_\mu - m)\,\psi = 0$$

we omit spinor subscripts whenever there is no danger of confusion

Dirac matrices

we have introduced four Dirac γ -matrices $\gamma^\mu \equiv (\beta, \beta\vec{\alpha})$
which satisfy the anticommutation relations

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$$

We can now unequivocally see that Dirac's equation

is actually 4 differential equations

$$\sum_{\sigma=1}^{4} \left\{ \sum_{\mu} i \, [\gamma_{\rho\sigma}]^\mu \, \partial_\mu - m \, \delta_{\rho\sigma} \right\} \psi_\sigma = 0$$

which couple the four components of a single Dirac spinor ψ

Lorentz invariance

General Lorentz transformation contains rotations and because $\psi(x)$ is supposed to describe a field with spin ☛ under the transformation

$x^\mu \to x'^\mu = \Lambda^\mu{}_\nu x^\nu$ we allow for a rearrangement of $\psi(x)$ components

Because both Dirac equation and Lorentz transformation of coordinates are themselves linear ☛ we ask transformation between ψ & ψ' be linear

$$\psi'(x') = \psi'(\Lambda x) = S(\Lambda)\,\psi(x) = S(\Lambda)\psi(\Lambda^{-1}x')$$

$S(\Lambda)$ is a 4×4 matrix which operates on ψ

To figure out S ☛ demand Dirac eq. has same form in any inertial frame

$$(i\gamma^\mu \partial_\mu{}' - m)\psi'(x') = 0$$

or equivalently

$$(i\gamma^\mu \Lambda_\mu{}^\nu \partial_\nu - m)S(\Lambda)\,\psi(x) = 0$$

Lorentz Invariance

If we multiply by $S^{-1}(\Lambda)$ from left we get

$$(i\, S^{-1}\, \gamma^\mu\, S\, \Lambda_\mu{}^\nu\, \partial_\nu - m)\, \psi(x) = 0$$

Dirac eq. is form-invariant provided we can find $S(\Lambda)$ such that

$$S^{-1}(\Lambda)\ \gamma^\mu\ S(\Lambda)\ \Lambda_\mu{}^\nu = \gamma^\nu$$ ❖

Consider infinitesimal Lorentz Transformation

$$S(\Lambda) = 1 - \frac{i}{2}\omega_{\mu\nu}\Sigma^{\mu\nu}$$

after a bit of algebra ❖ reduces to the condition

$$[\Sigma^{\mu\nu}, \gamma^\beta] = -i(g^{\mu\beta}\gamma^\nu - g^{\nu\beta}\gamma^\mu)$$

A solution is seen to be

$$\Sigma^{\mu\nu} \equiv \frac{1}{2}\sigma^{\mu\nu} = \frac{i}{4}[\gamma^\mu, \gamma^\nu]$$ ✧

See lecture notes for details

Hints for the calculation

$$(1+\frac{i}{2}\omega_{\rho\sigma}\Sigma^{\rho\sigma})\gamma^{\mu}(1+\frac{i}{2}\omega_{\rho\sigma}\Sigma^{\rho\sigma}) = (1-\frac{i}{2}\omega_{\rho\sigma}\mathcal{J}^{\rho\sigma})^{\mu}_{\nu}\gamma^{\nu}$$

with

$$(\mathcal{J}^{\rho}\sigma)_{\mu\nu} = i(\delta^{\rho}_{\mu}\,\delta^{\sigma}_{\nu} - \delta^{\rho}_{\nu}\,\delta^{\sigma}_{\mu})$$

This equation is just the infinitesimal form of

$$S^{-}1(\Lambda)\,\gamma^{\mu}\,S(\Lambda) = \Lambda^{\mu}_{\nu}\gamma^{\nu}$$

Lorentz Algebra

By repeated use of ☒ it is easily seen that

- satisfies commutation relations of Lorentz algebra

$$[\Sigma^{\mu\nu}, \Sigma^{\rho\sigma}] = i(g^{\nu\rho}\Sigma^{\mu\sigma} - g^{\mu\rho}\Sigma^{\nu\sigma} - g^{\nu\sigma}\Sigma^{\mu\rho} + g^{\mu\sigma}\Sigma^{\nu\rho})$$

Incidentally $S^\dagger(\Lambda) = \gamma^0 \; S^{-1}(\Lambda) \; \gamma^0$

The form for $S(\Lambda)$ when Λ is not infinitesimal is

$$S(\Lambda) = e^{-(i/2)\,\omega_{\mu\nu}\;\Sigma^{\mu\nu}}$$

Rotations & Boosts

For a rotation $\omega_{i0} = 0$ and $\omega_{ij} = \theta_k$ and because $\Sigma^{ij} = \frac{1}{2}\epsilon^{ijk}\,\sigma^k$

we get $S(\Lambda) = e^{-(i/2)\,\theta\,.\,\sigma}$ which shows the connection between ω_{ij}

and parameters characterizing rotation $(i,\ j,\ k = 1,\ 2,\ 3)$

For a pure Lorentz transformation $\omega_{ij} = 0$ and $\omega_{i0} = \vartheta_i$

and because $\Sigma^{0i} = \frac{i}{2}\alpha^i$ we have

$$
\begin{aligned}
S(\Lambda) &= e^{(1/2)\,\vartheta\,.\,\alpha} \\
&= 1 + \frac{1}{2}\vartheta\,.\alpha + \frac{\mathbf{1}}{\mathbf{2!}}\left(\frac{\vartheta^{\mathbf{2}}}{\mathbf{4}}\right) + \frac{\mathbf{1}}{\mathbf{3!}}\left(\frac{\vartheta^{\mathbf{2}}}{\mathbf{4}}\right)\frac{\vartheta\,.\alpha}{\mathbf{2}} + \ldots \\
&= \cosh\frac{\vartheta}{2} + \hat{\vartheta}\,.\alpha\,\sinh\frac{\vartheta}{\mathbf{2}}
\end{aligned}
$$

Hints for the calculation

$$\alpha\beta = \begin{pmatrix} 0 & \sigma \\ \sigma & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & -\sigma \\ \sigma & 0 \end{pmatrix}$$

$$\beta\alpha\beta = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -\sigma \\ \sigma & 0 \end{pmatrix} = \begin{pmatrix} 0 & -\sigma \\ -\sigma & 0 \end{pmatrix} = -\alpha$$

$$\frac{i}{4}\left[\gamma^0, \gamma^i\right] = \frac{i}{4}\left\{\beta^2\alpha - \beta\alpha\beta\right\} = \frac{i}{2}\alpha$$

$$\Sigma^{0i} = \frac{i}{2}\alpha_i \qquad \Sigma^{i0} = -\frac{i}{2}\alpha_i \qquad \omega_{0i} = -\omega_{i0}$$

$$\vartheta \,.\, \alpha = \vartheta_1\alpha_1 + \vartheta_2\alpha_2 + \vartheta_3\alpha_3$$

$$(\sigma_1)^2 = (\sigma_2)^2 = (\sigma_3)^2 \Rightarrow (\vartheta_1\alpha_1 + \vartheta_2\alpha_2 + \vartheta_3\alpha_3)^2 = \vartheta^2$$

Rotation through imaginary angle

For special case we may find connection between ϑ and velocity $\vec{v}$ characterizing pure Lorentz transformation by looking at

consider Lorentz transformation in which new -prime- frame moves with velocity v along x_3 axis of original -unprimed- frame

$$\begin{aligned} t' &= \cosh(\vartheta_3)\, t - \sinh(\vartheta_3)\, x_3 \\ x'_3 &= -\sinh(\vartheta_3)\, t + \cosh(\vartheta_3)\, x_3 \end{aligned}$$

with x and y unchanged

here ☛

$$\cosh(\vartheta_3) = \frac{1}{\sqrt{1-v^2}} \quad \text{and} \quad \hat{\vartheta} \equiv \frac{\vec{v}}{v} = \hat{k}$$

Because $\cos(i\vartheta_3) = \cosh(\vartheta_3)$ and $\sin(i\vartheta_3) = \sinh(\vartheta_3)$

we see that Lorentz transformation may be regarded as a rotation through imaginary angle $i\vartheta_3$ in $it - x_3$ plane

Adjoint Dirac Equation

- To construct currents ☛ we duplicate Klein Gordon calculation taking account that Dirac eq. is matrix Eq. and thus we must consider hermitian rather than complex conjugate eq.

- Dirac eq. hermitian conjugate is

$$-i\partial_t \psi^\dagger \gamma^0 - i\partial_{x^k} \psi^\dagger (-\gamma^k) - m\psi^\dagger = 0 \quad ❄$$

- To restore covariant form we need to flip plus sign in 2nd term while leaving 1st term unchanged
- Since $\gamma^0\gamma^k = -\gamma^k\gamma^0$ this can be accomplished by multiplying ❄ from the right by γ^0
- Introducing the adjoint --row-- spinor $\bar{\psi} \equiv \psi^\dagger\gamma^0$ we obtain ☛

$$i\partial_\mu \bar{\psi}\gamma^\mu + m\bar{\psi} = 0$$

Dirac Lagrangian

we pause to discuss transformation properties of

$$\bar{\psi}(x)\,\gamma^\mu\,\psi(x)$$

we have

$$\begin{aligned}\bar{\psi}'(x')\,\gamma^\mu\,\psi'(x') &= \bar{\psi}(x)\,S^{-1}(\Lambda)\,\gamma^\mu\,S(\Lambda)\,\psi(x)\\ &= \Lambda^\mu{}_\alpha\,\bar{\psi}(x)\,\gamma^\alpha\,\psi(x)\end{aligned}$$

under a Lorentz transformation

bilinear combination $\bar{\psi}(x)\,\gamma^\mu\,\psi(x)$

transforms like a contravariant four-vector

we can write down a Lagrangian for spin-1/2 relativistic particles

$$\mathcal{L}_{\text{Dirac}} = \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi$$

Continuity Equation

By adding ♠ multiplied from left by $\bar{\psi}$ and ❄ from right by ψ

we obtain ☛

$$\bar{\psi}\,\gamma^\mu\,\partial_\mu\psi + (\partial_\mu\bar{\psi})\gamma^\mu\psi = \partial_\mu(\bar{\psi}\gamma^\mu\psi) = 0$$

showing that probability and flux densities $j^\mu = \bar{\psi}\gamma^\mu\psi$

satisfy continuity equation

Moreover ☛

$$\rho \equiv j^0 = \bar{\psi}\gamma^0\psi = \psi^\dagger\psi = \sum_{i=1}^{4}|\psi_i|^2$$

is now positive definite

In this respect quantity $\psi(x)$ resembles Schrodinger wave function

Dirac eq. may serve as a one particle eq.

In that role --however-- coefficient of $-ix^0$ in Fourier decomposition

$$\psi(x) = \int dp\,\psi(p)\,e^{-ip\,.\,x}$$

plays role of energy

and there is no reason why negative energies should be excluded

Plane wave solutions

Next ☛ we discuss plane wave solutions of Dirac equation

we will treat positive and negative frequency terms separately and therefore write ☛

$$\psi(x) = u(p)e^{-ipx} + v(p)\,e^{ipx}$$

Since ψ also satisfies Klein Gordon equation ☛

it is necessary that $p^\mu p_\mu = m^2$ so that $p^0 = +\sqrt{\vec{p}^{\,2} + m^2} \equiv E$

we will call e^{-iEx_0} positive frequency solution

From Dirac eq. it follows that

$$[i\gamma^\mu(-ip_\mu) - m]\,u(p)\;e^{-ipx} + [i\gamma^\mu(ip_\mu) - m]\,v(p)\;e^{ipx} = 0$$

or equivalently

$$\begin{aligned}(\gamma^\mu p_\mu - m)\,u(p) &= 0\\ (\gamma^\mu p_\mu + m)\,v(p) &= 0\end{aligned}$$

because positive and negative frequency solutions are independent

Positron spinors

- two negative energy solutions $u^{(3,4)}$

  are to be associated with an antiparticle ☛ say the positron

- Using antiparticle prescription

  positron of energy E and momentum $\vec{p}$

  is described by one of $-E$ and $-\vec{p}$ electron solutions

$$u^{(3,4)}(-p)\, e^{-i[-p]\,\cdot\, x} \equiv v^{(2,1)}(p) e^{ip\,\cdot\, x}$$

  where $p^0 \equiv E > 0$

- positron spinors v are defined just for notational convenience

Solution for free particles at rest

- It is useful to introduce the notation $\gamma^\mu p_\mu = \gamma_\mu p^\mu = \not p$
- "slash" quantities satisfy $\{\not a, \not b\} = a_\mu b_\nu \{\gamma^\mu, \gamma^\nu\} = 2 a_\mu b^\mu \equiv 2a.b$
- Dirac eq. for a plane wave solution may thus be written as

$$\begin{aligned} (\not p - m)\ u(p) &= 0 \\ (\not p + m)\ v(p) &= 0 \end{aligned}$$

- It is easily seen that

$$\begin{aligned} \bar u(p)\ (\not p - m) &= 0 \\ \bar v(p)\ (\not p + m) &= 0 \end{aligned}$$

- When $\vec p = 0$ and $p_0 = m$ equations take form ☛

$$\begin{aligned} (\gamma^0 - 1)\ m\ u(0) &= 0 \\ (\gamma^0 + 1)\ m\ v(0) &= 0 \end{aligned}$$

- There are two positive and two negative frequency solutions ☛

$$u^{(1)}(0) = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad u^{(2)}(0) = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \quad v^{(2)}(0) = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \quad v^{(1)}(0) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

Solution for arbitrary momentum

- Since $(\not p + m)(\not p - m) = p^2 - m^2 = 0$

  we may write the solution for arbitrary p in the form

$$
\begin{aligned}
u^{(r)}(p) &= C\,(m + \not p)\,u^{(r)}(0) \\
v^{(r)}(p) &= C'\,(m - \not p)\,v^{(r)}(0)
\end{aligned}
$$

$r = 1, 2$ and C and C' are normalization constants

- For fermions we choose covariant normalization in which we have

  $2E$ particles/unit volume just as we did for bosons

$$
\int_{\text{unit vol.}} \rho\, dV = \int \psi^\dagger\, \psi\, dV = u^\dagger(p)\; u(p) = 2E
$$

where we have used

- $\rho \equiv j^0 = \bar{\psi}\gamma^0\psi = \psi^\dagger\psi = \sum_{i=1}^{4} |\psi_i|^2$
- $\psi(x) = u(p)e^{-ipx} + v(p)\,e^{ipx}$

Orthogonality Relations

- This leads to the orthogonality relations

$$u^{(r)\dagger}(p)\; u^{(s)}(p) = 2E\delta_{rs}\,, \quad v^{(r)\dagger}(p)\; v^{(s)}(p) = 2E\delta_{rs}$$

- By summing

$$\begin{cases} \bar{u}(p)\,\gamma^0(\gamma^\mu p_\mu - m)\,u(p) = 0 \\ \bar{u}(p)\,(\gamma^\mu p_\mu - m)\,\gamma^0\,u(p) = 0 \end{cases}$$

we obtain ☛ $2\,\bar{u}(p)\;p_0\;u(p) - 2\,m\,u^\dagger(p)\;u(p) = 0$

where we have used relation ☛ $\gamma^0\gamma^k = -\gamma^k\gamma^0$

- orthogonality relations then become

$$\bar{u}^{(r)}(p)\; u^{(s)}(p) = \frac{m}{E}\,u^{(r)\dagger}(p)\; u^{(s)}(p) = 2m\delta_{rs}$$

and

$$\bar{v}^{(r)}(p)\; v^{(s)}(p) = -\frac{m}{E}\,v^{(r)\dagger}(p)\; v^{(s)}(p) = -2m\delta_{rs}$$

Normalization Constant

- Using $\not{p}\,\not{p} = p^2$ we obtain

$$\begin{aligned}
\bar{u}^{(r)}(p)\, u^{(s)}(p) &= |C|^2\, \bar{u}^{(r)}(0)\, (m+\not{p})(m+\not{p})\, u^{(s)}(0) \\
&= 2m\,|C|^2\, \bar{u}^{(r)}(0)\, (m+\not{p}\, u^{(s)}(0) \\
&= 2m\,|C|^2\, \bar{u}^{(r)}(0)\, (m+\gamma^0 p_0 + \alpha^k p_k \beta)\, u^{(s)}(0) \\
&= 2m\,|C|^2\, (m+E)\, \bar{u}^{(r)}(0)\, u^{(s)}(0) \\
&= 2m\,|C|^2\, (m+E)\, \delta_{rs}
\end{aligned}$$

and determine the normalization constant

$$C = \frac{1}{\sqrt{m+E}}$$

- A straightforward calculation leads to ☛

$$C' = \frac{1}{\sqrt{m+E}}$$

Hints for the calculation

$$(\not p + m)(\not p + m) = m^2 + 2m \not p + \not p^2 = 2m^2 + 2m \not p$$

$$\begin{aligned}
\gamma^0 p_0 + \gamma^k p_k &= \begin{pmatrix} E & 0 & 0 & 0 \\ 0 & E & 0 & 0 \\ 0 & 0 & -E & 0 \\ 0 & 0 & 0 & -E \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & p_1 \\ 0 & 0 & p_1 & 0 \\ 0 & p_1 & 0 & 0 \\ p_1 & 0 & 0 & 0 \end{pmatrix} \\
&+ \begin{pmatrix} 0 & 0 & 0 & -ip_2 \\ 0 & 0 & ip_2 & 0 \\ 0 & -ip_2 & 0 & 0 \\ ip_2 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & p_3 & 0 \\ 0 & 0 & 0 & -p_3 \\ p_3 & 0 & 0 & 0 \\ 0 & -p_3 & 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} E & 0 & p_3 & p_1 - ip_2 \\ 0 & E & p_1 + ip_2 & -p_3 \\ p_3 & p_1 - ip_2 & -E & 0 \\ p_1 + ip_2 & -p_3 & 0 & -E \end{pmatrix}
\end{aligned}$$

Hints for the calculation (cont'd)

$$
\begin{aligned}
\bar{u}^{(1)}(0)(\gamma^0 p_0 + \gamma^k p_k) u^{(1)}(0) &= \begin{pmatrix} 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \\
&\times \begin{pmatrix} E & 0 & p_3 & p_1 - ip_2 \\ 0 & E & p_1 + ip_2 & -p_3 \\ p_3 & p_1 - ip_2 & -E & 0 \\ p_1 + ip_2 & -p_3 & 0 & -E \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} E \\ 0 \\ p_3 \\ p_1 + ip_2 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} E \\ 0 \\ -p_3 \\ -p_1 - ip_2 \end{pmatrix}
\end{aligned}
$$

General Solution of Dirac Equation

Introducing two-component spinors $\chi^{(r)}$ where $\chi^1 = \binom{1}{0}$ and $\chi^2 = \binom{0}{1}$

we may examine explicit form of solution of Dirac equation

For $E > 0$ we have

$$
\begin{aligned}
u^{(r)}(p) &= \frac{m + \not{p}}{\sqrt{m+E}}\,\chi^{(r)} \\
&= \frac{m + \sigma_3 E - i\sigma_2 \sigma\,.\vec{p}}{\sqrt{m+E}}\,\chi^{(r)} \\
&= \frac{1}{\sqrt{m+E}} \begin{pmatrix} m+E & -\sigma.\vec{p} \\ \sigma.\vec{p} & m-E \end{pmatrix} \begin{pmatrix} \chi^{(r)} \\ 0 \end{pmatrix} \\
&= \sqrt{E+m} \begin{pmatrix} \chi^{(r)} \\ (E+m)^{-1}\,\sigma.\vec{p}\;\chi^{(r)} \end{pmatrix}
\end{aligned}
$$

and so positive-energy four spinor solutions of Dirac's equation are

- $$u_1(E,\vec{p}) = \sqrt{m+E} \begin{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ (m+E)^{-1}\,\sigma.\vec{p} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \end{pmatrix}$$
- $$u_2(E,\vec{p}) = \sqrt{m+E} \begin{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ (m+E)^{-1}\,\sigma.\vec{p} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{pmatrix}$$

For low momenta

upper two components are a great deal larger than the lower ones

NEGATIVE ENERGY SOLUTIONS

- For the $E < 0$

$$u^{(r+2)}(p) = \frac{1}{\sqrt{m+E}} \begin{pmatrix} m+E & -\sigma.\vec{p} \\ \sigma.\vec{p} & m-E \end{pmatrix} \begin{pmatrix} 0 \\ \chi^{(r)} \end{pmatrix}$$

- solutions of Dirac eq. are

$$u_3(E,\vec{p}) = \sqrt{m-E} \begin{pmatrix} -(m-E)^{-1}\,\sigma.\vec{p} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \end{pmatrix}$$

$$u_4(E,\vec{p}) = \sqrt{m-E} \begin{pmatrix} -(m-E)^{-1}\,\sigma.\vec{p} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{pmatrix}$$

Projection operators & completeness

- To obtain completeness properties of solutions
  consider positive and negative solutions separately
  We use the explicit solutions already obtained ☛

$$
\begin{aligned}
(\Lambda_+)_{\alpha\beta} &\equiv \frac{1}{2m}\sum_{r=1}^{2} u_\alpha^{(r)}(p)\,\bar{u}_\beta^{(r)}(p) \\
&= \frac{1}{2m\,(m+E)}\left[\sum_r (\not{p}+m)u^{(r)}(0)\,\bar{u}^{(r)}(0)(\not{p}+m)\right]_{\alpha\beta} \\
&= \frac{1}{2m\,(m+E)}\left[(m+\not{p})\,\frac{1+\gamma^0}{2}\,(m+\not{p})\right]_{\alpha\beta} \\
&= \frac{1}{2m\,(m+E)}\left\{m(\not{p}+m)+\frac{1}{2}(\not{p}+m)[(m-\not{p})\gamma^0+2E]\right\}_{\alpha\beta} \\
&= \frac{1}{2m}\,(\not{p}+m)_{\alpha\beta}
\end{aligned}
$$

More on Projection Operators & Completeness

- Similarly if we define Λ_- by ☛ $(\Lambda_-)_{\alpha\beta} = -\frac{1}{2m} \sum_{r=1}^{2} v_\alpha^{(r)}(p)\ \bar{v}_\beta^{(r)}(p)$

  we get ☛ $(\Lambda_-)_{\alpha\beta} = \frac{1}{2m}(m - \not{p})_{\alpha\beta}$

- completness relation is that

$$\Lambda_+ + \Lambda_- = \frac{1}{2m} \sum_{r=1}^{2} [u_\alpha^{(r)}(p)\ \bar{u}_\beta^{(r)}(p) - v_\alpha^{(r)}(p)\ \bar{v}_\beta^{(r)}(p)] = 1$$

- matrices Λ_+ and Λ_- have properties of projection operators

  because $\Lambda_\pm^2 = \Lambda_\pm$ and $\Lambda_+\Lambda_- = \Lambda_-\Lambda_+ = 0$

- operators $\Lambda_\pm$ project positive and negative frequency solutions but because there are four solutions there ☛ must still be another projector operator which separates $r = 1, 2$ solutions

- This projector operator h must be such that

$$h^{(r)}\, h^{(s)} = \delta_{rs}\, h^{(r)} \quad \text{and} \quad [h^{(r)}, \Lambda_\pm] = 0$$

Helicity operator

Since two solutions have something to do with two possible polarization directions of spin-1/2 particle ☛ we may expect the operator to be some sort of generalization of non-relativistic operator that projects out the state polarized in a given direction for two component spinors

On inspection ☛ we see that the the helicity operator

$$h \equiv \hat{p} \, . \, \mathbf{\Sigma} = \frac{1}{2} \hat{p}_k \begin{pmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{pmatrix}$$

satisfies ✂ where $\hat{p} \equiv \vec{p}/|\vec{p}|$

Helicity operator commutes with H
so it shares its eigenstates with H
and its eigenvalues are conserved

Eigenvalues of helicity operator

To find the eigenvalues of the helicity operator we calculate

$$h^2 = \frac{1}{4}\begin{pmatrix} (\sigma.\hat{p})^2 & 0 \\ 0 & (\sigma.\hat{p})^2 \end{pmatrix} = \frac{1}{4}\begin{pmatrix} \hat{p}^2 & 0 \\ 0 & \hat{p}^2 \end{pmatrix}$$

Thus, the eigenvalues of the helicity operator are

$$h = \begin{cases} +\frac{1}{2} \ \text{positive helicity} & \Rightarrow\!\!\rightarrow \\ -\frac{1}{2} \ \text{negative helicity} & \Leftarrow\!\!\rightarrow \end{cases}$$

The spin component in the direction of motion $\frac{1}{2}\hat{p}\cdot\sigma$

is thus a good quantum number and can be used to label the solutions

Particle's spin up and spin down

Assuming a particle has a momentum $\vec{p}$ and choosing $x_3-\mathrm{axis}$ along the direction of $\vec{p}$ we can determine which of the four spinor u_1, u_2, v_1 and v_2 have spin up and spin down

With these assumptions $\sigma \cdot \vec{p} = \sigma_3 p_3, |\vec{p}| = p_3$

and the helicity operator simplifies to $h = \frac{1}{2}\begin{pmatrix} \sigma_3\hat{p}_3 & 0 \\ 0 & \sigma_3\hat{p}_3 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} \sigma_3 & 0 \\ 0 & \sigma_3 \end{pmatrix}$

We then find ☛

$$
\begin{aligned}
hu_1 &= \frac{\sqrt{E+m}}{2}\begin{pmatrix} 1 & & & \\ & -1 & & \\ & & 1 & \\ & & & -1 \end{pmatrix}\begin{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ (E+m)^{-1}\,\sigma.\vec{p}\,\begin{pmatrix} 1 \\ 0 \end{pmatrix} \end{pmatrix} \\
&= \frac{\sqrt{E+m}}{2}\begin{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ (E+m)^{-1}\,\sigma.\vec{p}\,\begin{pmatrix} 1 \\ 0 \end{pmatrix} \end{pmatrix} = \frac{1}{2}u_1 \\
hu_2 &= \frac{\sqrt{E+m}}{2}\begin{pmatrix} 1 & & & \\ & -1 & & \\ & & 1 & \\ & & & -1 \end{pmatrix}\begin{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ (E+m)^{-1}\,\sigma.\vec{p}\,\begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{pmatrix} \\
&= \frac{\sqrt{E+m}}{2}\begin{pmatrix} \begin{pmatrix} 0 \\ -1 \end{pmatrix} \\ (E+m)^{-1}\,\sigma.\vec{p}\,\begin{pmatrix} 0 \\ -1 \end{pmatrix} \end{pmatrix} = -\frac{1}{2}u_2
\end{aligned}
$$

Antiparticle's spin up and spin down

For antiparticles with negative energy and momentum $-\vec{p}, \sigma.\vec{p} = \sigma_3(-p_3)$

and the helicity operator simplifies to

$$h = \frac{1}{2}\begin{pmatrix} -\sigma_3 \hat{p}_3 & 0 \\ 0 & -\sigma_3 \hat{p}_3 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} -\sigma_3 & 0 \\ 0 & -\sigma_3 \end{pmatrix}$$

We then find ☛

$$hv_1 = \frac{\sqrt{E+m}}{2}\begin{pmatrix} -1 & & & \\ & 1 & & \\ & & -1 & \\ & & & 1 \end{pmatrix}\begin{pmatrix} (E+m)^{-1}\ \sigma.\vec{p}\begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{pmatrix}$$

$$= \frac{\sqrt{E+m}}{2}\begin{pmatrix} (E+m)^{-1}\ \sigma.\vec{p}\begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{pmatrix} = \frac{1}{2}v_1$$

$$hv_2 = \frac{\sqrt{E+m}}{2}\begin{pmatrix} -1 & & & \\ & 1 & & \\ & & -1 & \\ & & & 1 \end{pmatrix}\begin{pmatrix} (E+m)^{-1}\ \sigma.\vec{p}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \end{pmatrix}$$

$$= \frac{\sqrt{E+m}}{2}\begin{pmatrix} (E+m)^{-1}\ \sigma.\vec{p}\begin{pmatrix} -1 \\ 0 \end{pmatrix} \\ \begin{pmatrix} -1 \\ 0 \end{pmatrix} \end{pmatrix} = -\frac{1}{2}v_2$$

Homework

Use Euler-Lagrange equation to derive:

Klein-Gordon equation from ☛ $\mathcal{L}_{\rm KG} = \frac{1}{2}\partial_\mu\phi\ \partial^\mu\phi - \frac{1}{2}m^2\phi^2$

and Dirac equation from ☛ $\mathcal{L}_{\rm Dirac} = \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi$

Bilinear Covariants

To construct most general form of Lorentz covariant currents

need to tabulate bilinear quantities of form $\bar{\psi}(4 \times 4)\psi$

which have definite properties under Lorentz transformations

To simplify the notation ☛ we introduce $\gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3$

It follows that

$$\gamma^{5\dagger} = \gamma^5, \quad (\gamma^5)^2 = \mathbb{I}, \quad \gamma^5\gamma^\mu + \gamma^\mu\gamma^5 = 0$$

In Dirac-Pauli representation

$$\gamma^5 = \begin{pmatrix} 0 & \mathbb{I} \\ \mathbb{I} & 0 \end{pmatrix}$$

We are interested in behavior of bilinear quantities under:

proper Lorentz transformations --that is rotations and boosts--

and

under space invertion --parity operation--

Explicit form of Bilinear covariants

list arranged in increasing order of γ^μ matrices

that are sandwiched between $\bar{\psi}$ and ψ

pseudoscalar is the product of four matrices

If five matrices were used ☛ at least two would be the same

in which case product will be reduced to three

and be already included in axial vector

		No. of Compts.	Space Inversion, P
Scalar	$\bar{\psi}\psi$	1	$+$ under P
Vector	$\bar{\psi}\gamma^\mu\psi$	4	Space compts. $-$ under P
Tensor	$\bar{\psi}\sigma^{\mu\nu}\psi$	6	
Axial vector	$\bar{\psi}\gamma^5\gamma^\mu\psi$	4	Space compts. $+$ under P
Pseudoscalar	$\bar{\psi}\gamma^5\psi$	1	$-$ under P

Examples

Because of anticommutation relations, $\{\gamma^\mu, \gamma^\nu\} = 2\gamma^{\mu\nu}$

tensor is antisymmetric

$$\sigma^{\mu\nu} = \frac{i}{2}(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu)$$

From
$$\begin{aligned} \bar{\psi}'(x')\,\gamma^\mu\,\psi'(x') &= \bar{\psi}(x)\,S^{-1}(\Lambda)\,\gamma^\mu\,S(\Lambda)\,\psi(x) \\ &= \Lambda^\mu{}_\alpha\,\bar{\psi}(x)\,\gamma^\alpha\,\psi(x) \end{aligned}$$

It follows immediately that $\bar{\psi}\psi$ is a Lorentz scalar

The probability density $\rho = \psi^\dagger\psi$ is not a scalar, but is the timelike component of the four vector $\bar{\psi}\gamma^\mu\psi$

Because $\gamma^5 S_P = -S_P\gamma^5$ the presence of γ^5 gives rise to the pseudo-nature of the axial vector and pseudoscalar

i.e., a pseudoscalar is a scalar under proper Lorentz transformations but, unlike a scalar, changes sign under parity


That's all