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Problems set # 5 | **Physics 400** | March 14, 2019

1. A dart of mass 1 kg is dropped from a height of 1 m, with the intention to hit a certain point on the ground. Estimate the limitation set by the uncertainty principle of the accuracy that can be achieved. For simplicity, consider only motion in the xy plane (y vertical and x horizontal).

2. Using the uncertainty principle estimate how long a time a pencil can be balanced on its point.

3. The operator $A^\dagger$ is called the *hermitian conjugate* (or adjoint) of $\hat{A}$ if

$$\int_{-\infty}^{+\infty} (\hat{A}^\dagger \phi)^* \psi dx = \int_{-\infty}^{+\infty} \phi^* \hat{A} \psi \, dx \, ; \tag{1}$$

in bra-ket notation (1) becomes $\langle A^\dagger \phi | \psi \rangle = \langle \phi | A \psi \rangle$. Its easy to show that $(c\hat{A})^\dagger = c^* \hat{A}^\dagger$ and $(\hat{A} + \hat{B})^\dagger = \hat{A}^\dagger + \hat{B}^\dagger$ just from the properties of the dot product. Using (1) show that $(\hat{A}^\dagger)^\dagger = \hat{A}$ and $(\hat{A}\hat{B})^\dagger = \hat{B}^\dagger \hat{A}^\dagger$.

4. The operator $\hat{A}$ is called hermitian if $\hat{A}^\dagger = \hat{A}$, i.e.

$$\int_{-\infty}^{+\infty} (\hat{A} \phi)^* \psi dx = \int_{-\infty}^{+\infty} \phi^* \hat{A} \psi \, dx \, ; \tag{2}$$

in bra-ket notation (2) becomes $\langle A \phi | \psi \rangle = \langle \phi | A \psi \rangle$. Convince yourself that $\hat{x}$ and $\hat{p} = -i\hbar \partial / \partial x$ are hermitian operators.

5. A physical variable must have real expectation values (and eigenvalues). By computing the complex conjugate of the expectation value of a physical variable show that every operator corresponding to an observable is hermitian.

SOLUTIONS

1. If there was no uncertainty principle, then a dart released from a height h above the point $x = 0$ would strike the point $x = 0$ since we can set both $x(0) = 0$ and $v_x(0) = 0$ initially. Quantum mechanics does not let us do this, however. Assume that $y(0) = h = 1$ m and $v_y(0) = 0$ for the vertical motion. Any uncertainty principle effects will be negligible in the vertical direction. We also assume that $x(0) = \Delta x$ and $v_x(0) = p_x(0)/m = p_x(0) = \Delta p$, where $\Delta x \Delta p \approx \hbar$. We then have the following equations of motion:

$$\begin{cases} y(t) = y(0) + v_y(0)t - \frac{1}{2}gt^2 = h - \frac{1}{2}gt^2 \\ x(t) = x(0) + v_x(0)t = \Delta x + \Delta p t \Rightarrow t = \frac{x - \Delta x}{\Delta p} \end{cases} .$$

We then have

$$y = h - \frac{1}{2} g \left(\frac{x - \Delta x}{\Delta p} \right)^2 .$$

We are interested in finding the minimum value of x when $y = 0$ (hits the ground). Substituting $y = 0$, $h = 1$ m, $g = 10$ m/s^2, $\Delta p = \hbar/\Delta x$ and solving for x we get

$$x = \Delta x + \frac{0.45\hbar}{\Delta x} .$$

We find the minimum by computing

$$\frac{\partial x}{\partial \Delta x} = 0 = 1 - \frac{0.45\hbar}{(\Delta x)^2} \Rightarrow \Delta x = \sqrt{0.45\hbar} .$$

Therefore, the minimum x consistent with quantum mechanics is $x_{\text{min}} = 2\sqrt{0.45\hbar} \approx 10^{-17}$ m. This is a very small distance! An atomic nucleus has a diameter of 10^{15} m.

2. Consider the diagram in Fig. 1. Modeling the pencil as a uniform rod with mass m, length r, moment of inertia $I = \frac{1}{3}mr^2$, the torque τ when it is at an angle θ to the vertical is $\tau = mgr \sin\theta$. For small deflections, $\sin\theta \approx \theta$ and we will use that assumption below. Then the equation of motion for the center-of-mass of the pencil becomes

$$I\ddot{\theta} = mgr \sin\theta \Rightarrow \dot{\theta}\frac{d\dot{\theta}}{d\theta} = \frac{mgr}{I} \sin\theta .$$

Integrating we get the conservation equation

$$\frac{\dot{\theta}^2 - \dot{\theta}_0^2}{2} = -\frac{mgr}{I}(\cos\theta - \cos\theta_0) .$$

We then get

$$\frac{d\theta}{dt} = \sqrt{\dot{\theta}_0^2 - \frac{mgr}{I}(\cos\theta - \cos\theta_0)} ,$$

which implies (integrating)

$$t = K_1 \int_{\theta_0}^{\theta_f} \frac{d\theta}{\sqrt{K_2 - \cos\theta}} ,$$

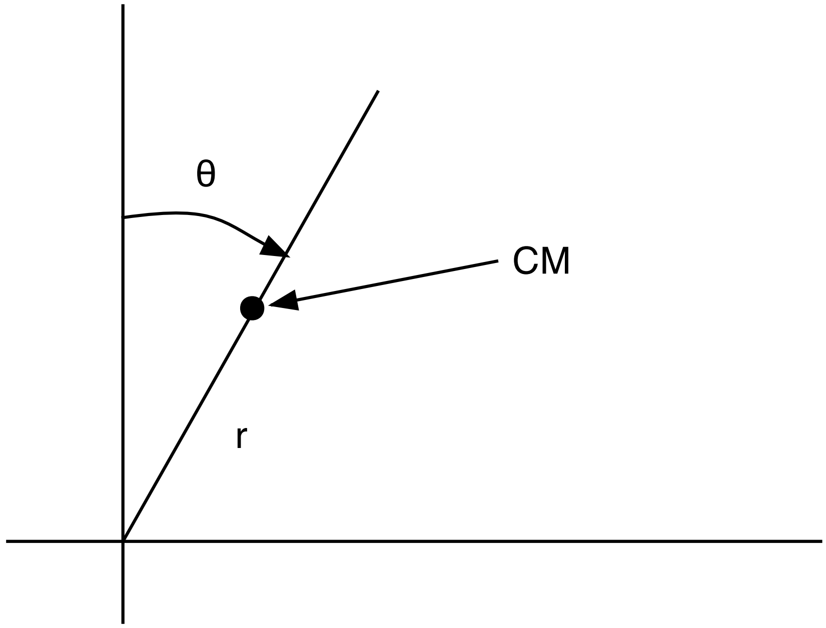

Figure 1: The situation in problem 5.

where

$$K_1 = \sqrt{\frac{I}{2mgr}} \quad \text{and} \quad K_2 = \frac{I\dot{\theta}_0^2 + 2mgr\cos\theta_0}{2mgr}\,.$$

Now we assume that $\theta_0 \approx (\Delta\theta)_0$ and $\dot{\theta}_0 \approx (\Delta\dot{\theta})_0$ which says that the best possible initial conditions are given by the uncertainties. If we are trying to balance the pencil for the longest time, then θ_0 and $\dot{\theta}_0$ must be as small as possible. Therefore we have $K_2 \approx 1 + \epsilon$, with $\epsilon \ll 1$. Therefore, small θ-values will dominate the integral, i.e., the pencil spends most of its time at small θ. You can see that this is true by trying an experiment!

Therefore we can write $\cos\theta \approx 1 - \theta^2/2$ which gives

$$t = K_1 \int_{\theta_0}^{\theta_f} \frac{d\theta}{\sqrt{K_3 + \theta^2/2}}\,,$$

where $K_3 = K_2 - 1 = \epsilon$. Now let

$$\theta = \sqrt{2K_3}\tan\phi \to d\theta = \sqrt{2K_3}\sec^2\phi\,.$$

We then have

$$t = K_1 \int_{\phi_0}^{\phi_f} \sec\phi\; d\phi\,,$$

where $\phi_0 = \tan^{-1}(\theta_0/\sqrt{2K_3})$ and $\phi_f = \tan^{-1}(\theta_f/\sqrt{2K_3})$. Now, $\int \sec\phi\, d\phi = \ln(\sec\phi + \tan\phi)$ and se we get

$$t = \sqrt{2}K_1 \ln \frac{\sec\phi_f + \tan\phi_f}{\sec\phi_0 + \tan\phi_0}\,.$$

The uncertainty principle says (at best) that initially

$$(\Delta x)_0(\Delta p_x)_0 = \hbar \Rightarrow [r(\Delta\theta)_0)][mr(\Delta\dot{\theta})_0] = \hbar = mr^2\theta_0\dot{\theta}_0,$$

which implies that

$$\dot{\theta}_0 = \frac{\hbar}{mr^2\theta_0}\,.$$

Thus,

$$K_2 \approx 1 + \frac{\theta_0^2}{2} + \alpha\frac{\hbar^2}{\theta_0^2}$$

where for a typical pencil

$$\alpha = \frac{I}{2m^3gr^5} \approx 0.005.$$

Therefore,

$$K_3 = \epsilon = \frac{\theta_0^2}{2} + \alpha\frac{\hbar^2}{\theta_0^2}$$

and

$$\phi_0 = \tan^{-1}\frac{\theta_0}{\sqrt{2K_3}} \Rightarrow \tan\phi_0 \approx 1 \approx \sec\phi_0.$$

Using $\theta_f = \pi/2$ (pencil on the floor), we have

$$\frac{\theta_f}{\sqrt{2K_3}} \gg 1 \Rightarrow \sec\phi_f \approx \tan\phi_f \approx \frac{\pi}{2\sqrt{2K_3}}\,.$$

Therefore,

$$t \approx \sqrt{2}K_1 \ln\frac{\pi}{2\sqrt{2K_3}}$$

where for a typical pencil

$$K_1 = \sqrt{\frac{I}{2mgr}} \approx 0.1\,.$$

Now, if we want maximum time (upright) we need to minimize K_3. Thus we have

$$\frac{\partial K_3}{\partial\theta_0} = \theta_0 - \frac{2\alpha\hbar^2}{\theta_0^3} = 0\,,$$

and we get

$$\theta_0 \approx (2\alpha)^{1/4}\sqrt{\hbar} \approx \frac{1}{3}\sqrt{\hbar} \Rightarrow K_3 \approx \frac{\hbar}{4}\,.$$

Finally, we have

$$t_{\max} \approx \frac{\sqrt{2}}{2}\frac{1}{10}(-\ln K_3) \approx 0.1(-\ln\hbar) \approx 0.1\times 34 \sim 3\,\text{to}\,4\text{ s.}$$

Again, try a few experiments!

3. *(i)* From the indentity (1) it follows that:

$$\langle (A^\dagger)^\dagger\phi|\psi\rangle = \langle\phi|A^\dagger\psi\rangle = \langle A^\dagger\psi|\phi\rangle^* = \langle\psi|\hat{A}\phi\rangle^* = (\langle A\phi|\psi\rangle^*)^* = \langle\hat{A}\phi|\psi\rangle \Rightarrow (\hat{A}^\dagger)^\dagger = \hat{A} \qquad (3)$$

and

$$\langle\phi|\hat{A}\hat{B}\psi\rangle = \langle\hat{A}^\dagger\phi|B\psi\rangle = \langle\hat{B}^\dagger\hat{A}^\dagger\phi|\psi\rangle \Rightarrow (AB)^\dagger = B^\dagger A^\dagger\,. \qquad (4)$$

4. *(i)* The operator $\hat{x}$ is hermitian because

$$\int_{-\infty}^{+\infty} (\hat{x}\phi)^* \, \psi \, dx = \int_{-\infty}^{+\infty} (x\phi(x))^* \, \psi(x) = \int_{-\infty}^{+\infty} \phi^* x \, \psi \, dx = \int_{-\infty}^{+\infty} \psi^* \, \hat{x}\psi \, dx \,. \tag{5}$$

(ii) The operatror $\hat{p}$ is hermitian because

$$\int (\hat{p}\phi)^* \psi dx = \int_{-\infty}^{+\infty} \left(-i\hbar \frac{\partial \phi}{\partial x} \right)^* \psi \, dx = i\hbar \int_{-\infty}^{+\infty} \left(\frac{\partial \phi}{\partial x} \right)^* \psi \, dx \tag{6}$$

and after integration by parts, recognizing that the wave function tends to zero as $x \to \infty$, the right-hand side of (6) becomes

$$-i\hbar \int_{-\infty}^{+\infty} \phi^* \, \frac{\partial \psi}{\partial x} \, dx = \int_{-\infty}^{+\infty} \phi^* \, \hat{p}\psi dx \,. \tag{7}$$

5. We may assert without proof that the expectation value of a physical observable is real, i.e. $\langle \psi | \hat{A}\psi \rangle = \langle \psi | \hat{A}\psi \rangle^*$. Now,

$$\langle \psi | \hat{A}\psi \rangle^* = \left[\int_{-\infty}^{+\infty} \psi^*(x) \hat{A}\psi(x) dx \right]^* = \int_{-\infty}^{+\infty} \psi(x) [\hat{A}\psi(x)]^* dx = \int_{-\infty}^{+\infty} [\hat{A}\psi(x)]^* \psi \, dx = \langle \hat{A}\psi | \psi \rangle \,, \tag{8}$$

so from (2) it follows that physical observables are represented by hermitian operators.