Prof. Anchordoqui

Problems set # 6 **Physics 541-735** October 18, 2011

1. Show that the charge $Q = \int d^3x\, j^0$ must be a conserved quantity because of $U(1)$ phase invariance.

2. The Lagrangian for three interacting real fields ϕ_1, ϕ_2, ϕ_3 is

$$\mathscr{L} = \frac{1}{2}(\partial_\mu \phi_i)^2 - \frac{1}{2}\mu^2 {\phi_i}^2 - \frac{1}{4}\lambda({\phi_i}^2)^2, \tag{1}$$

where $\mu^2 < 0$, $\lambda > 0$, and a summation of ${\phi_i}^2$ over i is implied. Show that it describes a massive field of mass $\sqrt{-2\mu^2}$ and two massless Goldstone bosons.

3. Rather than (2.4.60), take instead ϕ to be an $SU(2)$ triplet of real scalar fields. For $\mu^2 < 0$ and $\lambda > 0$, show that in this case two gauge bosons acquire mass but that the third remains massless. [Hint: Verify, and use, $(T_k)_{ij} = -i\epsilon_{ijk}$ for the triplet representation of $SU(2)$.]

4. Generic extensions of the standard model predic the existence of extra $U(1)$ gauge symmetries beyond hypercharge. However, in order to avoid long range forces other than gravity and Coulomb forces, only $U(1)$ of hypercharge must survive as a massless gauge boson and all other extra Abelian gauge bosons must grow a mass. To understand the basis of the Stuckelberg mechanism giving mass to the $U(1)$'s, consider the following Lagrangian

$$\mathscr{L} \; = \; -\frac{1}{12} H^{\mu\nu\rho} H_{\mu\nu\rho} - \frac{1}{4g^2} F^{\mu\nu} F_{\mu\nu} + \frac{c}{4}\, \epsilon^{\mu\nu\rho\sigma} B_{\mu\nu}\, F_{\rho\sigma}$$

coupling an Abelian gauge field A_μ to an antisymmetric tensor $B_{\mu\nu}$, where

$$H_{\mu\nu\rho} = \partial_\mu B_{\nu\rho} + \partial_\rho B_{\mu\nu} + \partial_\nu B_{\rho\mu}, \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu,$$

and g, c are arbitrary constants. This corresponds to the kinetic term for the fields $B_{\mu\nu}$ and A_μ together with the $B \wedge F$ term.
(i) Imposing the constraint $H = dB$ by the standard introduction of a Lagrange multiplier field η, rewrite the Lagrangian in terms of the (arbitrary) field $H_{\mu\nu\rho}$.
(ii) Show that the Lagrangian is dual to

$$\mathscr{L} = -\frac{1}{4g^2}\; F^{\mu\nu}\; F_{\mu\nu} - \frac{c^2}{2}\,(A_\sigma + \partial_\sigma \eta)^2\;,$$

which is just a Stuckelberg mass term for the gauge field A_μ after "eating" the scalar η to acquire a mass $m^2 = g^2 c^2$.

5. Show that

$$\frac{1}{2}\tau^3 L_L = T^3 L_L,$$

where

$$L_L = \begin{pmatrix} \nu_L \\ e_L \end{pmatrix}, \quad \tau^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

and

$$T^3 = \frac{1}{2} \int \psi_L^\dagger \tau^3 \psi_L d^3x \, .$$