Special Relativity

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VI: Relativistic Dynamics

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Equation of Motion

- Newton's first law of motion holds in special relativistic mechanics as well as nonrelativistic mechanics
- In the absence of forces, a body is at rest or moves in a straight line at constant speed
- This is summarized by $$\frac{d\mathbf{U}}{d\tau} = 0$$ ☛ Newton's First Law

  since this equation implies $\vec{u}$ is constant in any inertial frame
- The objective of relativistic mechanics is to introduce the analog of Newton's second law $\vec{F} = m\vec{a}$
- There is nothing from which this law can be derived, but plausibly it must satisfy certain properties:

1- It must satisfy the principle of relativity, i.e., take the same form in every inertial frame

2- It must reduce to the above equation when the force is zero

3- It must reduce to $\vec{F} = m\vec{a}$ in any inertial frame when the speed of the particle is much less than the speed of the light

Equation of Motion (cont'd)

- The choice

$$m\frac{d\mathbf{U}}{d\tau} = \mathbf{f}$$

☛ Newton's Second Law

naturally suggests itself

- The constant m, which characterizes the particle's inertial properties, is called the rest mass, and $\mathbf{f}$ is called the 4-force
- Requirement (1) is satisfied because this is a 4-vector equation, (2) is evident, and (3) is satisfied with a proper choice of $\mathbf{f}$
- This is the correct law of motion for special relativistic mechanics and the special relativistic generalization of Newton's second law
- By introducing the 4-acceleration 4-vector $\mathbf{A}$

$$\mathbf{A} \equiv \frac{d\mathbf{U}}{d\tau}$$

☛ 4-acceleration

the equation of motion can be written in the evocative form

$$\mathbf{f} = m\mathbf{A}$$

Equation of Motion (cont'd)

- Although this represents 4-equations, they are not all independent
- The normalization of the 4-velocity means ☛ $m\frac{d(\mathbf{U}\cdot\mathbf{U})}{d\tau}=0$

  which from Newton's second law implies $\mathbf{U}\cdot\mathbf{A}=0$ or $\mathbf{f}\cdot\mathbf{U}=0$
- This relation shows that there are only three independent equations of motion the same number as in Newtonian mechanics
- The connection is discussed in more detail soon, and Newton's third law will be discussed as well

Example

The 4-acceleration $\mathbf{A}$ for the world line described last class has components $A^0 \equiv dU^0/d\tau = a\sinh(a\tau)$ $A^1 \equiv dU^1/d\tau = a\cosh(a\tau)$

the magnitude of this acceleration is

so the constant a is aptly named

$$(\mathbf{A}\cdot\mathbf{A})^{\frac{1}{2}} = a \equiv \alpha$$

The 4-force required to accelerate the particle along this world line is $\mathbf{f}=m\mathbf{A}$ where m is the particle's rest mass

Conservation laws

suppose we shoot 2 balls with mass m_A and m_B with velocity u_A and u_B onto each other. During the collision, some mass is transferred from A onto B and we end up with two particles with mass m_C and m_D with velocities u_C and u_D

Assume that the 3-momentum is conserved in frame S

(a) Now go to frame S-bar that is moving with v relative to S

Prove that the 3-momentum is also conserved in S-bar, if you use the Galilean velocity addition rule

(b) Use Einstein's velocity addition rule. Is the 3-momentum still conserved in S-bar?

(a) Galilean velocity addition

$$m_a\vec{u}_a + m_b u_b = m_c\vec{u}_c + m_d\vec{u}_d \qquad \vec{u}_{a,b,c,d} = \overline{\vec{u}}_{a,b,c,d} + \vec{v}$$

in $\overline{\mathrm{S}}$:

$$m_a\left(\overline{\vec{u}}_a + \vec{v}\right) + m_b\left(\overline{\vec{u}}_b + \overline{\vec{u}}_b\right) = m_c\left(\overline{\vec{u}}_c + \vec{v}\right) + m_d\left(\overline{\vec{u}}_d + \vec{v}\right)$$

$$m_a\overline{\vec{u}}_a + m_b\overline{\vec{u}}_b + \vec{v}\underbrace{\left(m_a + m_b\right)}_{\text{mass conserved}} = m_c\overline{\vec{u}}_c + m_d\overline{\vec{u}}_d + \vec{v}\underbrace{\left(m_c + m_d\right)}_{\text{mass conserved}}$$

$$m_a\overline{\vec{u}}_a + m_b\overline{\vec{u}}_b = m_c\overline{\vec{u}}_c + m_d\overline{\vec{u}}_d$$

3-momentum also conserved in S-bar

(b) now with Einstein's velocity addition rule v and u only in x-direction

$$u_{a,b,c,d} = \frac{\bar{u}_{a,b,c,d} + v}{1 + \frac{\bar{u}_{a,b,c,d} v}{c^2}}$$

$$m_A \frac{\bar{u}_A + v}{1 + \frac{\bar{u}_A v}{c^2}} + m_B \frac{\bar{u}_B + v}{1 + \frac{\bar{u}_B v}{c^2}} = m_C \frac{\bar{u}_C + v}{1 + \frac{\bar{u}_C v}{c^2}} + m_A \frac{\bar{u}_D + v}{1 + \frac{\bar{u}_D v}{c^2}}$$

o.k. let's investigate with the equation a head on collision of two equal masses A and B with equal velocity $u_A = -u_B = v$. After they collide, they get stuck together, hence the collision is "completely inelastic"

In S: $$\underbrace{m_A u - m_A u}_{before} = \underbrace{\overbrace{2m_A}^{2m_A \text{ stuck}} \overbrace{u_{CD}}^{=0}}_{after}$$ 3-momentum conserved now with Einstein's velocity summation in S-bar:

$$\bar{u}_A = \frac{\overbrace{u_A}^{v} - v}{1 - \frac{uv}{c^2}} = 0; \quad \bar{u}_B = \frac{\overbrace{-u_A}^{-v} - v}{1 - \frac{uv}{c^2}} = \frac{-2v}{1 - \frac{uv}{c^2}}; \; \bar{u}_C = \frac{0 - v}{1 - \frac{0v}{c^2}} = -v; \; \bar{u}_D = \frac{0 - v}{1 - \frac{0v}{c^2}} = -v$$

$$\underbrace{m_A\bar{u} - m_A\bar{u}}_{before} = \underbrace{\overbrace{2m_A}^{\text{2m}_\text{A}\text{ stuck}} \; \bar{u}_{CD}}_{after}$$

$$0 - m_A\left(\frac{2v}{1 + \frac{v^2}{c^2}}\right) \neq -2m_A v$$

3-momentum is not conserved !!!

What about the 4-momentum?

Energy-Momentum

- The equation of motion leads naturally to the relativistic ideas of energy and momentum
- If the 4-momentum is defined by

$$\mathbf{p} = m\mathbf{U}$$ ☛ 4-momentum

then the equation of motion can be written $$\frac{d\mathbf{p}}{d\tau} = \mathbf{f}$$

- An important property of the 4-momentum follows from its definition and the normalization of the 4-velocity

$$\mathbf{p}^2 \equiv \mathbf{p}\cdot\mathbf{p} = -m^2$$ ☛ rest mass square

- In view of $U^\alpha = (\gamma, \gamma\vec{u})$ the components of the 4-momentum are related to the 3-velocity $\vec{u}$ in an inertial frame by

$$p^0 = \frac{m}{\sqrt{1-\vec{u}^2}} \qquad \vec{p} = \frac{m\vec{u}}{\sqrt{1-\vec{u}^2}}$$

Energy-Momentum (cont'd)

- For small speeds $u \ll 1$

$$p^0 = m + \frac{1}{2}m\vec{u}^2 + ...$$

$$\vec{p} = m\vec{u} + ...$$

- Thus, at small velocities $\vec{p}$ reduces to the usual momentum, and p^0 reduces to the kinetic energy plus the rest mass
- For this reason $\mathbf{p}$ is also called the energy momentum 4-vector and its components in an inertial frame are written

$$p^\alpha = (E, \vec{p}) = (m\gamma, m\gamma\vec{u})$$

where $E \equiv p^0$ is the energy and $\vec{p}$ is the 3-momentum

- Equation (rest mass squared) can be solved for the energy in terms of the 3-momentum to give

$$E = (m^2 + \vec{p}^2)^{\frac{1}{2}}$$

which shows how rest mass is a part of the energy of a relativistic particle

Energy-Momentum (cont'd)

- Indeed, for a particle at rest, the previous equation reduces to $E = mc^2$ in more usual units
- This must be the most famous equation in relativity if not one of the most famous ones in all of physics
- An important application of special relativistic kinematics occurs in particle reactions, where the total 4-momentum is conserved in particle collisions, corresponding to the law of energy conservation and the conservation of total 3-momentum
- In a particular inertial frame the connection between the relativistic equation of motion and Newton's laws can be made more explicit by defining the 3-force F as $\frac{d\vec{p}}{dt} \equiv \vec{F}$
- This has the same form as Newton's law but with the relativistic expression for the 3-momentum
- Solving problems in the mechanics of special relativity is, therefore, essentially the same as solving Newton's equation of motion

Energy-Momentum (cont'd)

- The only difference arises from the different relation of momentum to velocity
- Newton's third law applies to the force $\vec{F}$ just as it does in Newtonian mechanics because $\frac{d\vec{p}}{dt} \equiv \vec{F}$, it implies that the total 3-momentum of a system of particles is conserved in all inertial frames
- Evidently $\vec{f} = d\vec{p}/d\tau = (d\vec{p}/dt)(dt/d\tau) = \gamma\vec{F}$
- The 4-force can be written in terms of the 3-force as

$$\mathbf{f} = (\gamma\vec{F}\cdot\vec{u}, \gamma\vec{F})$$

☛ 4-force in terms of 3-force

where $\vec{u}$ is the particle's 3-velocity

Energy-Momentum (cont'd)

- The time component of the equation of motion is

$$\frac{dE}{dt} = \vec{F} \cdot \vec{u}$$

which is a familiar relation from Newtonian mechanics

- This time component of the equation of motion is a consequence of the other three

- Thus, in terms of the 3-force, the equation of motion take the same form as they do in usual Newtonian mechanics but with the relativistic expressions for energy and momentum

- Newtonian mechanics is the low-velocity approximation to special relativistic mechanics

Particle Collisions

- ✔ In a classical collision process, momentum and mass are always conserved, whereas kinetic energy is (in general) not
- ✔ A "sticky" collision generates heat at the expense of kinetic energy
- ✔ An "explosive" collision generates kinetic energy at the expense of chemical energy (or some other kind)
- ✔ If the kinetic energy is conserved, as in the ideal collision of two billiard balls, we call the process elastic

Relativistic Particle Collisions

✔ In the relativistic case momentum and total energy are always conserved but mass and kinetic energy (in general) are not

✔ Once again we call the process elastic if the kinetic energy is conserved

✔ In such a case the rest energy (being the total minus the kinetic) is also conserved, and therefore so too is the mass

Note: Particle physicists have come to regard a collision as elastic only when the very same particles comes out as went in therefore ☛ $e^+\pi^- \rightarrow e^+\pi^-$ would be considered elastic

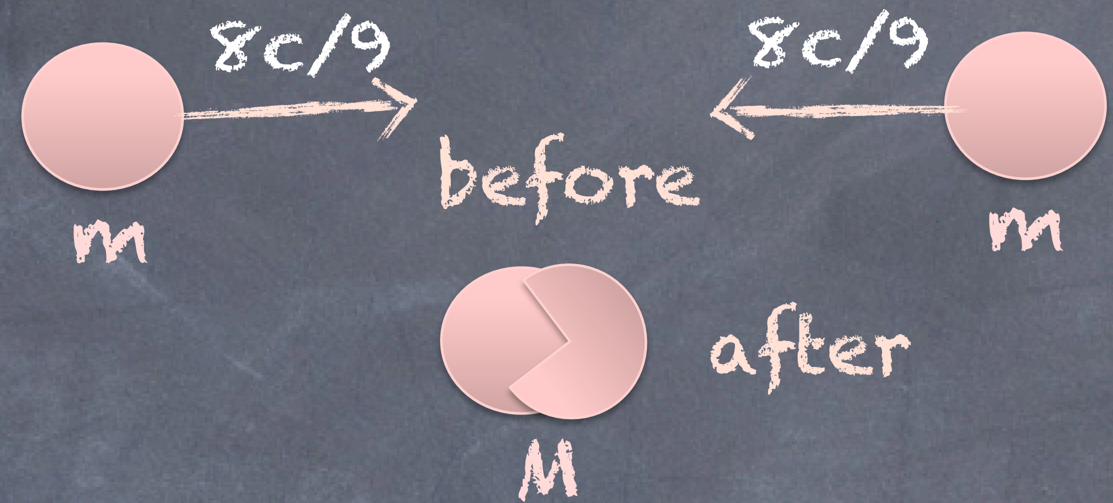
but $e^+\pi^- \rightarrow e^-\pi^+$ would not

☛ even though the masses are identical in the two reactions

Example I ☛ Inelastic collision

Two lumps of clay (with rest mass m each) collide head-on with 8c/9

What is the mass of the composite lump?



In this case conservation of momentum is trivial: zero before, zero after

The energy of each lump prior to the collision is

$$\frac{mc^2}{\sqrt{1-\frac{(8/9)^2c^2}{c^2}}} = \frac{mc^2}{\sqrt{1-(8/9)^2}} = 2.2mc^2$$

and the energy of the composite lump after the collision is Mc^2 (since it's at rest)

So conservation of energy says ☛ $2.2\ mc^2 + 2.2\ mc^2 = Mc^2$

Hence $M = 4.4\ m$

Notice that this is greater than the sum of the initial masses!

In the classical analysis of such a collision, we say that kinetic energy was converted into thermal energy ☛ the composite lump is hotter than the two colliding pieces

- This is of course true in the relativistic picture too ☛ but what is the thermal energy?
- It's the sum of the total random kinetic and potential energies of all the atoms and molecules in the substance
- Relativity tell us that these microscopic energies are represented in the mass of the object: a hot potato is heavier than a cold potato, and a compressed spring is heavier than a relaxed spring
- Not by much ☛ it's true internal energy U contributes U/c^2 to the mass and c^2 is a very large number by everyday standards
- You can never get two lumps of clay going anywhere near fast enough to detect the non-conservation of mass in their collision
- In the realm of elementary particles, however, the effect can be very striking, e.g. ☛ $\pi^0 \to e^+ e^-$

  $m_{\pi^0} = 2.4 \times 10^{-28}$ kg and $m_{e^\pm} = 9.11 \times 10^{-31}$ kg

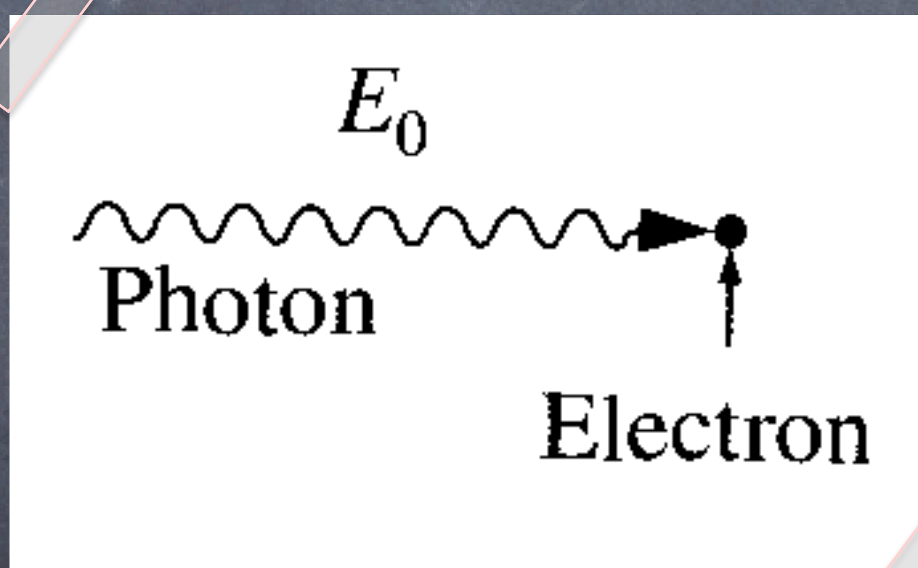
  the rest energy is converted almost entirely into kinetic energy ☛ less than 1% of the original mass remains

Example II ☛ Elastic collision: Compton scattering

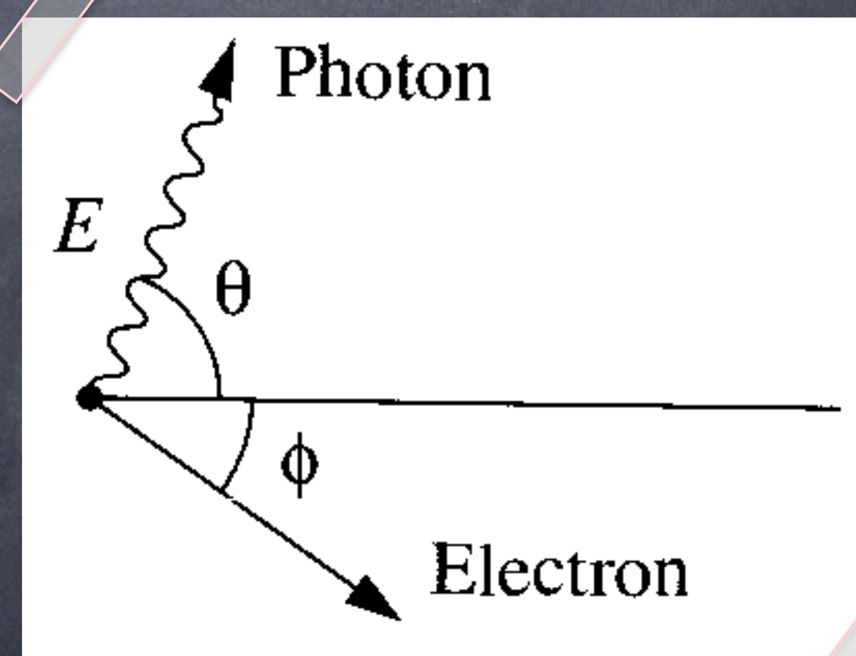
A photon of energy E_0 bounces off an electron that is initially at rest

Find the energy E of the outgoing photon, as a function of the scattering angle θ

before ☛



after ☛



momentum conservation

b ☛ before
a ☛ after

$$\vec{p}_{p,b} + \vec{p}_{e,b} = \vec{p}_{p,a} + \vec{p}_{e,a}$$

Conservation of momentum in the vertical direction gives

$$p^{\perp}_{p,b} + p^{\perp}_{e,b} = p^{\perp}_{p,a} + p^{\perp}_{e,a}$$

$$0 + 0 = p_{p,a}\sin\theta + p_{e,a}\sin(-\phi)$$

$$p_{p,a}\sin\theta = p_{e,a}\sin(\phi)$$

$$\sin(-\phi) = -\sin(\phi)$$

after the collision the photon has an energy E smaller than E_0
And the momentum is given by E=pc

$$\sin\theta\frac{E}{p_{e,a}c} = \sin(\phi)$$

Conservation of momentum in the horizontal direction gives

$$p^{\parallel}_{p,b} + p^{\parallel}_{e,b} = p^{\parallel}_{p,a} + p^{\parallel}_{e,a}$$

$$E_0/c + 0 = p_{p,a}\cos\theta + p_{e,a}\overbrace{\cos(-\phi)}^{=\cos(\phi)} = \frac{E}{c}\cos\theta + p_{e,a}\sqrt{1-\sin^2\phi}$$

$\cos^2(\phi) = 1 - \sin^2\phi$

and we have an expression for the $\sin\theta$ ☛ $\sin\theta \dfrac{E}{p_{e,a}c} = \sin(\phi)$

$$E_0/c = \frac{E}{c}\cos\theta + p_{e,a}\sqrt{1-\frac{E^2}{p^2_{e,a}c^2}\sin^2\theta}$$

solving for p²c², reason will become clear soon

$$E_0 - E\cos\theta = p_{e,a}c\sqrt{1-\frac{E^2}{p^2_{e,a}c^2}\sin^2\theta}$$

$$(E_0 - E\cos\theta)^2 = p^2_{e,a}c^2\left(1-\frac{E^2}{p^2_{e,a}c^2}\sin^2\theta\right)$$

$$(E_0 - E\cos\theta)^2 = p^2_{e,a}c^2 - E^2\sin^2\theta$$

$$p^2_{e,a}c^2 = (E_0 - E\cos\theta)^2 + E^2\sin^2\theta = E_0^2 - 2E_0E\cos\theta + \overbrace{E^2\cos^2\theta + E^2\sin^2\theta}^{\sin^2+\cos^2=1} = E_0^2 - 2E_0E\cos\theta + E^2$$

finally ☛ conservation of energy says that

$$E_0 + mc^2 = E + \sqrt{m^2c^4 + p_{e,a}^2c^2}$$

$$p_{e,a}^2c^2 = E_0^2 - 2E_0E\cos\theta + E^2$$

replace p^2c^2 and solve for E

$$\begin{aligned}(E_0 + mc^2 - E)^2 &= E_0^2 - E_0E + E_0mc^2 - E_0E + E^2 - Emc^2 + E_0mc^2 - Emc^2 + m^2c^4 \\ &= m^2c^4 + E_0^2 - 2E_0E\cos\theta + E^2\end{aligned}$$

$$2E_0mc^2 - 2E_0E - 2Emc^2 = -2E_0E\cos\theta$$

$$E_0mc^2 = E_0E + Emc^2 - E_0E\cos\theta = E\left(E_0 + mc^2 - E_0\cos\theta\right)$$

$$\Rightarrow E = \frac{E_0mc^2}{\left(E_0 + mc^2 - E_0\cos\theta\right)} = \frac{1}{\frac{E_0}{E_0mc^2} + \frac{mc^2}{E_0mc^2} - \frac{E_0\cos\theta}{E_0mc^2}} =$$

$$= \frac{1}{\frac{1}{mc^2} + \frac{1}{E_0} - \frac{\cos\theta}{mc^2}} = \frac{1}{\frac{1}{mc^2}\left(1 - \cos\theta\right) + \frac{1}{E_0}}$$

that is it!

the answer looks nicer when expressed in terms of photon wavelength

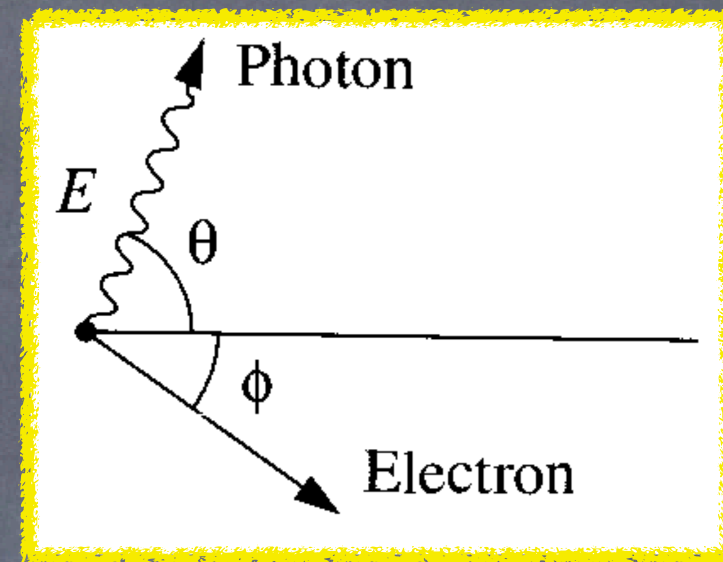
$$E = h\nu = h\frac{c}{\lambda}$$

$$E = \frac{1}{\frac{1}{mc^2}(1-\cos\theta) + \frac{1}{E_0}}$$

$$\frac{hc}{\lambda} = \frac{1}{\frac{1}{mc^2}(1-\cos\theta) + \frac{\lambda_0}{hc}}$$

$$\Rightarrow \frac{\lambda}{hc} = \frac{1}{mc^2}(1-\cos\theta) + \frac{\lambda_0}{hc}$$

$$\Rightarrow \lambda = \lambda_0 + \frac{h}{mc}(1-\cos\theta)$$



$\frac{h}{mc}$ is called the Compton wavelength of an electron

Two obvious cases:

case 1 ☛ $\theta = 0$

photon goes right through, does not interact with electron, no scattering, no loss of energy

case 2 ☛ $\theta = 180^\circ$ head on collision

maximum energy transfer $\lambda = \lambda_0 + \frac{h}{mc}(1-(1-1)) \Rightarrow \lambda = \lambda_0 + \frac{2h}{mc}$

wavelength of outgoing photon maximal (energy minimal)

Summary

time	not invariant	
proper time	invariant	
velocity	not invariant	not conserved
proper velocity	not invariant	not conserved
3-momentum	not invariant	not conserved
relativistic 4- momentum	not invariant	conserved
non-relativistic energy	not invariant	not conserved
relativistic energy	not invariant	conserved
rest mass	invariant	not conserved
charge	invariant	conserved