Special Relativity

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V: Space-time

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4-dimensional space-time

An event in the 4-dimensional continuum space-time is a point with coordinates

$$x^0 = ct \qquad x^1 = x \qquad x^2 = y \qquad x^3 = z$$

or equivalently x^μ (with $\mu = 0, 1, 2, 3$)

Recall the summation convention

Wherever there are repeated upper and lower indices summation is implied, e.g.

$$a_\mu b^\mu = \sum_{\mu=0}^{3} a_\mu b^\mu$$

The metric of space-time is given by $ds^2 = \eta_{\mu\nu}\, dx^\mu\, dx^\nu$

$$\eta_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\text{Inverse} \equiv \eta^{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Hence the metric $ds^2 = -(dx^0)^2 + (dx^1)^2 + (dx^2)^2 + (dx^3)^2$

This metric is unusual for a geometry in that it is not positive definite:

for spacelike displacements it is positive

and for timelike displacements it is negative

This metric is related to your proper time τ by $ds^2 = -c^2 d\tau^2$

Indices are raised and lowered with $\eta_{\mu\nu}$, e.g. if a_μ is a vector then $a_\mu = \eta_{\mu\nu} a^\nu$

This extends to tensors in space-time etc. Upper indices are referred to as contravariant and lower indices as covariant

Representation of a Lorentz transformation

A Lorentz transformation preserves ds^2

We represent a Lorentz transformation by

$$x'^{\mu} = \Lambda^{\mu}{}_{\nu} x^{\nu} \qquad\qquad \eta_{\mu\nu} = \Lambda^{\sigma}{}_{\mu}\Lambda^{\tau}{}_{\nu}\eta_{\sigma\tau}$$

That is, a Lorentz transformation is the equivalent of an orthogonal matrix in the 4-dimensional space-time with indefinite metric

Conditions

- det $\Lambda = 1$ - rules out reflections $(x \to -x)$
- $\Lambda^{0}{}_{0} > 0$ - isochronous

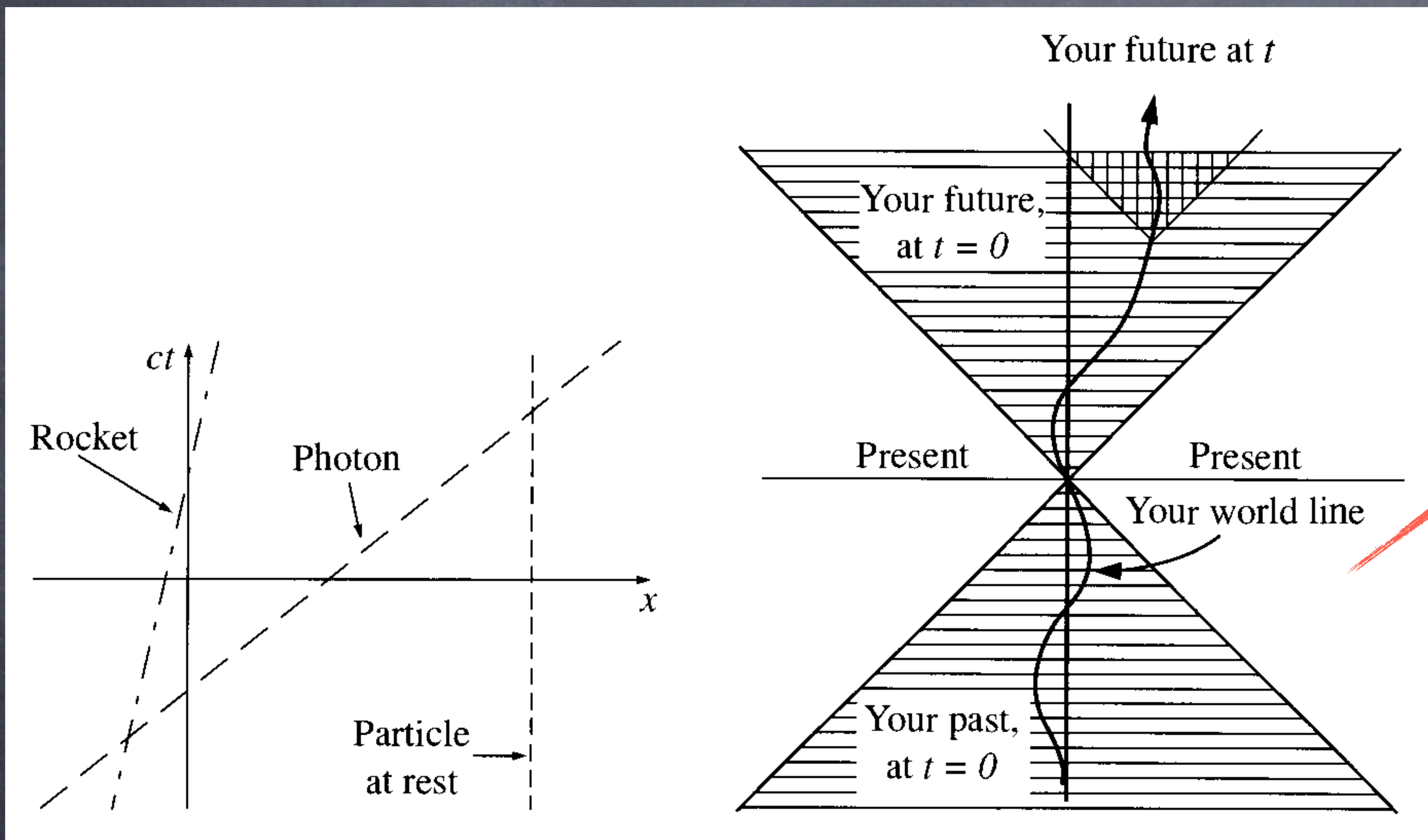
For the special case of a Lorentz transformation involving a boost along the x-axis

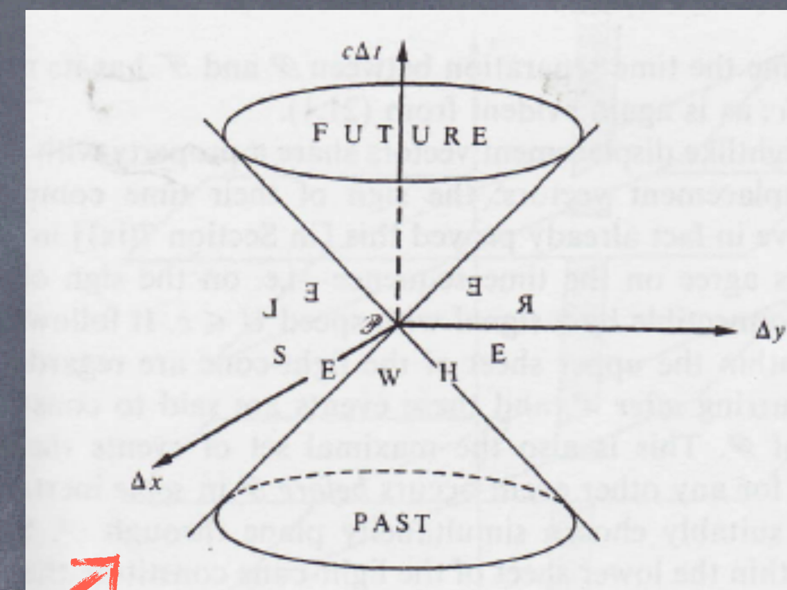
$$\Lambda^{\mu}{}_{\nu} = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Space-time diagrams

Minkowski-Diagrams (and world lines)

Time-like component (ct) is vertical position, space is horizontal and velocity is reciprocal of slope





By living (=making decisions) we permanently narrow the accessibility of our future. The motion through space-time is called a world line.
You see that in order to go to the past, you need to reach infinite velocity (rather than only light velocity) and this is really impossible

Relativistic Kinematics and Space-Time

Having introduced the idea of 4-vectors, let's now turn to their use for describing the motion of a particle in space-time terms.
A particle follows a timelike word line through spacetime. This curve can be specified by giving three spatial coordinates x^i as a function of time in a particular inertial frame. However, the 4-dimensional way of describing a worldline is to give all 4 coordinates of the particle x^α as a single-valued function of a parameter that varies along the worldline.

Many parameters are possible, but a natural one is the proper time that gives the spacetime distance τ along the world line measured both positively and negatively from some arbitrary starting point

A world line is then described by the equations

$$x^\alpha = x^\alpha(\tau)$$

4-velocity

The 4-velocity is the four vector $\mathbf{U}$ whose components U^α are the derivatives of the position along the world line with respect to the proper time parameter

$$U^\alpha = \frac{dx^\alpha}{d\tau}$$

The 4-velocity $\mathbf{U}$ is thus tangent to the world line at each point because a displacement is given by $\Delta x^\alpha = u^\alpha \Delta\tau$

The four components of the 4-velocity can be expressed in terms of the 3-velocity ☛ $\vec{u} = \frac{d\vec{l}}{dt}$

$$\vec{U} = \frac{d\vec{l}}{d\tau}$$

now replace $d\tau$ ☛

$$\vec{U} = \frac{d\vec{l}}{\underbrace{\sqrt{1 - \frac{u^2}{c^2}}dt}_{d\tau}} = \frac{1}{\sqrt{1 - \frac{u^2}{c^2}}}\vec{u}$$

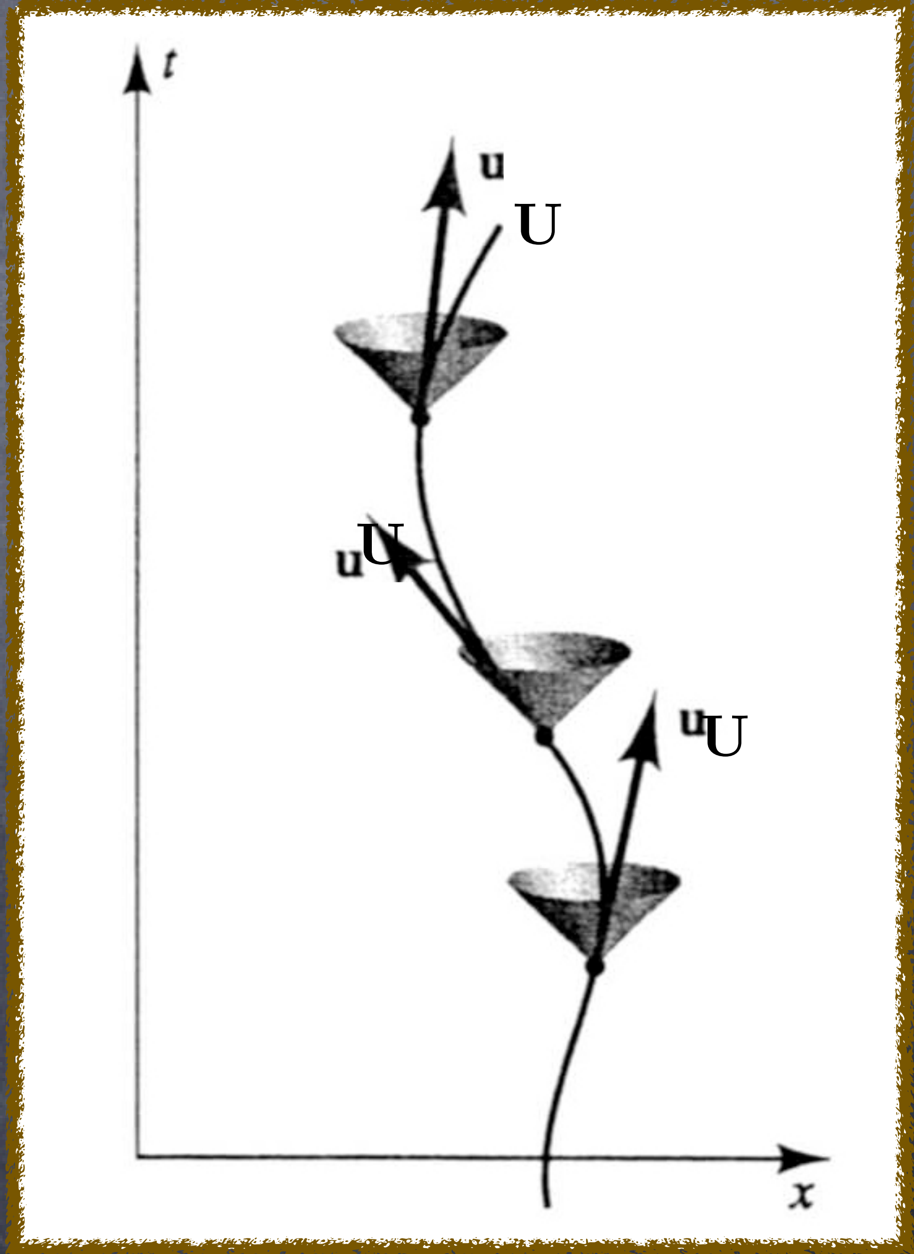
What is the 0^{th} component for this 4-vector (the 0^{th} component for the displacement 4-vector was ct):

$$\frac{dx^0}{d\tau} = U^0 = c\frac{dt}{d\tau} = c\frac{dt}{\sqrt{1-\frac{u^2}{c^2}}dt} = \frac{c}{\sqrt{1-\frac{u^2}{c^2}}}$$

Therefore: $\mathbf{U} \equiv U^\alpha = (\gamma c, \gamma \vec{u})$

The 4-velocity $\mathbf{U}(\tau)$ at any point along a particle's worldline is the unit timelike tangent 4-vector at that point

It lies inside the light cone of that point

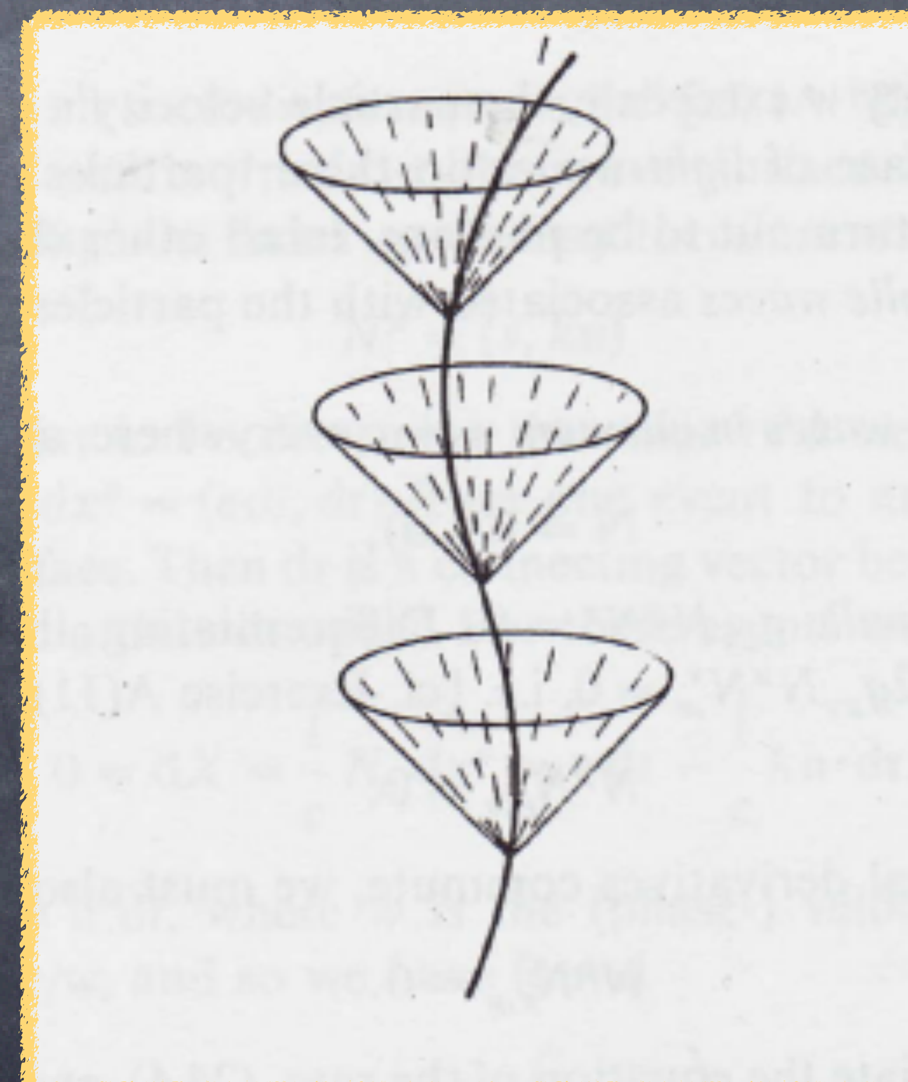


An immediate consequence of this result is that

$$\mathbf{U}\,.\,\mathbf{U} = \eta_{\alpha\beta}\frac{dx^\alpha}{d\tau}\,\frac{dx^\beta}{d\tau} = -c^2$$

so that the 4-velocity is always a time-like 4-vector whose Lorentz transformation is given by

$$\begin{aligned}
\bar{U}^0 &= \gamma(U^0 - \beta U^1)\\
\bar{U}^1 &= \gamma(U^1 - \beta U^0)\\
\bar{U}^2 &= U^2\\
\bar{U}^3 &= U^3
\end{aligned}$$



If you change from a system S to S-bar, that moves with V relative to S along a common x-axis, only the velocities along the x-axis are affected, the velocity components in the y and z directions are not affected.

let us first write down the transformation rules for the dx^μ (the components of the displacement)

$$\begin{aligned} d\bar{x}^0 &= \gamma(dx^0 - \beta\, dx^1) \\ d\bar{x}^1 &= \gamma(dx^1 - \beta\, dx^0) \\ d\bar{x}^2 &= dx^2 \\ d\bar{x}^3 &= dx^3 \end{aligned}$$

$$\frac{d\bar{x}^0}{d\tau} = \gamma\left(\frac{dx^0}{d\tau} - \beta\frac{dx^1}{d\tau}\right)$$

$$\frac{d\bar{x}^1}{d\tau} = \gamma\left(\frac{dx^1}{d\tau} - \beta\frac{dx^0}{d\tau}\right)$$

$$\frac{d\bar{x}^2}{d\tau} = \frac{dx^2}{d\tau}$$

$$\frac{d\bar{x}^3}{d\tau} = \frac{dx^3}{d\tau}$$

since proper time is invariant (your own clock goes the same, it does not matter whether you are in S or S-bar)

$$\gamma = \frac{1}{\sqrt{1 - V^2/c^2}}$$

4-acceleration

The relation between the 4-acceleration $\mathbf{A} = \dfrac{d^2 x^\mu}{d\tau^2} = \dfrac{d\mathbf{U}}{d\tau}$

and the 3-acceleration $\mathbf{a} = \dfrac{d^2 x^i}{dt^2}$ is more complicated

$$\mathbf{A} = \frac{d\mathbf{U}}{d\tau} = \gamma \frac{d\mathbf{U}}{dt} = \gamma \frac{d}{dt}(\gamma c, \gamma \mathbf{u}) = \gamma \left(\frac{d\gamma}{dt} c, \frac{d\gamma}{dt} \mathbf{u} + \gamma \mathbf{a} \right)$$

$$\gamma = \gamma(u)$$

But in the instantaneous rest frame of the particle $(u = 0)$

this expression simplifies to $\mathbf{A} = (0, \mathbf{a})$

since the derivative of γ contains a factor u

Thus $\mathbf{A} = 0$ if and only if the proper acceleration α ☛

magnitude of the three-acceleration in the rest frame vanishes

Similarly

$$\mathbf{A}\cdot\mathbf{A} = \alpha^2 \qquad\qquad \mathbf{U}\cdot\mathbf{A} = \mathbf{0}$$

$\mathbf{U}$ and $\mathbf{A}$ have analogues not only in classical kinematics, namely $\mathbf{u}$ and $\mathbf{a}$, but also in the differential geometry of curves:

They are the analogues (with respect to the particle's worldline) of the unit tangent vector dx^i/dl

and principal normal vector d^2x^i/dl^2 of a space curve $x^i = x^i(l)$

Thus, α is a measure of the curvature of the worldline (because we have taken τ rather than l as the parameter, the actual curvature of the worldline is α/c^2)

$$\mathbf{A}^2 = \gamma^2\left[(\dot\gamma\mathbf{u} + \gamma\mathbf{a})^2 - \dot\gamma^2 c^2\right]$$

Using ☛ $\dot\gamma = \gamma^3 u\dot u/c^2 \qquad \mathbf{u}^2 = u^2 \qquad \mathbf{u}\,.\,\dot{\mathbf{u}} = u\dot u$

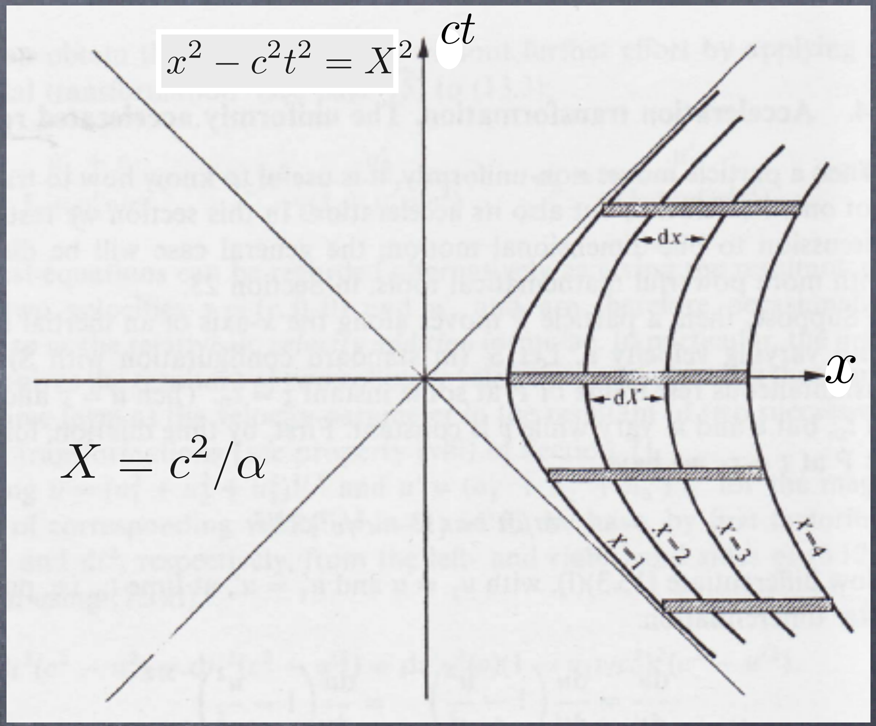
$$\alpha^2 = \gamma^2\left[\dot\gamma^2 u^2 + 2\gamma\dot\gamma u\dot u + \gamma^2 a^2 - \dot\gamma^2 c^2\right] = \gamma^6 u^2 \dot u^2/c^2 + \gamma^4 a^2 = \gamma^6\left[a^2 - c^{-2}(\mathbf{u}\times\mathbf{a})^2\right]$$

Rectilinear motion with constant proper acceleration

We can integrate $\frac{d}{dt}[\gamma(u)u]$ at once, choosing $t = 0$ when $u = 0$

$$\alpha t = \gamma(u)u$$

Squaring, solving for u, integrating once more and setting the constant of integration equal to zero, yields the following equation for the motion: $x^2 - c^2t^2 = c^4/\alpha^2$



Thus, rectilinear motion with constant proper acceleration is called hyperbolic motion

The corresponding classical calculation gives

$x = \frac{1}{2}\alpha t^2$ i.e. parabolic motion

Note that $\alpha = \infty$ implies $x = \pm ct$

hence the proper acceleration of a photon can be taken to be infinite

Note also, that a photon emitted a distance c^2/α behind the particle when the latter is momentarily at rest, cannot catch up with it (its graph is the asymptote to the particle's hyperbola)

Uniformly accelerated rod

Consider next the equation $$x^2 - c^2t^2 = X^2 \qquad ☻$$

for a continuous range of positive values of the parameter X

For each fixed X it represents a particle moving with constant proper acceleration c^2/X in the x-direction

Altogether it represents, a rigidly moving rod

By the rigid motion of a body one understands a motion during which every small volume element of the body shrinks always in the direction of its motion in proportion to its instantaneous Lorentz factor relative to a given inertial frame

Thus every small volume element preserves its dimensions in its own instantaneous rest frames, which shows that the definition is intrinsic, i.e. frame-independent

We can find by implicit differentiation of ☻ the velocity u and the corresponding factor of a point moving so that X=constant:

$$u = \frac{dx}{dt} = \frac{c^2t}{x}, \qquad \gamma(u) = \frac{x}{X}$$

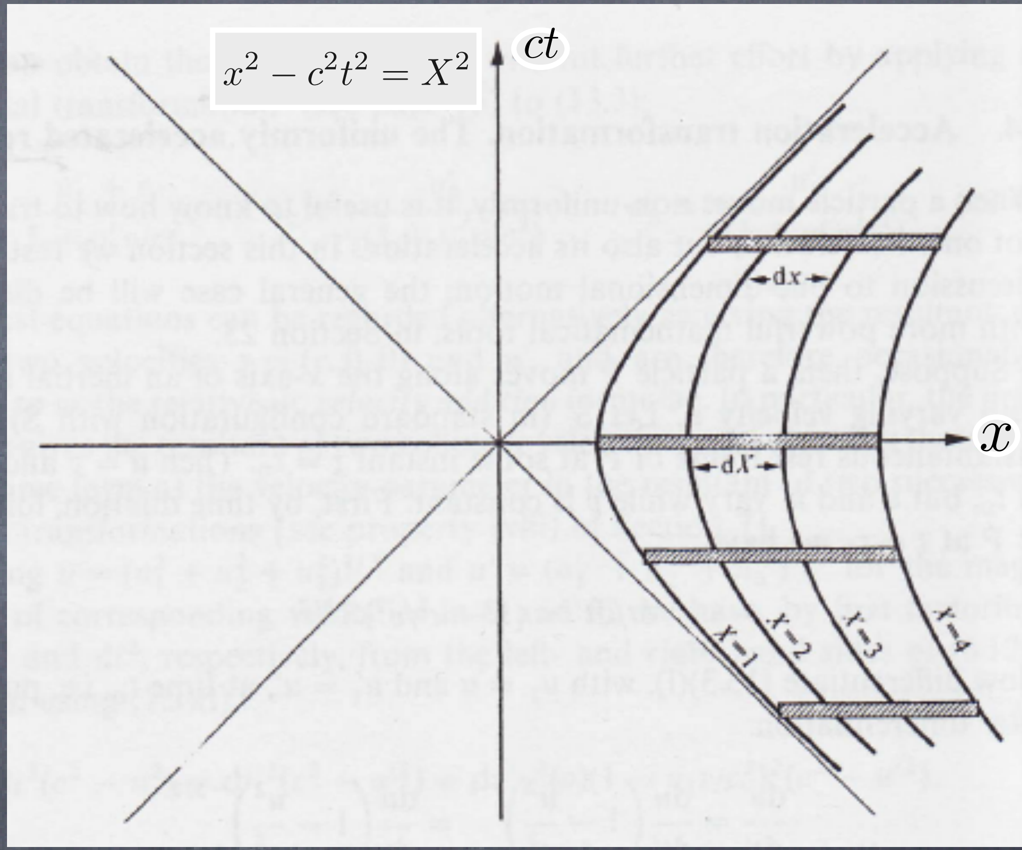
We can find by implicit differentiation of ☻ the velocity u and the corresponding Lorentz factor of a point moving so that X = constant:

$$u = \frac{dx}{dt} = \frac{c^2 t}{x}, \qquad \gamma(u) = \frac{x}{X}$$

Consider the motion of 2 points whose parameters X differ by dX

☛ at any fixed time t we have

$$dx = \frac{X\,dX}{x} = \frac{dX}{\gamma(u)}$$



At every instant t = constant the 2 points are separated by a coordinate distance dx inversely proportional to their Lorentz factor and consequently the element bounded by these points moves rigidly ➤ dX is recognized as the proper length

(Lorentz-Fitzgerald contraction shortens it)

A simple accelerated wordline

A particle moves on the x-axis along a worldline described parametrically by (we adopt natural units where the $c = 1$)

$$t(\sigma) = a^{-1} \sinh \sigma \qquad x(\sigma) = a^{-1} \cosh \sigma$$

where a is a constant with the dimension of inverse length

The parameter σ ranges from $-\infty$ to $+\infty$

For each value of σ, the parametric equations determine a point (t, x) in spacetime

(The y- and z- dimensions are unimportant for this example and will be suppressed in what follows)

As σ varies, the world line is swept out

A simple accelerated wordline (cont'd)

The wordline is the hyperbola ☛ $x^2 - t^2 = a^{-2}$

It could alternatively specified by giving $x(t) = (t^2 + a^{-2})^{\frac{1}{2}}$

but the parametric specification is more evenhanded between x and t

The wordline is accelerated because is not straight

Proper time τ along the wordline is related to σ ☛

$$d\tau^2 = dt^2 - dx^2 = (a^{-1}\cosh\sigma d\sigma)^2 - (a^{-1}\sinh\sigma d\sigma)^2 = (a^{-1}d\sigma)^2$$

Fixing τ to be zero when σ is zero, $\tau = a^{-1}\sigma$, and the wordline can be expressed with proper time as the parameter

$$t(\tau) = a^{-1}\sinh(a\tau) \qquad x(\tau) = a^{-1}\cosh(a\tau)$$

4-velocity of a simple wordline

The 4-velocity $\mathbf{U}$ of the wordline

$$U^0 \equiv dt/d\tau = \cosh(a\tau)$$

$$U^1 \equiv dx/d\tau = \sinh(a\tau)$$

This is correctly normalized

$$\mathbf{U} \cdot \mathbf{U} = -(U^0)^2 + (U^1)^2 = -\cosh^2(a\tau) + \sinh^2(a\tau) = -1$$

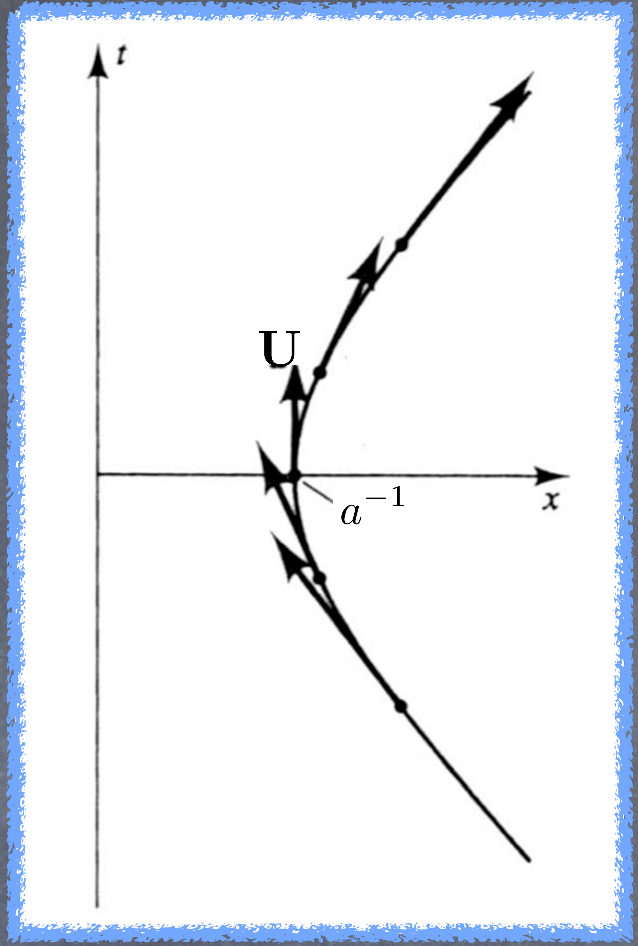
The particle's 3-velocity is

$$u^1 = \frac{dx}{dt} = \frac{dx/d\tau}{dt/d\tau} = \tanh(a\tau)$$

This never exceeds the speed of light $(|u^1| = 1)$

but approaches it at $\tau = \pm\infty$

A simple accelerated wordline (cont'd)



This spacetime diagram shows the worldline specified parametrically in terms of proper time τ

The points label values of $a\tau$ from -1 to 1 in steps of ½

4-velocity vectors $\mathbf{U}$ are shown for these points at half size

The next values of $a\tau$ of 1.5 and -1.5 are off the graph

The points are equidistant along the curve in the geometry of spacetime and the 4-vectors are all of equal length