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Problems set # 10 **Physics 400** April 18, 2019

1. The number of stars in our galaxy is about $N = 10^{11}$. Assume that the probability that a star has planets is $p = 10^{-2}$, the probability that the conditions on the planet are suitable for life is $q = 10^{-2}$, and the probability of life evolving, given suitable conditions, is $r = 10^{2}$. These numbers are rather arbitrary. *(i)* What is the probability of life existing in an arbitrary solar system (a star and planets, if any)? *(ii)* What is the probability that life exists in at least one solar system?

2. *(i)* Show that the following relation applies for any operator O that lacks an explicit dependence on time:

$$\frac{\partial}{\partial t}\langle O\rangle = \frac{i}{\hbar}\langle[H,O]\rangle$$

(ii) Use this result to derive Ehrenfest's relations, which show that classical mechanics still applies to expectation values:

$$m\frac{\partial}{\partial t}\langle\vec{x}\rangle = \langle\vec{p}\rangle \quad \text{and} \quad \frac{\partial}{\partial t}\langle\vec{p}\rangle = -\langle\vec{\nabla}V\rangle$$

[*Hint:* Remember that the Hamiltonian, H, is a Hermitian operator, and that H appears in the time-dependent Schrödinger equation.]

3. Suppose that the wave function of a (spinless) particle of mass m is

$$\psi(r,\theta,\phi) = A\frac{e^{-\alpha r} - e^{-\beta r}}{r},$$

where A, α and β are constants such that $0 < \alpha < \beta$. Find the potential $V(r,\theta,\phi)$ and the energy E of the particle.

4. Consider a particle of mass μ constrained to move on a circle of radius a. Show that

$$H = \frac{L^2}{2\mu a^2}$$

Solve the eigenvalue/eigenvector problem of H and interpret the degeneracy.

SOLUTIONS

1.(*i*) The probability of life in the vicinity of some arbitrarily selected star is equal to $pqr = 10^{-6}$, assuming that the three conditions are *independent*.

(*ii*) The probability P that life exists in the vicinity of at least one star is given by $P = 1 - P_0$, where P_0 is the probability that no stars have life about them.

The probability of no life about some arbitrarily selected star is $1 - pqr$. Therefore, we have

$$P_0 = (1 - pqr)^N$$

Now

$$\log P_0 = N \log(1 - pqr) \approx N(-pqr) = -10^5$$

so that

$$P_0 = e^{-pqrN} \approx e^{-10^5} \approx 0$$

Thus,

$$P = 1 - P_0 \approx 1$$

This says that even a very rare event is almost certain to occur in a large enough sample.

NOTE: A naive argument against a purely natural origin of life is sometimes based on the smallness of the probability (*i*), whereas it is the probability (*ii*) that is relevant!

2. (*i*) We have

$$\begin{aligned}\frac{\partial}{\partial t}\langle O\rangle &= \frac{\partial}{\partial t}\langle\psi|\,O\,|\psi\rangle \\ &= \left[\frac{\partial}{\partial t}\langle\psi|\right]O\,|\psi\rangle + \langle\psi|\,O\left[\frac{\partial}{\partial t}|\psi\rangle\right] \\ &= \left[-\langle\psi|\,\frac{1}{i\hbar}H\right]O\,|\psi\rangle + \langle\psi|\,O\left[\frac{1}{i\hbar}H\,|\psi\rangle\right] \\ &= \frac{i}{\hbar}\langle\psi|\,(HO-OH)\,|\psi\rangle\end{aligned}$$

which says that

$$\frac{\partial}{\partial t}\langle O\rangle = \frac{i}{\hbar}\langle[H,O]\rangle$$

(*ii*) We have

$$\begin{aligned}\frac{d}{dt}\langle x\rangle &= \frac{i}{\hbar}\langle[H,x]\rangle = \frac{i}{\hbar}\langle\frac{1}{2m}[p_x^2,x]\rangle \\ &= \frac{i}{2m\hbar}\langle(p_xp_xx - xp_xp_x)\rangle = \frac{i}{2m\hbar}\langle(p_xp_xx - p_xxp_x + p_xxp_x - xp_xp_x)\rangle \\ &= \frac{i}{2m\hbar}\langle(p_x[p_x,x] + [p_x,x]p_x)\rangle = \frac{i}{2m\hbar}\langle(-2i\hbar p_x)\rangle \\ &= \langle\frac{p_x}{m}\rangle\end{aligned}$$

Similarly for y and z components. Therefore we have

$$m\frac{\partial}{\partial t}\langle\vec{x}\rangle = \langle\vec{p}\rangle$$

Now we have

$$\begin{aligned}\frac{d}{dt}\langle p_x\rangle &= \frac{i}{\hbar}\langle[H,p_x]\rangle = \frac{i}{\hbar}\langle[V(x),p_x]\rangle \\ &= \frac{i}{\hbar}\langle i\hbar\frac{\partial V(x)}{\partial x}\rangle = -\langle\frac{\partial V(x)}{\partial x}\rangle\end{aligned}$$

Similarly for y and z components. Therefore we have

$$\frac{\partial}{\partial t}\langle\vec{p}\rangle = -\langle\nabla V\rangle$$

3. Write the wave function as

$$\psi(r,\theta,\phi) = A\frac{e^{-\alpha r} - e^{-\beta r}}{r} = \frac{u(r)}{r}$$

Since ψ depends only on r, we have $\ell = m = 0$, and the Schrodinger equation in spherical coordinates is

$$-\frac{\hbar^2}{2m}\frac{d^2u}{dr^2} + (V(r) - E)u = 0 \rightarrow V(r) - E = \frac{\hbar^2}{2m}\frac{1}{u}\frac{d^2u}{dr^2}$$

Differentiate $u(r) = A(e^{-\alpha r} - e^{-\beta r})$:

$$\frac{du}{dr} = A(-\alpha e^{-\alpha r} + \beta e^{-\beta r}) \quad , \quad \frac{d^2u}{dr^2} = A(\alpha^2 e^{-\alpha r} - \beta^2 e^{-\beta r})$$

This gives

$$V(r) - E = \frac{\hbar^2}{2m}\frac{\alpha^2 e^{-\alpha r} - \beta^2 e^{-\beta r}}{e^{-\alpha r} - e^{-\beta r}}$$

Since the potential must vanish at $r \rightarrow \infty$, we get

$$E = -\lim_{r\rightarrow\infty}\frac{\hbar^2}{2m}\frac{\alpha^2 e^{-\alpha r} - \beta^2 e^{-\beta r}}{e^{-\alpha r} - e^{-\beta r}} = -\frac{\hbar^2\alpha^2}{2m}$$

and the potential is

$$V(r) = \frac{\hbar^2}{2m}\left(\frac{\alpha^2 e^{-\alpha r} - \beta^2 e^{-\beta r}}{e^{-\alpha r} - e^{-\beta r}} - \alpha^2\right)$$

For small r:

$$V(r) \approx -\frac{\hbar^2}{2m}\frac{(\alpha^2 - \beta^2)e^{-\beta r}}{(\beta - \alpha)r} = -\frac{\hbar^2(\alpha+\beta)e^{-\beta r}}{2mr}$$

This is a screened Coulomb potential.

The same procedure to find V, E also works if ψ has an angular dependence, by including the centrifugal barrier term $\ell(\ell+1)\hbar^2/2mr^2$ in the radial Schrodinger equation.

4. We have the potential $V = 0$ and the kinetic energy

$$T = \frac{1}{2}\mu v^2 = \frac{1}{2}\mu a^2 \dot{\phi}^2 \quad , \quad v = a\dot{\phi}$$

In addition,

$$L_z = \mu a v = \mu a^2 \dot{\phi}$$

so that

$$H = T + V = \frac{L_z^2}{2\mu a^2}$$

Now we have

$$H\left|\psi\right\rangle = \frac{L_z^2}{2\mu a^2}\left|\psi\right\rangle = E\left|\psi\right\rangle$$

or

$$\begin{aligned}
&\left\langle\phi\right| H \left|\psi\right\rangle = \left\langle\phi\right| \tfrac{L_z^2}{2\mu a^2} \left|\psi\right\rangle = \left\langle\phi\right| E \left|\psi\right\rangle \\
&\tfrac{1}{2\mu a^2}\left(\tfrac{\hbar}{i}\tfrac{\partial}{\partial\phi}\right)^2 \left\langle\phi \mid \psi\right\rangle = E\left\langle\phi \mid \psi\right\rangle \\
&-\tfrac{\hbar^2}{2\mu a^2}\tfrac{\partial^2\psi(\phi)}{\partial\phi^2} = E\psi(\phi)
\end{aligned}$$

so that we have the solution

$$\psi(\phi) = Ae^{im\phi} \quad , \quad E = \frac{\hbar^2 m^2}{2\mu a^2}$$

Now, imposing single-valuedness, we have

$$\begin{aligned}
&\psi(\phi) = Ae^{im\phi} = \psi(\phi + 2\pi) = Ae^{im\phi}e^{i2\pi m} \\
&e^{i2\pi m} = 1 \to m = \text{ integer}
\end{aligned}$$

Since m and $-m$ give the same energy, each level is 2-fold degenerate, corresponding to rotation CW and CCW.