Special Relativity

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VII: Relativistic Electrodynamics (part 2)

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The Field Tensor

Recall that a 4-vector transforms by the rule

$$\bar{a}^\mu = \Lambda^\mu_\nu a^\nu$$

$$\begin{pmatrix} \bar{a}^0 \\ \\ \\ \\ \end{pmatrix} = \begin{pmatrix} \gamma\ \Lambda^0_0 & -\gamma\beta\ \Lambda^0_1 & 0\ \Lambda^0_2 & 0\ \Lambda^0_3 \\ -\gamma\beta\ \Lambda^1_0 & \gamma\ \Lambda^1_1 & 0\ \Lambda^1_2 & 0\ \Lambda^1_3 \\ 0\ \Lambda^2_0 & 0\ \Lambda^2_1 & 1\ \Lambda^2_2 & 0\ \Lambda^2_3 \\ 0\ \Lambda^3_0 & 0\ \Lambda^3_1 & 0\ \Lambda^3_2 & 1\ \Lambda^3_3 \end{pmatrix} \begin{pmatrix} a^0 \\ a^1 \\ a^2 \\ a^3 \end{pmatrix}$$

However ☛ from last class

the components of E and B are stirred together when you go from one inertial system to another

$$\bar{E}_x = E_x \quad \bar{E}_y = \gamma(E_y - vB_z) \quad \bar{E}_z = \gamma(E_z + vB_y)$$

$$\bar{B}_x = B_x \quad \bar{B}_y = \gamma\left(B_y + \frac{v}{c^2}E_z\right) \quad \bar{B}_z = \gamma\left(B_z - \frac{v}{c^2}E_y\right)$$

☛ 6 transformation rules

A tensor (in 4 dimensions) has $4 \times 4 = 16$ components which we can display in a 4×4 array

The Field Tensor (cont'd)

What type of matrix? the 16 elements need not all be different

In a symmetric tensor

$$F^{\mu\nu}_{sym} = \begin{pmatrix} F^{00} & F^{01} & F^{02} & F^{03} \\ F^{01} & F^{11} & F^{12} & F^{13} \\ F^{02} & F^{12} & F^{22} & F^{23} \\ F^{03} & F^{13} & F^{23} & F^{33} \end{pmatrix}$$

there are 10 distinct components ☛ 6 of the 16 are repeated

An antisymmetric tensor is more likely

$$F^{\mu\nu} = \begin{pmatrix} 0 & F^{01} & F^{02} & F^{03} \\ -F^{01} & 0 & F^{12} & F^{13} \\ -F^{02} & -F^{12} & 0 & F^{23} \\ -F^{03} & -F^{13} & -F^{23} & 0 \end{pmatrix}$$

such an object has only 6 distinct elements

The Field Tensor (cont'd)

Let's see how the transformation rule works

$$\bar{t}^{\mu\nu} = \Lambda^{\mu}_{\lambda}\Lambda^{\nu}_{\sigma}t^{\lambda\sigma}$$

where our matrix is the antisymmetric tensor

$$t^{\mu\nu} = \begin{pmatrix} 0 & t^{01} & t^{02} & t^{03} \\ -t^{01} & 0 & t^{12} & t^{13} \\ -t^{02} & -t^{12} & 0 & t^{23} \\ -t^{03} & -t^{13} & -t^{23} & 0 \end{pmatrix}$$

The Field Tensor (cont'd)

Example ☛ $\bar{t}^{\mu\nu} = \Lambda^{\mu}_{\lambda}\Lambda^{\mu}_{\sigma}t^{\lambda\sigma}$

$$\bar{t}^{00} = \Lambda^{0}_{\lambda=0..3}\Lambda^{0}_{\sigma=0..3}t^{\lambda\sigma} =$$

$$= \Lambda^0_0\Lambda^0_0 t^{00} + \Lambda^0_0\Lambda^0_1 t^{01} + \Lambda^0_0\Lambda^0_2 t^{02} + \Lambda^0_0\Lambda^0_3 t^{03} +$$
$$\Lambda^0_1\Lambda^0_0 t^{10} + \Lambda^0_1\Lambda^0_1 t^{11} + \Lambda^0_1\Lambda^0_2 t^{12} + \Lambda^0_1\Lambda^0_3 t^{13} +$$
$$\Lambda^0_2\Lambda^0_0 t^{20} + \Lambda^0_2\Lambda^0_1 t^{21} + \Lambda^0_2\Lambda^0_2 t^{22} + \Lambda^0_2\Lambda^0_3 t^{23} +$$
$$\Lambda^0_3\Lambda^0_0 t^{30} + \Lambda^0_3\Lambda^0_1 t^{31} + \Lambda^0_3\Lambda^0_2 t^{32} + \Lambda^0_3\Lambda^0_3 t^{33}$$

$$\bar{t}^{00} = \Lambda^0_0\Lambda^0_1 t^{01} + \Lambda^0_1\Lambda^0_0 t^{10} = \Lambda^0_0\Lambda^0_1 t^{01} - \Lambda^0_0\Lambda^0_1 t^{01} = 0$$

$$\begin{pmatrix} \gamma\ \Lambda^0_0 & -\gamma\beta\ \Lambda^0_1 & 0\ \Lambda^0_2 & 0\ \Lambda^0_3 \\ -\gamma\beta\ \Lambda^1_0 & \gamma\ \Lambda^1_1 & 0\ \Lambda^1_2 & 0\ \Lambda^1_3 \\ 0\ \Lambda^2_0 & 0\ \Lambda^2_1 & 1\ \Lambda^2_2 & 0\ \Lambda^2_3 \\ 0\ \Lambda^3_0 & 0\ \Lambda^3_1 & 0\ \Lambda^3_2 & 1\ \Lambda^3_3 \end{pmatrix}$$

$$\bar{t}^{01} = \Lambda^{0}_{\lambda=0..3}\Lambda^{1}_{\sigma=0..3}t^{\lambda\sigma} =$$

$$= \Lambda^0_0\Lambda^1_0 t^{00} + \Lambda^0_0\Lambda^1_1 t^{01} + \Lambda^0_0\Lambda^1_2 t^{02} + \Lambda^0_0\Lambda^1_3 t^{03} +$$
$$\Lambda^0_1\Lambda^1_0 t^{10} + \Lambda^0_1\Lambda^1_1 t^{11} + \Lambda^0_1\Lambda^1_2 t^{12} + \Lambda^0_1\Lambda^1_3 t^{13} +$$
$$\Lambda^0_2\Lambda^1_0 t^{20} + \Lambda^0_2\Lambda^1_1 t^{21} + \Lambda^0_2\Lambda^1_2 t^{22} + \Lambda^0_2\Lambda^1_3 t^{23} +$$
$$\Lambda^0_3\Lambda^1_0 t^{30} + \Lambda^0_3\Lambda^1_1 t^{31} + \Lambda^0_3\Lambda^1_2 t^{32} + \Lambda^0_3\Lambda^1_3 t^{33}$$

$$\bar{t}^{01} = \gamma^2 t^{01} + \gamma^2\beta^2 t^{10} = \underbrace{\left(\gamma^2 - \gamma^2\beta^2\right)}_{=1\ (\text{see right side})} t^{01} = t^{01}$$

$$\gamma^2\beta^2 = \frac{\frac{v^2}{c^2}}{1-\frac{v^2}{c^2}} = \frac{\frac{v^2}{c^2}}{1-\frac{v^2}{c^2}} - 1 + 1 =$$

$$\frac{\frac{v^2}{c^2}}{1-\frac{v^2}{c^2}} - 1 + \frac{1-\frac{v^2}{c^2}}{1-\frac{v^2}{c^2}} =$$

$$\frac{\frac{v^2}{c^2} + 1 - \frac{v^2}{c^2}}{1-\frac{v^2}{c^2}} - 1 = \gamma^2 - 1$$

The Field Tensor (cont'd)

More examples

$$\bar{t}^{23} = \Lambda^2_{\lambda=0..3}\Lambda^3_{\sigma=0..3}t^{\lambda\sigma} =$$

$$= \Lambda^2_0\Lambda^3_0 t^{00} + \Lambda^2_0\Lambda^3_1 t^{01} + \Lambda^2_0\Lambda^3_2 t^{02} + \Lambda^2_0\Lambda^3_3 t^{03} +$$

$$\Lambda^2_1\Lambda^3_0 t^{10} + \Lambda^2_1\Lambda^3_1 t^{11} + \Lambda^2_1\Lambda^3_2 t^{12} + \Lambda^2_1\Lambda^3_3 t^{13} +$$

$$\Lambda^2_2\Lambda^3_0 t^{20} + \Lambda^2_2\Lambda^3_1 t^{21} + \Lambda^2_2\Lambda^3_2 t^{22} + \Lambda^2_2\Lambda^3_3 t^{23} +$$

$$\Lambda^2_3\Lambda^3_0 t^{30} + \Lambda^2_3\Lambda^3_1 t^{31} + \Lambda^2_3\Lambda^3_2 t^{32} + \Lambda^2_3\Lambda^3_3 t^{33}$$

$$\bar{t}^{23} = t^{23}$$

$$\begin{pmatrix} \gamma\ \Lambda^0_0 & -\gamma\beta\ \Lambda^0_1 & 0\ \Lambda^0_2 & 0\ \Lambda^0_3 \\ -\gamma\beta\ \Lambda^1_0 & \gamma\ \Lambda^1_1 & 0\ \Lambda^1_2 & 0\ \Lambda^1_3 \\ 0\ \Lambda^2_0 & 0\ \Lambda^2_1 & 1\ \Lambda^2_2 & 0\ \Lambda^2_3 \\ 0\ \Lambda^3_0 & 0\ \Lambda^3_1 & 0\ \Lambda^3_2 & 1\ \Lambda^3_3 \end{pmatrix}$$

$$\bar{t}^{31} = \Lambda^3_{\lambda=0..3}\Lambda^1_{\sigma=0..3}t^{\lambda\sigma} =$$

$$= \Lambda^3_0\Lambda^1_0 t^{00} + \Lambda^3_0\Lambda^1_1 t^{01} + \Lambda^3_0\Lambda^1_2 t^{02} + \Lambda^3_0\Lambda^1_3 t^{03} +$$

$$\Lambda^3_1\Lambda^1_0 t^{10} + \Lambda^3_1\Lambda^1_1 t^{11} + \Lambda^3_1\Lambda^1_2 t^{12} + \Lambda^3_1\Lambda^1_3 t^{13} +$$

$$\Lambda^3_2\Lambda^1_0 t^{20} + \Lambda^3_2\Lambda^1_1 t^{21} + \Lambda^3_2\Lambda^1_2 t^{22} + \Lambda^3_2\Lambda^1_3 t^{23} +$$

$$\Lambda^3_3\Lambda^1_0 t^{30} + \Lambda^3_3\Lambda^1_1 t^{31} + \Lambda^3_3\Lambda^1_2 t^{32} + \Lambda^3_3\Lambda^1_3 t^{33}$$

$$\bar{t}^{31} = -\gamma\beta t^{30} + \gamma t^{31} = \gamma\left(t^{31} + \beta t^{03}\right)$$

The Field Tensor (cont'd)

So far, we have produced 3 of the 6 components of our antisymmetric tensor

I'll let you work out the others

The complete set of transformation rules is

$$\bar{t}^{01} = t^{01} \quad \bar{t}^{02} = \gamma\left(t^{02} - \beta t^{12}\right) \quad \bar{t}^{03} = \gamma\left(t^{03} + \beta t^{31}\right)$$

$$\bar{t}^{23} = t^{23} \quad \bar{t}^{31} = \gamma\left(t^{31} + \beta t^{03}\right) \quad \bar{t}^{12} = \gamma\left(t^{12} - \beta t^{02}\right)$$

$$\bar{E}_x = E_x \quad \bar{E}_y = \gamma\left(E_y - vB_z\right) \quad \bar{E}_z = \gamma\left(E_z + vB_y\right)$$

$$\bar{B}_x = B_x \quad \bar{B}_y = \gamma\left(B_y + \frac{v}{c^2}E_z\right) \quad \bar{B}_z = \gamma\left(B_z - \frac{v}{c^2}E_y\right)$$

$$\frac{\bar{E}_x}{c} = \frac{E_x}{c} \quad \frac{\bar{E}_y}{c} = \gamma\left(\frac{E_y}{c} - \beta B_z\right) \quad \frac{\bar{E}_z}{c} = \gamma\left(\frac{E_z}{c} + \beta B_y\right)$$

$$\bar{B}_x = B_x \quad \bar{B}_y = \gamma\left(B_y + \beta\frac{E_z}{c}\right) \quad \bar{B}_z = \gamma\left(B_z - \beta\frac{E_y}{c}\right)$$

These are precisely the rules we derived on physical grounds for the electromagnetic fields

In fact we can construct the field tensor $F^{\mu\nu}$ by direct comparison

$$t^{01} = E_x/c \quad t^{02} = E_y/c \quad t^{12} = B_z \quad t^{03} = E_z/c \quad t^{31} = B_y \quad t^{23} = B_x$$

Summary

$$t^{01} = E_x / c \quad t^{02} = E_y / c \quad t^{12} = B_z$$

$$t^{23} = B_x \quad t^{31} = B_y \quad t^{03} = E_z / c$$

written as an array

$$F^{\mu\nu} = \begin{pmatrix} 0 & F^{01} & F^{02} & F^{03} \\ -F^{01} & 0 & F^{12} & F^{13} \\ -F^{02} & -F^{12} & 0 & F^{23} \\ -F^{03} & -F^{13} & -F^{23} & 0 \end{pmatrix} = \begin{pmatrix} 0 & E_x/c & E_y/c & E_z/c \\ -E_x/c & 0 & B_z & -B_y \\ -E_y/c & -B_z & 0 & B_x \\ -E_z/c & B_y & -B_x & 0 \end{pmatrix}$$

The dual field tensor

If you follow that argument with exquisite care, you may have noticed that there was a different way of imbedding $\mathbf{E}$ and $\mathbf{B}$ in an antisymmetric tensor

We could have done the comparison also the other way around:

$$\bar{t}^{01} = t^{01} \quad \bar{t}^{02} = \gamma\left(t^{02} - \beta t^{12}\right) \quad \bar{t}^{03} = \gamma\left(t^{03} + \beta t^{31}\right)$$

$$\bar{t}^{23} = t^{23} \quad \bar{t}^{31} = \gamma\left(t^{31} + \beta t^{03}\right) \quad \bar{t}^{12} = \gamma\left(t^{12} - \beta t^{02}\right)$$

$$\bar{B}_x = B_x \quad \bar{B}_y = \gamma\left(B_y + \beta\frac{E_z}{c}\right) \quad \bar{B}_z = \gamma\left(B_z - \beta\frac{E_y}{c}\right)$$

$$-\frac{\bar{E}_x}{c} = -\frac{E_x}{c} \quad -\frac{\bar{E}_y}{c} = \gamma\left(-\frac{E_y}{c} + \beta B_z\right) \quad -\frac{\bar{E}_z}{c} = \gamma\left(-\frac{E_z}{c} - \beta B_y\right)$$

and we would have gotten

$$t^{01} = B_x \quad t^{02} = B_y \quad t^{12} = -E_z / c \quad t^{03} = B_z \quad t^{31} = -E_y / c \quad t^{23} = -E_x / c$$

The dual field tensor (cont'd)

With this we obtain the dual tensor

$$ {}^*F^{\mu\nu} = \begin{pmatrix} 0 & B_x & B_y & B_z \\ -B_x & 0 & -E_z/c & E_y/c \\ -B_y & E_z/c & 0 & -E_x/c \\ -B_z & -E_y/c & E_x/c & 0 \end{pmatrix} $$

${}^*F^{\mu\nu}$ can be obtained directly from $F^{\mu\nu}$ with the substitution

$$ \mathbf{E}/c \rightarrow \mathbf{B} \quad \mathbf{B} \rightarrow -\mathbf{E}/c $$

Both tensors generate the correct transformation rules for $\mathbf{E}$ and $\mathbf{B}$

Electrodynamics in Tensor notation

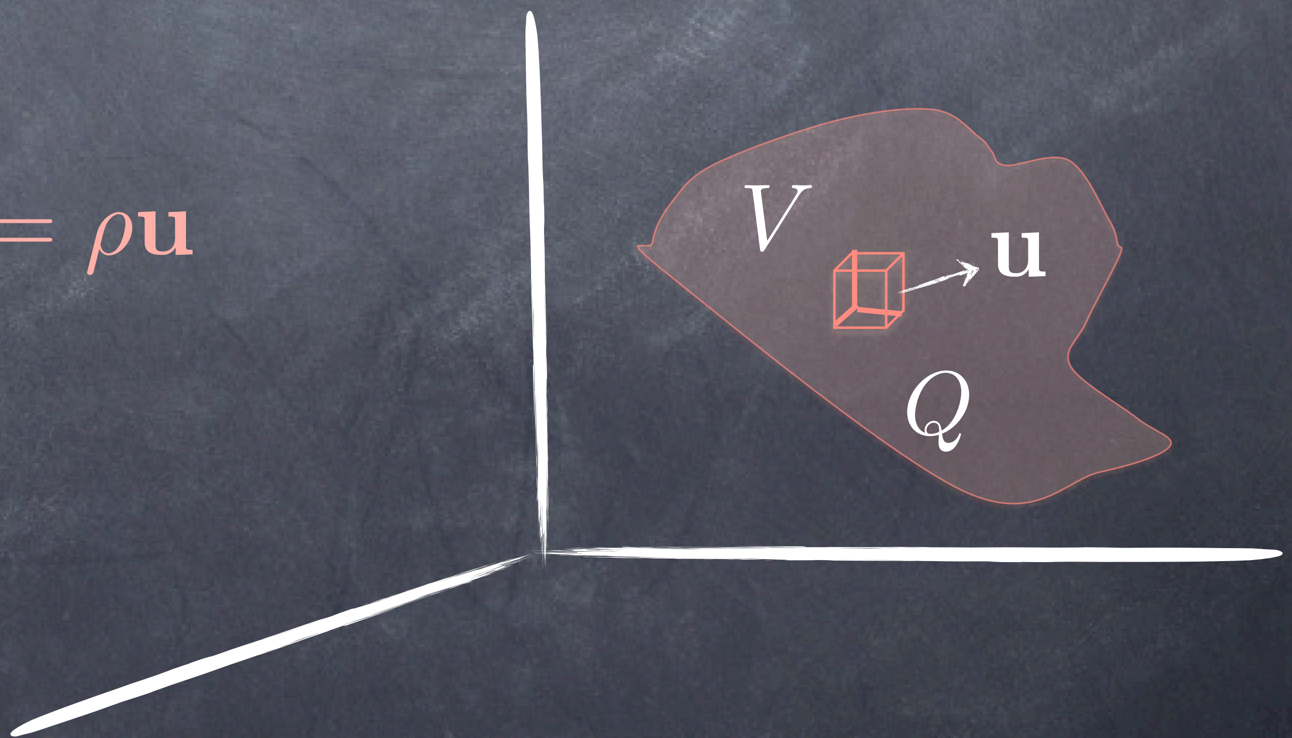
Now that we know how to represent the fields in relativistic notation, it is time to reformulate the laws of electrodynamics (Maxwell's equations and the Lorentz force law) in that language

To begin with, we must determine how the sources of the fields, ρ and $\mathbf{J}$, transform

Imagine a cloud of charge drifting by; we concentrate on an infinitesimal volume V, which contains charge Q moving at velocity $\mathbf{u}$

The charge density is ☛ $\rho = \dfrac{Q}{V}$

and the current density is ☛ $\mathbf{J} = \rho\mathbf{u}$



Electrodynamics in Tensor notation (cont'd)

I would like to express these quantities in terms of the proper charge density ρ_0, the density in the rest system of the charge:

$$\rho_0 = \frac{Q}{V_0}$$

☛ is the rest volume of the chunk

Because one dimension (the one along the direction of motion) is Lorentz-contracted

$$V = \sqrt{1 - u^2/c^2}V_0$$

and hence

$$\rho = \rho_0 \frac{1}{\sqrt{1 - u^2/c^2}}, \quad \mathbf{J} = \rho_0 \frac{\mathbf{u}}{\sqrt{1 - u^2/c^2}}$$

We recognize here the components of proper velocity, multiplied by the invariant ρ_0

Evidently charge density and current density go together to make a 4-vector ☛

$$J^\mu = \rho_0 U^\mu$$

whose components are ☛

$$J^\mu = (c\rho, J_x, J_y, J_z)$$

We'll call it the current density 4-vector

Electrodynamics in Tensor notation (cont'd)

The continuity equation ☛ $\nabla \cdot \mathbf{J} = -\dfrac{\partial \rho}{\partial t}$

expressing the local conservation of charge, takes on a nice compact form when written in terms of J^μ

For $$\nabla \cdot \mathbf{J} = \frac{\partial J_x}{\partial x} + \frac{\partial J_y}{\partial y} + \frac{\partial J_z}{\partial z} = \sum_{i=1}^{3} \frac{\partial J^i}{\partial x^i}$$

while $$\frac{\partial \rho}{\partial t} = \frac{1}{c}\frac{\partial J^0}{\partial t} = \frac{\partial J^0}{\partial x^0}$$

Thus, bringing $\partial\rho/\partial t$ over to the left side (in the continuity equation), we have

$$\frac{\partial J^\mu}{\partial x^\mu} = 0$$

with summation over μ implied

Incidentally $\partial J^\mu/\partial x^\mu$ is the 4-dimensional divergence of J^μ, so the continuity equation states that the current density 4-vector is divergenceless

Electrodynamics in Tensor notation (cont'd)

As for Maxwell's equations, they can be written

$$\frac{\partial F^{\mu\nu}}{\partial x^\nu} = \mu_0 J^\mu, \quad \frac{\partial^* F^{\mu\nu}}{\partial x^\nu} = 0$$

with summation over ν implied

Each of these stands for 4-equations -one for every value of μ

If $\mu = 0$, the first equation reads

$$\begin{aligned}
\frac{\partial F^{0\nu}}{\partial x^\nu} &= \frac{\partial F^{00}}{\partial x^0} + \frac{\partial F^{01}}{\partial x^1} + \frac{\partial F^{02}}{\partial x^2} + \frac{\partial F^{03}}{\partial x^3} \\
&= \frac{1}{c}\left(\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}\right) = \frac{1}{c}(\nabla \cdot \mathbf{E}) \\
&= \mu_0 J^0 = \mu_0 c \rho
\end{aligned}$$

or

$$\nabla \cdot \mathbf{E} = \frac{1}{\epsilon_0}\rho$$

This, of course, is Gauss's law

Electrodynamics in Tensor notation (cont'd)

If $\mu = 1$, we have

$$\frac{\partial F^{1\nu}}{\partial x^\nu} = \frac{\partial F^{10}}{\partial x^0} + \frac{\partial F^{11}}{\partial x^1} + \frac{\partial F^{12}}{\partial x^2} + \frac{\partial F^{13}}{\partial x^3}$$

$$= -\frac{1}{c^2}\frac{\partial E_x}{\partial t} + \frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z} = \left(-\frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t} + \nabla \times \mathbf{B}\right)_x$$

$$= \mu_0 J^1 = \mu_0 J_x$$

Combining this with the corresponding results for $\mu = 2$ and $\mu = 3$ gives

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}$$

which is Ampere's law with Maxwell's correction

Electrodynamics in Tensor notation (cont'd)

Meanwhile $\frac{\partial^* F^{\mu\nu}}{\partial x^\nu} = 0$ with $\mu = 0$ becomes

$$\frac{\partial^* F^{0\nu}}{\partial x^\nu} = \frac{\partial^* F^{00}}{\partial x^0} + \frac{\partial^* F^{01}}{\partial x^1} + \frac{\partial^* F^{02}}{\partial x^2} + \frac{\partial^* F^{03}}{\partial x^3}$$
$$= \frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z} = \nabla \cdot \mathbf{B} = 0$$

(the third of Maxwell's equations), whereas $\mu = 1$ yields

$$\frac{\partial^* F^{1\nu}}{\partial x^\nu} = \frac{\partial^* F^{10}}{\partial x^0} + \frac{\partial^* F^{11}}{\partial x^1} + \frac{\partial^* F^{12}}{\partial x^2} + \frac{\partial^* F^{13}}{\partial x^3}$$
$$= -\frac{1}{c}\frac{\partial B_x}{\partial t} - \frac{1}{c}\frac{\partial E_z}{\partial y} + \frac{1}{c}\frac{\partial E_y}{\partial z} = -\frac{1}{c}\left(\frac{\partial \mathbf{B}}{\partial t} + \nabla \times \mathbf{E}\right)_x = 0$$

So, combining this with the corresponding results for $\mu = 2$ and $\mu = 3$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

which is Faraday's law

Electrodynamics in Tensor notation (cont'd)

In relativistic notation, then, Maxwell's four rather cumbersome equations reduce to two delightfully simples ones

In terms of $F^{\mu\nu}$ and the proper velocity U^{μ}, the Minkowski force on a charge q is given by

$$f^{\mu} = qU_{\nu}F^{\mu\nu}$$ ❂

For if $\mu = 1$, we have

$$\begin{aligned} f^1 &= qU_{\nu}F^{1\nu} = q(-U^0F^{10} + U^1F^{11} + U^2F^{12} + U^3F^{13}) \\ &= q\left[\frac{-c}{\sqrt{1-u^2/c^2}}\left(\frac{-E_x}{c}\right) + \frac{u_y}{\sqrt{1-u^2/c^2}}(B_z) + \frac{u_z}{\sqrt{1-u^2/c^2}}(-B_y)\right] \\ &= \frac{q}{\sqrt{1-u^2/c^2}}[\mathbf{E} + (\mathbf{u} \times \mathbf{B})]_x \end{aligned}$$

with a similar formula for $\mu = 2$ and $\mu = 3$

Thus, $\mathbf{f} = \frac{q}{\sqrt{1-u^2/c^2}}[\mathbf{E} + (\mathbf{u} \times \mathbf{B})]$ and therefore, $\mathbf{F} = q[\mathbf{E} + (\mathbf{u} \times \mathbf{B})]$

which is the Lorentz force law

❂ then represents the Lorentz force law in relativistic notation

Relativistic Potentials

The electric and magnetic fields can be expressed in terms of a scalar potential V and a vector potential $\mathbf{A}$:

$$\mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t}, \;\; \mathbf{B} = \nabla \times \mathbf{A}$$ ✸

As you might guess, V and $\mathbf{A}$ together constitute a 4-vector:

$$A^{\mu} = (V/c, A_x, A_y, A_z)$$

In terms of this 4-vector potential the field tensor can be written

$$F^{\mu\nu} = \frac{\partial A^{\nu}}{\partial x_{\mu}} - \frac{\partial A^{\mu}}{\partial x_{\nu}}$$ ✻

(observe that the differentiation is with respect to the covariant vectors x_{μ} and x_{ν} ; remember this changes the sign of the zeroth component: $x_0 = -x^0$)

Relativistic Potentials (cont'd)

To check that ✸ is equivalent to ⁂, let's evaluate a few terms explicitly For $\mu = 0, \nu = 1$

$$F^{01} = \frac{\partial A^1}{\partial x_0} - \frac{\partial A^0}{\partial x_1} = -\frac{\partial A_x}{\partial (ct)} - \frac{1}{c}\frac{\partial V}{\partial x}$$

$$= -\frac{1}{c}\left(\frac{\partial \mathbf{A}}{\partial t} + \nabla V\right)_x = \frac{E_x}{c}$$

That (and its companions with $\nu = 2$ and $\nu = 3$) is the first equation in ✸ For $\mu = 1,\ \nu = 2$,we get

$$F^{12} = \frac{\partial A^2}{\partial x_1} - \frac{\partial A^1}{\partial x_2} = \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} = (\nabla \times \mathbf{A})_z = \mathbf{B}_z$$

which (together with the corresponding results for F^{23} and F^{13} is the second equation in ✸

Relativistic Potentials (cont'd)

The potential formulation automatically takes care of the homogeneous Maxwell equation $(\partial^* F^{\mu\nu}/\partial x^\nu = 0)$

As for the inhomogeneous equation $(\partial F^{\mu\nu}/\partial x^\nu = \mu_0 J^\mu)$, that becomes

$$\frac{\partial}{\partial x_\mu}\left(\frac{\partial A^\nu}{\partial x^\nu}\right) - \frac{\partial}{\partial x_\nu}\left(\frac{\partial A^\mu}{\partial x^\nu}\right) = \mu_0 J^\mu \quad *$$

This is an intractable equation as it stands

However, you will recall that the potentials are not uniquely determined by the fields- in fact, it's clear from * that you could add to A^μ the gradient of any scalar function λ :

$$A^\mu \rightarrow A^{\mu'} = A^\mu + \frac{\partial \lambda}{\partial x_\mu}$$

without changing $F^{\mu\nu}$

Relativistic Potentials (cont'd)

This is precisely the gauge invariance; we can exploit it to simplify ❃

In particular ☛ $\nabla \cdot \mathbf{A} = -\frac{1}{c^2}\frac{\partial V}{\partial t}$

becomes, in relativistic notation ☛ $\frac{\partial A^\mu}{\partial x^\mu} = 0$

In the Lorentz gauge, therefore, ❃ reduces to

$$\Box^2 A^\mu = -\mu_0 J^\mu$$ ❄

where $\Box^2$ is the d'Alembertian

$$\Box^2 \equiv \frac{\partial}{\partial x_\nu}\frac{\partial}{\partial x^\nu} = \nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}$$

❄ combines our previous results into a single 4-vector equation- it represents the most elegant (and the simplest) formulation of Maxwell's equations