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Problems set # 4 **Physics 400** February 28, 2019

1. As a consequence of the Heisenberg uncertainty principle the more closely an electron is confined to a region of space the higher its kinetic energy will be. In an atom the electrons are confined by the Coulomb potential of the nucleus. The competition between the confining nature of the potential and the liberating tendency of the uncertainty principle gives rise to various quantum mechanical effects. Some of these microscopic effects have repercussions in the way this universe is structured. Estimate the size of the hydrogen atom in its ground state by minimizing the total energy as a function of the orbital radius of the electron.

2. While estimating the size of the hydrogen atom in its ground state we have seen that when we consider small radii, the electron is present very close to the nucleus. Pushing the electron any closer to the nucleus results in increased energies. The electron may even gain enough energy to fly away from the nucleus. This is when the atom will ionize and hence the useful rule, "it is impossible to squish atoms." When atoms are subjected to a high enough pressure they become ionized. This will happen, for example, at the center of a sufficiently massive gravitating body. *(i)* In order to ionize an atom a certain minimum energy must be supplied to it. Estimate the reduction in atomic radius required to ionize a hydrogen atom. *(ii)* What pressure P is needed to bring this about? [Hint: $P = -dE/dV$, where E is energy and V is the volume.] *(iii)* A planet is defined as a body in which the atoms resist the compressive force of gravity. Estimate the maximum mass and size of a planet composed of hydrogen. (You will need to estimate the pressure required at the center of the planet to support a column of mass against its weight.) This turns out to be of the order of the mass of Jupiter. Thus, Jupiter is not only the largest planet composed of hydrogen in the solar system but anywhere in the universe!

3. According to general principles of classical electrodynamics, accelerated charged particles always radiate electromagnetic waves. This is the basic rule upon which all radiation sources are based. At the end of the last century Larmor calculated the total power radiated by an accelerated non-relativistic electron ($v \ll c$). His well known result is (in Gaussian units)

$$P = \frac{2}{3} \frac{q^2 a^2}{c^3} \,, \tag{1}$$

where a is the acceleration and q is the charge. In the Rutherford model of the hydrogen atom's ground state, the electron moves in a circular orbit of radius $a_0 = 0.529$ $\AA$ around the proton, which is assumed to be rigidly fixed in space. Since the electron is accelerating, a classical analysis suggests that it will continuously radiate energy, and therefore the radius of the orbit would shrink with time. Assuming that the electron is always in a nearly circular orbit and that the rate of radiation of energy is sufficiently well approximated by classical, nonrelativistic electrodynamics, how long is the fall time of the electron, i.e., the time for the electron to spiral into the origin?

4. The neutrons produced in a reactor are known as thermal neutrons, because their kinetic energies have been reduced (by collisions) until $K = \frac{3}{2}kT$, where T is room temperature (about

293 K). *(i)* What is the kinetic energy of such neutrons? *(ii)* What is their de Broglie wavelength?

5. Position x and momentum p_x are conjugate variables. Consider a harmonic oscillator, oscillating in the x direction. The potential energy of this oscillator $V(x)$ is quadratic in x. Using the Sommerfeld-Wilson quantization rule, find the energy levels of any oscillator defined by a potential energy function $V(x)$.

SOLUTIONS

1. Consider a single hydrogen atom: an electron of charge $-e$ free to move around in the electric field of a fixed proton of charge $+e$. (The proton is $\sim$ 2000 times heavier than electron, so we consider it fixed.) The electron has a potential energy due to the attraction to the proton of $V(r) = -\frac{e^2}{4\pi\epsilon_0 r}$, where r is the electron proton separation. The electron has a kinetic energy of $K = \frac{p^2}{2m}$. The total energy is then

$$E(r) = \frac{p^2}{2m} - \frac{e^2}{4\pi\epsilon_0 r}.$$

Classically, the minimum energy of the hydrogen atom is $-\infty$ the state in which the electron is on top of the proton $p = 0$, $r = 0$. Quantum mechanically, the uncertainty principle forces the electron to have non-zero momentum and non-zero expectation value of position. If a is an average electron-proton distance, the uncertainty principle informs us that the minimum electron momentum is on the order of $\hbar/a$. The enrgy as a function of a is then

$$E(a) = \frac{\hbar^2}{2ma^2} - \frac{e^2}{4\pi\epsilon_0 a}.$$

If we insist on placing the electron right on top of the proton ($a = 0$), the potential energy is still $-\infty$, just as it is classically, but the total energy is:

$$\begin{aligned} E(0) &\approx \lim_{a\to 0}\left[\frac{\hbar^2}{2ma^2} - \frac{e^2}{4\pi\epsilon_0 a}\right] \\ &= \lim_{a\to 0}\left[\frac{2\pi\epsilon_0\hbar^2 - me^2 a}{4\pi\epsilon_0 m a^2}\right] \\ &\to +\infty\,. \end{aligned}$$

Hence, quantum mechanics tells us that an atom could never collapse as it would take an infinite energy to locate the electron on top of the proton. The minimum energy state, quantum mechanically, can be estimated by calculating the value of $a = a_0$ for which $E(a)$ is minimized:

$$\left.\frac{\partial E(a)}{\partial a}\right|_{a_0} = -\frac{\hbar^2}{ma^3} + \frac{e^2}{4\pi\epsilon_0 a_0^2} = 0,$$

and so

$$a_0 = \frac{4\pi\epsilon_0\hbar^2}{me^2} = \frac{10^{-10}\cdot 10^{-68}}{10^{-30}\cdot 2\cdot 10^{-38}}\mathrm{m} \approx 0.5\mathring{A}\,.$$

By preventing localization of the electron near the proton, the uncertainty principle retards the classical collapse of the atom. The state of minimum energy corresponds to $E(a_0) = -13.6$ eV; see Fig. 1.

2. *(i)* The momentum of an electron confined within a radius r is approximately $p \sim \hbar/r$. The total energy is,

$$E = \frac{\hbar^2}{2mr^2} - \frac{e^2}{4\pi\epsilon_0 r}\,.$$

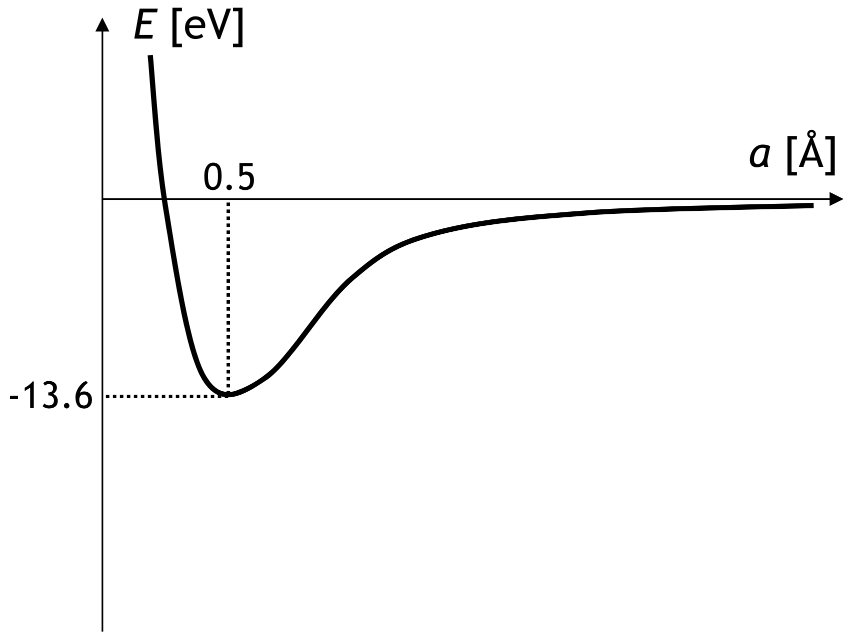

Figure 1: As a function of a the total energy looks like this.

Ionization occurs when the energy of the electron approached zero, the energy of the vacuum state. We calculate the radius $r_{\rm ion}$ when $E = 0$; namely

$$\frac{\hbar^2}{2mr^2} - \frac{e^2}{4\pi\epsilon_0 r} \Rightarrow r_{\rm ion} = \frac{2\pi\epsilon_0\hbar^2}{me^2} \simeq 0.24\mathring{A}\,.$$

The radius $r_{\rm ion}$ is smaller than the a_0, as we expect. Excessive pressure inside a planet can push the electron to this radius. At this point, the atoms will ionize and the planet will not be stable. *(ii)* The pressure is given by

$$P = -\frac{dE}{dV} = -\frac{dE}{dr}\frac{dr}{dV}\,.$$

We have $V = \frac{4}{3}\pi r^2$, $dV = 4\pi r^2 dr$ and $dr/dV = (4\pi r^2)^{-1}$. Differentiating the energy expression

$$\begin{aligned} \frac{dE}{dr} &= \frac{\hbar^2}{2m}\left(\frac{-2}{r^3}\right) - \frac{1}{4\pi\epsilon_0}e^2\left(\frac{-1}{r^2}\right) \\ &= -\frac{\hbar^2}{mr^3} + \frac{e^2}{4\pi\epsilon_0 r^2}\,. \end{aligned}$$

We now substitute the value of the radius, $r = r_{\rm ion}$,

$$\left.\frac{dE}{dr}\right|_{r_{\rm ion}} = -3.9\times 10^7\ \text{J/m},$$

resulting in the ionizing pressure

$$P_{\rm ion} = -\frac{dE}{dr}\frac{1}{4\pi r_{\rm ion}^2} = 5.2\times 10^{13}\ \text{Pa}\,.$$

(iii) We assume a spherical planet of radius R and mass M. We determine the parameters that result in ionizing pressures at the centre of the planet. First of all, we assume a constant density ρ of the planet throughout the interior. An estimate of the density is the proton mass divided by the volume of the atom,

$$\rho = \frac{3\, m_p}{4\pi r_{\rm ion}^3} \simeq 2.8 \times 10^4 \ \mathrm{kg/m^3}\,.$$

The pressure exerted by a fluid of length R at its base is given by $\rho g R$. However, the value of g on this planet is unknown, but from Newton's law of gravitation, we know that $g = GM/R^2$. Therefore,

$$P_{\rm ion} = \rho g R = \frac{\rho G M}{R} \Rightarrow R = \frac{\rho G M}{P_{\rm ion}} \simeq 3.5 \times 10^{-20} M \ \mathrm{m}\,.$$

Now the density ρ can also be equated to the mass of the planet divided by its volume,

$$\rho = 2.8 \times 10^4 \ \mathrm{kg/m^3} = \frac{3M}{4\pi R^3} \Rightarrow M = \frac{4}{3}\pi \rho R^3.$$

Substituting results in $M = 4 \times 10^{26}$ kg and $R = 1.6 \times 10^7$ m. The measured mass and radius of Jupiter are 1.9×10^{27} kg and 7×10^7 m.

3. The dominant energy loss is from electric dipole radiation, which obeys the Larmor formula. For an electron of charge $-e$ and mass m_e in an orbit of radius r about a fixed nucleus of charge $+e$, the radial component of the nonrelativistic force law, $\vec{F} = m_e \vec{a}$, tells us that $e^2/r^2 = m_e a_r \approx m_e\, v_\theta^2/r$, in the adiabatic approximation that the orbit remains nearly circular at all times. In the same approximation, $a_\theta \ll a_r$, i.e., $a \approx a_r$, and hence,

$$\frac{dE}{dt} = -\frac{2e^6}{3r^4 m_e^2 c^3} = -\frac{2}{3}\frac{r_0^3}{r^4} m_e c^3\,, \tag{2}$$

where $r_0 = e^2/(m_e c^2) = 2.8 \times 10^{-15}$ m is the classical electron radius. The total nonrelativistic energy (kinetic plus potential) is

$$E = -\frac{e^2}{r} + \frac{1}{2} m_e v^2 = -\frac{e^2}{2r} = -\frac{r_0}{r} m_e c^2\,. \tag{3}$$

Equating the time derivative of (3) to (2), we have

$$\frac{dE}{dt} = \frac{r_0}{2r^2}\dot{r} m_e c^2 = -\frac{2}{3}\frac{r_0^3}{r^4} m_e c^3, \quad \text{or equivalently} \quad r^2 \dot{r} = \frac{1}{3}\frac{dr^3}{dt} = -\frac{4}{3} r_0^2 c, \quad \text{yielding} \quad r^3 = a_0^3 - 4 r_0^2 c t\,. \tag{4}$$

The time to fall to the origin is then

$$t_{\rm fall} = \frac{a_0^3}{4 r_0^2 c} = 1.6 \times 10^{-11} \ \mathrm{s}\,. \tag{5}$$

This is of the order of magnitude of the lifetime of an excited hydrogen atom, whose ground state, however, appears to have infinite lifetime.

4. *(i)* The average kinetic energy of the neutrons is $K = 3kT/2 = 0.0379$ eV. *(ii)* The neutrons are non-relativistic so the momentum is given by $p = \sqrt{2mK} = \sqrt{2mc^2 K}/c = 8.44 \times 10^3$ eV/c, yielding $\lambda = hc/(pc) = 0.147$ nm.

5. Let's say the oscillator has energy E and amplitude x_{max}. Then

$$E = \frac{p^2}{2m} + \frac{1}{2}kx = \frac{1}{2}kx_{\text{max}}^2 \, .$$

The oscillator has frequency $\omega = \sqrt{k/m}$, so $k = m\omega^2$. Therefore,

$$p = \sqrt{\frac{1}{2}m\omega^2(x_{\text{max}}^2 - x^2)2m} = m\omega\sqrt{x_{\text{max}}^2 - x^2} \, .$$

We want to integrate $\oint p \cdot dx$ along one oscillation. We do this by integrating this on a quarter-cycle ($x = 0$ to $x = x_{\text{max}}$) and multiplying by 4 (details somewhat omitted):

$$nh = m\omega \oint_{\text{one period}} \sqrt{x_{\text{max}}^2 - x^2}\, dx = 4m\omega \int_0^{x_{\text{max}}} \sqrt{x_{\text{max}}^2 - x^2}\, dx = m\omega\pi x_{\text{max}}^2 \, .$$

Putting this together gives us

$$nh = \pi m\omega x_{\text{max}}^2 = \frac{2\pi}{\omega}\left(\frac{1}{2}m\omega^2 x_{\text{max}}^2\right) \, .$$

But, $\frac{1}{2}m\omega^2 x_{\text{max}}^2$ is the energy E, so

$$\frac{2\pi}{\omega}E = nh \Rightarrow E = \frac{nh\omega}{2\pi} = nh\nu \, .$$

This tells us that the energy levels of this oscillator are quantized, and equally spaced.