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Problems set # 3 **Physics 400** February 21, 2019

1. A pendulum of length $l = 0.1$ m and mass $m = 0.01$ kg swings up to a maximum angle of $\theta = 0.1$ rad. If its energy is quantized, the discontinuous jumps in energy are very small. For this angle of swing, what is the quantum number n that corresponds with the total kinetic energy of the system? What does this mean?

2. *(i)* Stars behave approximately like blackbodies. Use Wien's displacement formula to obtain a rough estimate of the surface temperature of the Sun, assuming that it is an ideal blackbody and that evolution on Earth worked well (i.e., that the human eye uses optimal the light from the Sun). *(ii)* The solar constant (radiant flux at the surface of the Earth) is about 1.365 kW/m^2. Find the effective surface temperature of the Sun. [*Hint:* Astronomical data which may be helpful: radius of Sun $R_\odot = 7 \times 10^5$ km; mean Sun-Earth distance $r_{\rm SE} = 1$ AU $= 1.5 \times 10^8$ km.]

3. For a typical case of photoemission from sodium, show that classical theory predicts that: *(i)* $K_{\rm max}$ depends on the incident light intensity I; *(ii)* $K_{\rm max}$ does not depend on the frequency of the incident light; *(iii)* there is a long time lag between the start of illumination and the beginning of the photocurrent. The work function for sodium is $\varphi = 2.28$ eV and an absorbed power per unit area of 1.00×10^{-7} mW/cm^2 produces a measurable photocurrent in sodium.

4. A metal surface has a photoelectric cutoff wavelength of 325.6 nm. It is illuminated with light of wavelength 259.8 nm. What is the stopping potential?

5. Another effect that revealed the quantized nature of radiation is the (elastic) scattering of light on particles shown in Fig. 1, called the Compton effect. *(i)* Using conservation of energy and momentum, derive the Compton shift formula,

$$\lambda - \lambda_0 = \lambda_{\rm c}(1 - \cos\theta)\,, \tag{1}$$

where $\lambda_{\rm c}$ is the Compton wavelength of the particle, which is equivalent to the wavelength of a photon whose energy is the same as the mass of the particle. *(ii)* In the experiment by Compton, X-rays are scattered by nearly free electrons ($\lambda_{\rm c} = 2.43\times10^{-10}$ cm) in carbon (graphite). (Although no scattering target contains actual "free" electrons, the outer or valence electrons in many materials are very weakly attached to the atom and behave like nearly free electrons. The binding energies of these electrons in the atom are so small compared with the energies of the incident X-ray photons that they can be regarded as nearly "free" electrons.) A movable detector measured the energy of the scattered X rays at various angles θ. At each angle, two peaks appear, corresponding to scattered X-ray photons with two different energies or wavelengths. The wavelength of one peak does not change as the angle is varied; this peak corresponds to scattering that involves "inner" electrons of the atom, which are more tightly bound to the atom so that the photon can scatter with no loss of energy. The wavelength of the other peak, however, varies strongly with angle. This

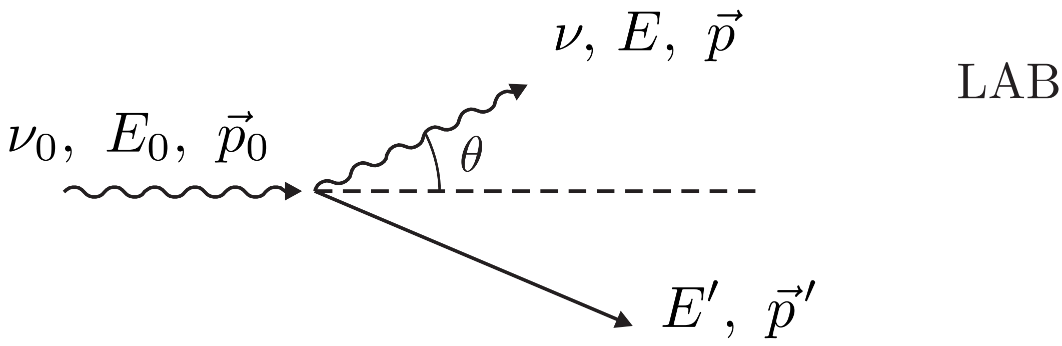

Figure 1: The geometry of Compton scattering.

variation is exactly as the Compton formula predicts. Show that for a maximal scattering angle the fractional change $\Delta\lambda/\lambda_0$ is about 7%. In summary, the particle character of light is confirmed in Compton's experiment and we assign the energy of $E = \hbar\omega$ and the momentum $\vec{p} = \hbar\vec{k}$ to the (undivisible) photon, where $\hbar = h/(2\pi)$. The Compton shift formula (1) reveals a proportionality to $\hbar$, a quantum mechanical property that is confirmed by experiment. Classically no change of the wavelength is to be expected.

SOLUTIONS

1. We first need to find the energy and frequency of the pendulum. To find the energy we need to apply some geometry to the situation to figure out the height h that the pendulum is lifted to. Constructing a triangle, with hypotenuse of length l and height of length $(l - h)$ we get the relationship: $\cos\theta = \frac{l-h}{l}$ and so $h = l - l\cos\theta = l(1-\cos\theta)$. Given the acceleration of gravity $g \approx 9.8$ m/s^2, the total (classical) energy of the swinging pendulum and the frequency ν of oscillation is thus $E_{\rm class} = mgh = mgl(1-\cos\theta)$ and $\nu = \frac{1}{2\pi}\sqrt{\frac{g}{l}}$. If this system were to be described by quantum physics, the energy must satisfy the relation $E = nh\nu$, with $n = 0, 1, 2, \cdots$, where $h = 6.6 \times 10^{34}$ Js is Planck's constant. So if we set $E_{\rm class} = E$ we get $mgl(1-\cos\theta) = nh\nu = \frac{nh}{2\pi}\sqrt{\frac{g}{l}}$ and solving for n gives us

$$n = \frac{mgl}{h\nu}(1-\cos\theta) = \frac{2\pi m}{h}\sqrt{\frac{l}{g}}(1-\cos\theta) = \frac{2\pi m}{h}\sqrt{gl^3}(1-\cos\theta) \approx 4.7 \times 10^{28}\,,$$

which is clearly a HUGE number. The adjacent energy levels for this system are only about $\Delta E \approx 10^{-33}$ J apart, which fits with the calculation above. Of course variations for the values of m, g, l, θ, etc. make it impossible to confirm the discrete quantum nature of this type of system. According to quantum theory, n must be an integer, but of course for this calculation, there is no way that our physical measurements of the of the system could be accurate enough to determine if the calculated value for n is an integer or not.

2. *(i)* We can obtain a first estimate of the surface temperature of the Sun from the sensitivity of the human eye to light in the range $400 - 700$ nm. Assuming that the evolution worked well, i.e. that the human eye uses optimal the light from the Sun, and that the atmosphere is for all frequencies in the visible range similarly transparent, we identify the maximum in Wien's law with the center of the frequency range visible for the human eye. Thus we set $\lambda_{\rm max,\odot} \approx 550$ nm, and obtain $T_\odot \approx 5270$ K for the surface temperature of the Sun. *(ii)* The bolometric luminosity L of a star is given by the product of its surface $A = 4\pi R^2$ and the radiation emitted per area σT^4, i.e., $L = 4\pi R^2 \sigma T^4$. The radiant flux is defined by $F = L/A$, so that we recover the well known inverse-square law for the energy flux at the distance $r > R$ outside of the star, $F = L/(4\pi r^2)$. The validity of the inverse-square law $F(r) \propto r^{-2}$ relies on the assumptions that no radiation is absorbed and that relativistic effects can be neglected. The later condition requires, in particular, that the relative velocity of observer and source is small compared to the velocity of light. The energy flux received from the Sun at the distance of the Earth, $r_{\rm SE} = 1$ AU, is equal to $F = 1365$ W/m^2. The solar luminosity follows then as $L_\odot = 4\pi d^2 F = 4 \times 10^{33}\,{\rm erg\,s^{-1}}$, and serves as a convenient unit in stellar astrophysics. The Stefan-Boltzmann law can then be used to define, with $R_\odot \approx 7 \times 10^{10}$ cm, the effective temperature of the Sun, $T_\odot \approx 5780$ K.

3. *(i)* According to classical theory, the energy in a light wave is spread out uniformly and continuously over the wavefront. Assuming that all absorption of light occurs in the top atomic layer of the metal, that each atom absorbs an equal amount of energy proportional to its cross sectional area, A, and that each atom somehow funnels this energy into one of its electrons, we find that each electron absorbs an energy K in time t given by $K = \epsilon I A t$ where ϵ is a fraction accounting for less than 100% light absorption. Because the most energetic electrons are held in the metal by a surface energy barrier φ, these electrons will be emitted with $K_{\rm max}$ once they have absorbed enough energy to overcome the barrier. We can express this as $K_{\max} = \epsilon IAt - \varphi$. Thus, classical theory predicts that for a fixed absorption period, t, at low light intensities when $\epsilon IAt < \varphi$, no electrons must be emitted. At higher intensities, when $\epsilon IAt > \varphi$ electrons should be emitted with higher kinetic energies the higher the light intensity. Therefore, classical predictions contradict experiment at both very low and very high light intensities. *(ii)* According to classical theory, the intensity of a light wave is proportional to the square of the amplitude of the electric field, $|\vec{E}_0|^2$, and it is this electric field amplitude that increases with increasing intensity and imparts an increasing acceleration and kinetic energy to an electron. Substituting I with a quantity proportional to $|\vec{E}_0|^2$ in part (i) shows that $K_{\max}$ should not depend at all on the frequency of the classical light wave, again contradicting the experimental results. *(iii)* To estimate the time lag between the start of illumination and the emission of electrons, we assume that an electron must accumulate just enough light energy to overcome the work function. Setting $K_{\max} = 0$ gives $0 = \epsilon IAt - \varphi$ or $t = \varphi/(\epsilon IA)$. Taking $\epsilon = 1$ and the cross sectional area of the atom $A = \pi r^2$, where $r = 1.0 \times 10^{-8}$ cm is a typical atomic radius, we have $t = 1.2 \times 10^7$ s ≈ 130 days. Thus we see that the classical calculation of the time lag for photoemission does not agree with the experimental result, disagreeing by a factor of 10^{16}!

4. At the threshold wavelength the photoelectrons have just enough energy to overcome the work function, so $K_{\max} = 0$. Hence we have $\varphi = hc/\lambda_0 = 3.808$ eV. When 259.8 nm light is used, $eV_0 = hc/\lambda - \varphi = 0.964$ eV, so $V_0 = 0.964$ V.

5. *(i)* Consider an incident photon of frequency ν_0 which is scattered by a stationary electron to give a photon of frequency ν at an angle θ with respect to the original photon. Conservation of energy gives

$$E_0 + mc^2 = E + E', \tag{2}$$

while conservation of 3-momentum gives $\vec{p}_0 = \vec{p} + \vec{p}'$ or $\vec{p}' = \vec{p}_0 - \vec{p}$. Squaring this $\vec{p}'^2 = \vec{p}_o^{\,2} + \vec{p}^{\,2} - 2\vec{p}_0 \cdot \vec{p}$ and using $E_0 = |\vec{p}_0 c|$, $E = |\vec{p}c|$, and $E'^2 = (\vec{p}'c)^2 + m^2c^4$, we get $E'^2 - m^2c^4 = E_0^2 + E^2 - 2E_0E\cos\theta$. Now, from (2) the left-hand side of the previous relation equals $(E_0 - E)^2 + 2mc^2(E_0 - E)$, thus yielding

$$2mc^2(E_0 - E) = 2E_0E - 2E_0E - 2E_0E\cos\theta\,. \tag{3}$$

Using $E_0 = hc/\lambda_0$ and $E = hc/\lambda$ we obtain the well known formula for the change in wavelength as a function of angle

$$\lambda - \lambda_0 = \frac{h}{mc}(1 - \cos\theta) \tag{4}$$

The quantity $h/(mc)$ is known as the Compton wavelength (in this case, of the electron). *(ii)* Since λ_c is very small, high energy radiation (X-rays) is needed to observe the effect. If we choose a wavelength of 7×10^{-9} cm for the X-rays we estimate for a maximal scattering angle an effect of $\Delta\lambda/\lambda_0 = 2\lambda_{\rm c}/\lambda_{\rm Xray} \approx 0.07$.