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Problems set # 9 | **Physics 400** | April 11, 2019

1. Let $\mathbb{A}$ be a square finite-dimensional matrix (real elements) such that $\mathbb{A}\mathbb{A}^T = \mathbb{1}$. *(i)* Show that $\mathbb{A}^T\mathbb{A} = \mathbb{1}$ *(ii)* Does this result hold for infinite dimensional matrices?

2. Let us define a state using a hardness basis $\{|h\rangle, |s\rangle\}$, where

$$\hat{O}_{\rm HARDNESS}|h\rangle = |h\rangle \quad \text{and} \quad \hat{O}_{\rm HARDNESS}|s\rangle = -|s\rangle.$$

Suppose that we are in the state

$$|A\rangle = \cos\theta|h\rangle + e^{i\phi}\sin\theta|s\rangle$$

(i) Is this state normalized? Show your work. If not, normalize it. *(ii)* Find the state $|B\rangle$ that is orthogonal to $|A\rangle$. Make sure $|B\rangle$ is normalized. *(iii)* Express $|h\rangle$ and $|s\rangle$ in the $\{|A\rangle, |B\rangle\}$ basis. *(iv)* What are the possible outcomes of a hardness measurement on state $|A\rangle$ and with what probability will each occur? [*Hint:* Recall that eigenstates of hermitian operators with different eigenvalues are orthogonal to each other.]

3. If the states $\{|1\rangle, |2\rangle|3\rangle\}$ form an orthonormal basis and if the operator $\hat{G}$ has the properties

$$\begin{aligned} \hat{G}|1\rangle &= 2|1\rangle - 4|2\rangle + 7|3\rangle \\ \hat{G}|2\rangle &= -2|1\rangle + 3|3\rangle \\ \hat{G}|3\rangle &= 11|1\rangle + |2\rangle - 6|3\rangle \end{aligned}$$

What is the matrix representation of $\hat{G}$ in the $|1\rangle, |2\rangle|3\rangle$ basis?

4. Given particles in state

$$|\alpha\rangle = \frac{1}{\sqrt{83}}(-3|1\rangle + 5|2\rangle + 7|3\rangle)$$

where $\{|1\rangle, |2\rangle, |3\rangle\}$ form an orthonormal basis, what are the possible experimental results for a measurement of

$$\hat{Y} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -6 \end{pmatrix}$$

(written in this basis) and with what probabilities do they occur?

SOLUTIONS

1.(*i*) Since

$$\det(AB) = \det(A)\det(B)\ ,\ \det(A) = \det(A^T)\ ,\ \det(I) = 1$$

we have

$$\det(AA^T) = \det(A)\det(A^T) = \det(A^2) = \det(I) = 1$$

Therefore the inverse of A exists and we have $A^T = A^{-1}$ with $A^{-1}A = AA^{-1} = I$.

(*ii*) The answer is no. We have a counterexample. Let

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & \dots \\ 0 & 0 & 1 & 0 & 0 & \dots \\ 0 & 0 & 0 & 1 & 0 & \dots \\ . & . & . & . & . & . \\ . & . & . & . & . & . \end{pmatrix}$$

Then the transpose matrix A^T of A is given by

$$A^T = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \dots \\ 1 & 0 & 0 & 0 & 0 & \dots \\ 0 & 1 & 0 & 0 & 0 & \dots \\ 0 & 0 & 1 & 0 & 0 & . \\ . & . & . & . & . & . \end{pmatrix}$$

It follows that

$$AA^T = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \dots \\ 0 & 1 & 0 & 0 & 0 & \dots \\ 0 & 0 & 1 & 0 & 0 & \dots \\ 0 & 0 & 0 & 1 & 0 & . \\ . & . & . & . & . & . \end{pmatrix} = I$$

and

$$A^TA = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 1 & 0 & 0 & 0 & \dots \\ 0 & 0 & 1 & 0 & 0 & \dots \\ 0 & 0 & 0 & 1 & 0 & . \\ . & . & . & . & . & . \end{pmatrix} \neq I$$

Consequently,

$$AA^T \neq A^TA$$

2.(i)

$$\begin{aligned}\langle A \mid A\rangle &= \left(\cos\theta\,\langle h| + e^{-i\varphi}\sin\theta\,\langle s|\right)\left(\cos\theta\,|h\rangle + e^{i\varphi}\sin\theta\,|s\rangle\right)\\ &= \cos^2\theta\,\langle h \mid h\rangle + e^{i\varphi}\sin\theta\cos\theta\,\langle h \mid s\rangle + e^{-i\varphi}\sin\theta\cos\theta\,\langle s \mid h\rangle + \sin^2\theta\,\langle s \mid s\rangle\\ &= \cos^2\theta + \sin^2\theta = 1\end{aligned}$$

which says that the vector is normalized.

(ii)

$$|B\rangle = \alpha\,|h\rangle + \beta\,|s\rangle$$
$$\langle A \mid B\rangle = \left(\cos\theta\,\langle h| + e^{-i\varphi}\sin\theta\,\langle s|\right)\left(\alpha\,|h\rangle + \beta\,|s\rangle\right) = 0$$
$$0 = \alpha\cos\theta + e^{-i\varphi}\beta\sin\theta \Rightarrow \beta = -e^{i\varphi}\alpha\cot\theta$$
$$\langle B \mid B\rangle = \left(\alpha^*\,\langle h| + \beta^*\,\langle s|\right)\left(\alpha\,|h\rangle + \beta\,|s\rangle\right) = |\alpha|^2 + |\beta|^2 = 1$$
$$|\alpha|^2 + \cot^2\theta\,|\alpha|^2 = 1 \Rightarrow |\alpha|^2 = \tfrac{1}{1+\cot^2\theta} = \sin^2\theta$$
$$\alpha = \sin\theta \quad , \quad \beta = -e^{i\varphi}\cos\theta$$
$$|B\rangle = \sin\theta\,|h\rangle - e^{i\varphi}\cos\theta\,|s\rangle$$

(iii)

$$|A\rangle = \cos\theta\,|h\rangle + e^{i\varphi}\sin\theta\,|s\rangle$$
$$|B\rangle = \sin\theta\,|h\rangle - e^{i\varphi}\cos\theta\,|s\rangle$$
$$\langle h \mid A\rangle = \cos\theta = \langle A \mid h\rangle \quad , \quad \langle h \mid B\rangle = \sin\theta = \langle B \mid h\rangle$$
$$\langle s \mid A\rangle = e^{i\varphi}\sin\theta = \langle A \mid s\rangle^* \quad , \quad \langle s \mid B\rangle = -e^{i\varphi}\cos\theta = \langle B \mid s\rangle^*$$
$$|h\rangle = \langle A \mid h\rangle\,|A\rangle + \langle B \mid h\rangle\,|B\rangle = \cos\theta\,|A\rangle + \sin\theta\,|B\rangle$$
$$\begin{aligned}|s\rangle &= \langle A \mid s\rangle\,|A\rangle + \langle B \mid s\rangle\,|B\rangle\\ &= e^{-i\varphi}\sin\theta\,|A\rangle - e^{-i\varphi}\cos\theta\,|B\rangle = e^{-i\varphi}\left(\sin\theta\,|A\rangle - \cos\theta\,|B\rangle\right)\end{aligned}$$
$$|s\rangle = \sin\theta\,|A\rangle - \cos\theta\,|B\rangle$$

since overall phase factors are irrelevant.

(iv)

$$P(h|A) = |\langle h \mid A\rangle|^2 = \cos^2\theta$$
$$P(s|A) = |\langle s \mid A\rangle|^2 = \sin^2\theta$$

3.

$$\langle 1|\hat{G}|1\rangle = 2\langle 1 \mid 1\rangle - 4\langle 1 \mid 2\rangle + 7\langle 1 \mid 3\rangle = 2 = G_{11}$$
$$\langle 2|\hat{G}|1\rangle = 2\langle 2 \mid 1\rangle - 4\langle 2 \mid 2\rangle + 7\langle 2 \mid 3\rangle = -4 = G_{21}$$
$$\langle 3|\hat{G}|1\rangle = 2\langle 3 \mid 1\rangle - 4\langle 3 \mid 2\rangle + 7\langle 3 \mid 3\rangle = 7 = G_{31}$$
$$\langle 1|\hat{G}|2\rangle = -2\langle 1 \mid 1\rangle + 3\langle 1 \mid 3\rangle = -2 = G_{12}$$
$$\langle 2|\hat{G}|2\rangle = -2\langle 2 \mid 1\rangle + 3\langle 2 \mid 3\rangle = 0 = G_{22}$$
$$\langle 3|\hat{G}|2\rangle = -2\langle 3 \mid 1\rangle + 3\langle 3 \mid 3\rangle = 3 = G_{32}$$
$$\langle 1|\hat{G}|3\rangle = 11\langle 1 \mid 1\rangle + 2\langle 1 \mid 2\rangle - 6\langle 1 \mid 3\rangle = 11 = G_{13}$$
$$\langle 2|\hat{G}|3\rangle = 11\langle 2 \mid 1\rangle + 2\langle 2 \mid 2\rangle - 6\langle 2 \mid 3\rangle = 2 = G_{23}$$
$$\langle 3|\hat{G}|3\rangle = 11\langle 3 \mid 1\rangle + 2\langle 3 \mid 2\rangle - 6\langle 3 \mid 3\rangle = -6 = G_{33}$$

$$G = \begin{pmatrix} 2 & -2 & 11 \\ -4 & 0 & 2 \\ 7 & 3 & -6 \end{pmatrix}$$

4. We have

$$|\alpha\rangle = \frac{1}{\sqrt{83}}\left(-3\,|1\rangle + 5\,|2\rangle + 7\,|3\rangle\right)$$

where the $\{|1\rangle, |2\rangle, |3\rangle\}$ basis is the set of vectors

$$|1\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad , \quad |2\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad , \quad |3\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

The observable

$$\hat{Y} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -6 \end{pmatrix}$$

has eigenvectors $\{|1\rangle, |2\rangle, |3\rangle\}$ and eigenvalues $2, 3, -6$. The possible values of any measurement are the eigenvalues and the probabilities are given by

$$\begin{aligned}
P(2|\alpha) &= |\langle 1 \mid \alpha\rangle|^2 = \tfrac{1}{83}\left|-3\,\langle 1 \mid 1\rangle + 5\,\langle 1 \mid 2\rangle + 7\,\langle 1 \mid 3\rangle\right|^2 = \tfrac{9}{83} \\
P(3|\alpha) &= |\langle 2 \mid \alpha\rangle|^2 = \tfrac{1}{83}\left|-3\,\langle 2 \mid 1\rangle + 5\,\langle 2 \mid 2\rangle + 7\,\langle 2 \mid 3\rangle\right|^2 = \tfrac{25}{83} \\
P(-6|\alpha) &= |\langle 3 \mid \alpha\rangle|^2 = \tfrac{1}{83}\left|-3\,\langle 3 \mid 1\rangle + 5\,\langle 3 \mid 2\rangle + 7\,\langle 3 \mid 3\rangle\right|^2 = \tfrac{49}{83}
\end{aligned}$$