Astronomy, Astrophysics, and Cosmology

Luis A. Anchordoqui

Department of Physics and Astronomy
Lehman College, City University of New York

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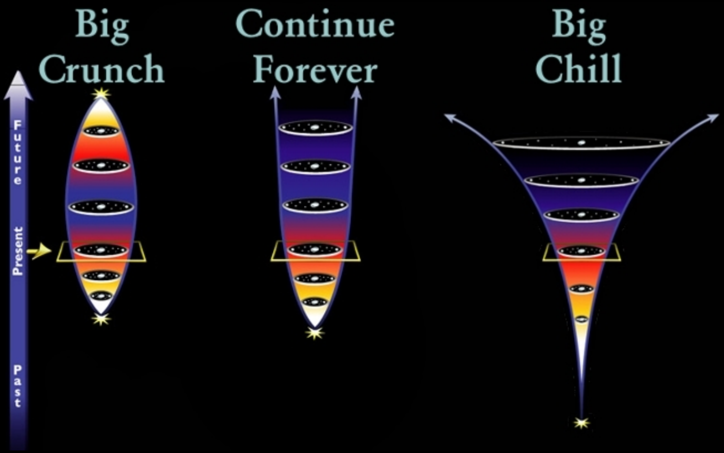
Big
Crunch
Continue
Forever
Big
Chill
Future
Present
Past

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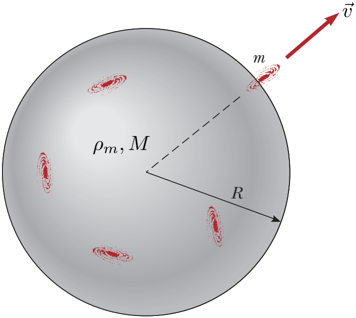
- In 1917 Einstein presented model of universe which describing geometrically symmetric (spherical) space with finite volume but no boundary
- In accordance with Cosmological Principle model is homogeneous and isotropic
- It is also static ☞ volume of space does not change
- To obtain static model Einstein introduced new repulsive force in his equations
- Size of this cosmological term is given by Λ
- Einstein presented model before redshifts of galaxies were known taking universe to be static was then reasonable
- When expansion of universe was discovered argument in favor of cosmological constant vanished
- Most recent observations seem to indicate that non-zero Λ has to be present



- Consider spherical region of galaxies
  with larger radius than distance between clusters of galaxies
  but smaller radius than any distance characterizing the universe
- Assume ☞ $\Lambda = 0$
- Mass of this sphere

$$M = \frac{4\,\pi\,R^3}{3}\,\rho_m \tag{1}$$

- Consider the motion of a galaxy at edge of spherical region

- According to Hubble's law ☞ $v = HR$
- Galaxy kinetic energy

$$K = \frac{1}{2} m v^2 = \frac{1}{2} m H^2 R^2 \tag{2}$$

- Potential energy at edge of sphere

$$U = -\frac{GMm}{R} = -\frac{4\pi m R^2 \rho_m G}{3} \tag{3}$$

- Total energy

$$E = K + U \;=\; \frac{1}{2} m H^2 R^2 - Gm \frac{4\pi}{3} R^2 \rho_m \tag{4}$$

  has to remain constant as universe expands

- Rewrite (4) as

$$\frac{2E}{mR^2} = H^2 - \frac{8\pi}{3} G \rho_m \tag{5}$$

- Since we assume that universe is homogeneous
  H and ρ_m cannot be functions of R
- LHS of
  $$\frac{2E}{mR^2} = H^2 - \frac{8\pi}{3} G\rho_m \tag{6}$$
  cannot depend on chosen distance R to coordinate center
- However ☞ $2E/(mR^2)$ is time-dependent
  because galaxy-Earth distance changes as universe expands
- Since mass of test galaxy is arbitrary
  choose $|2E/(mc^2)| = 1$ holds at an arbitrary moment (with $E \neq 0$)
- For different times ☞ LHS of (6) scales as R^{-2}

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi}{3} G\rho_m - \frac{kc^2}{a^2 R_0^2} \tag{7}$$

Friedmann equation

- Since E is constant ☞ k is constant too
- Actually ☞ $k = 0, \pm 1$ is known as curvature constant
- We account for equivalence of mass and energy
  by including not only mass but also energy density

$$\rho = \rho_m c^2 + \cdots \tag{8}$$

- Friedmann equation (without Λ) in Newtonian limit

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi}{3} G\rho_m - \frac{kc^2}{a^2 R_0^2} \tag{9}$$

becomes

$$H^2 \equiv \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi}{3} G\frac{\rho}{c^2} - \frac{kc^2}{a(t) R_0^2} \tag{10}$$

- (10) agrees exactly with the one derived from general relativity

- H_0 fixes present density ρ_0 for $k = 0$ as

$$\rho_0(k=0) \equiv \rho_c = \frac{3H_0^2 c^2}{8\pi G} \tag{11}$$

- Most cosmological quantities depend on actual value of H_0
- One “hides” this dependence by introducing h

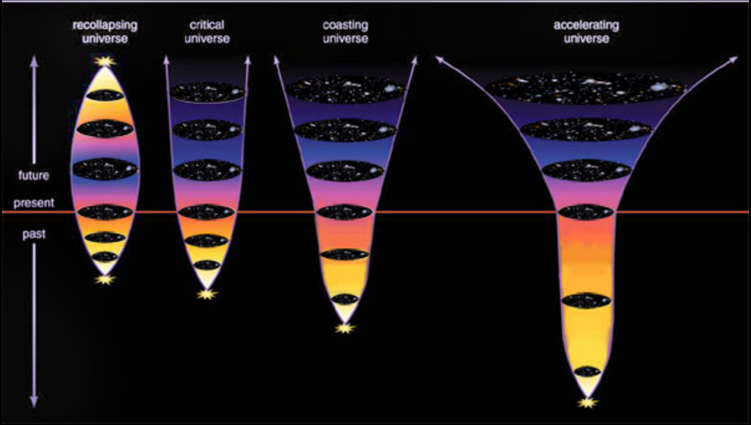
$$H_0 = 100\, h \ \text{km}\,\text{s}^{-1}\,\text{Mpc}^{-1} \tag{12}$$

- Critical density can be written in terms of Hubble parameter

$$\begin{aligned} \rho_c &= 2.77 \times 10^{11} h^2 M_\odot/\text{Mpc}^3 \\ &= 1.88 \times 10^{-29} h^2 \text{g}/\text{cm}^3 \\ &= 1.05 \times 10^{-5} h^2 \ \text{GeV}/\text{cm}^3 \end{aligned} \tag{13}$$

- $h \approx 0.7$ ☞ flat universe requires energy density ~ 10 protons/m^3

- Expansion of universe can be compared to motion of mass launched vertically from surface of celestial body
- Form of orbit depends on initial energy
- To compute complete orbit mass of main body and initial velocity have to be known
- In cosmology corresponding parameters are: ✧ mean density ✧ Hubble constant
- If density exceeds critial density expansion of any spherical region will turn to a contraction and it will collapse to a point
- This corresponds to closed Friedmann model
- If $\rho_m < \rho_c$ ☞ ever-expanding hyperbolic model is obtained
- These 3 models of universe are called standard models
- They are simplest relativistic cosmological models for $\Lambda = 0$
- Models with $\Lambda \neq 0$ are mathematically more complicated but show same behaviour



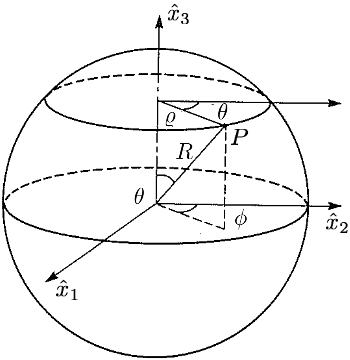
- Define abundance Ω_i of different players in cosmology as their energy density relative to ρ_c
- Examples

$$\Omega_m = \frac{\rho_m}{\rho_c} = \frac{8\pi G}{3H_0^2}\,\rho_m \quad \text{and} \quad \Omega_\Lambda = \frac{\Lambda\, c^2}{3H_0^2} \tag{14}$$

- By solving Einstein equations Robertson and Walker showed that k-hypersurfaces ☞ (hyper-sphere, hyper-plane, hyper-pseudosphere) are possible geometries for homegeneous and isotropic expanding universe
- FRW line element is most generally written in the form

$$ds^2 = c^2 dt^2 - a^2(t) \left[\frac{d\varrho^2}{1 - k\varrho^2/R^2} + \varrho^2 (d\theta^2 + \sin^2\theta d\phi^2) \right] \quad (15)$$

- Spatial component of the FRW metric ☞ consists of uniformly curved space of radius R scaled by square of scale factor $a(t)$



- Equation of S^2 of radius R

$$x_1^2 + x_2^2 + x_3^2 = R^2 \tag{16}$$

- Line element in 3-dimensional Euclidean space

$$ds^2 = dx_1^2 + dx_2^2 + dx_3^2 \tag{17}$$

- If x_3 is taken as fictitious third spatial coordinate
  it can be eliminated from ds^2 by use of (16)

$$ds^2 = dx_1^2 + dx_2^2 + \frac{(x_1 dx_1 + x_2 dx_2)^2}{R^2 - x_1^2 - x_2^2}\,. \tag{18}$$

- Introduce coordinates ϱ and θ defined in terms of x_1 and x_2 by

$$x_1 = \varrho \cos\theta \quad \text{and} \quad x_2 = \varrho \sin\theta\,. \tag{19}$$

- ϱ and θ ☞ polar coordinates in the x_3-plane ➽ $x_3^2 = R^2 - \varrho^2$
- In terms of new coordinates ☞ (18) becomes

$$ds^2 = \frac{R^2 d\varrho^2}{R^2 - \varrho^2} + \varrho^2 d\theta^2 \tag{20}$$

- For S^3 ☞ fictitious fourth spatial dimension is introduced
- In cartesian coordinates S^3 defined by

$$R^2 = x_1^2 + x_2^2 + x_3^2 + x_4^2 \quad (21)$$

- Spatial metric of 4-dimeniosnal Euclidean space

$$ds^2 = dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2 \quad (22)$$

- Fictitious coordinate can be removed to give

$$ds^2 = dx_1^2 + dx_2^2 + dx_3^2 + \frac{(x_1 dx_1 + x_2 dx_2 + x_3 dx_3)^2}{R^2 - x_1^2 - x_2^2 - x_3^2} \quad (23)$$

- In terms of coordinates

$$x_1 = \varrho \sin\theta \cos\phi \quad x_2 = \varrho \sin\theta \sin\phi \quad x_3 = \varrho \cos\theta \quad (24)$$

metric is given by spatial part of (15) with $k = 1$

- Equivalent formulas for space of constant negative curvature can be obtained with replacement $R \to iR$ in (16)

- Metric corresponding to form of (20) for negative curvature case

$$ds^2 = \frac{R^2 d\varrho^2}{R^2 + \varrho^2} + \varrho^2 d\theta^2 \qquad (25)$$

- Embedding of hyperbolic plane H^2 in Euclidean space requires three fictitious extra dimensions and such embedding is of little use in visualizing geometry

- While H^2 cannot be globally embedded in $\mathbb{R}^3$ it can be partailly represented by pseudosphere

- Embedding H^3 in Euclidean space requires four fictitious extra dimensions

- If universe had positive curvature $k = 1$
  then universe would be closed or finite in volume
- This would not mean galaxies extended out to certain boundary
  beyond which there is empty space
- There is no boundary or edge in such a universe
- If a particle were to move in straight line in particular direction
  it would eventually return to starting point
  perhaps eons of time later

- Using substitution

$$\varrho = S_k(r) = \begin{cases} R\sin(r/R) & \text{for } k = +1 \\ r & \text{for } k = 0 \\ R\sinh(r/R) & \text{for } k = -1 \end{cases} \tag{26}$$

FRW line element can be rewritten as

$$ds^2 = c^2 dt^2 - a^2(t)\left[dr^2 + S_k^2(r)\left(d\theta^2 + \sin^2\theta d\phi^2\right)\right] \tag{27}$$

- Time variable t in FRW metric
  is cosmological proper time ☞ called cosmic time for short
- t measured by observer who sees universe
  expanding uniformly around him
- Spatial variables (ϱ, θ, ϕ) or (r, θ, ϕ) are comoving coordinates
  of point in space
- If expansion of universe is homogeneous and isotropic
  comoving coordinates of any point remain constant with time

Expansion of homogeneous universe is adiabatic

- To describe time evolution of $a(t)$ ☞ need equation describing how energy content of universe is affected by expansion
- First law of thermodynamics

$$dU = TdS - PdV \tag{28}$$

with $dQ = 0$ becomes

$$dU = -PdV \Rightarrow \frac{dU}{dt} + P\frac{dV}{dt} = 0 \tag{29}$$

(no heat exchange to outside ☞ since no outside exists)

- Caveat ☞ when particles annihilate (e.g. electrons and positrons) this adds heat and makes expansion temporarily non-adiabatic
- This matters at some specific epochs in the very early universe

Fluid equation

- For sphere of comoving radius R_0

$$V = \frac{4}{3}\pi R_0^3 a^3(t) \Rightarrow \dot{V} = 4\pi R_0^3 a^2 \dot{a} = 3\frac{\dot{a}}{a}V \tag{30}$$

- Since $U = \rho V$

$$\dot{U} = \dot{\rho}V + \rho\dot{V} = V\left(\dot{\rho} + 3\frac{\dot{a}}{a}\rho\right) \tag{31}$$

- Substituting (30) and (31) into (29) we have

$$V\left(\dot{\rho} + 3\frac{\dot{a}}{a}\rho + 3\frac{\dot{a}}{a}P\right) = 0 \Rightarrow \dot{\rho} = -3(\rho + P)\frac{\dot{a}}{a} \tag{32}$$

- Expansion decreases energy density by dilution
  and by work required to expand gas with pressure $P \geq 0$

- To solve fluid equation ☞ need equation of state relating P and ρ
- Suppose we write this in form

$$P = w\rho \tag{33}$$

- In principle ☞ w could change with time
- Assume any time derivatives of w
  are negligible compared to time derivatives of ρ
- Reasonable assumption if w is determined by microphysics
  that is not directly tied to universe expansion
- Fluid equation implies

$$\frac{\dot{\rho}}{\rho} = -3(1+\omega)\frac{\dot{a}}{a} \tag{34}$$

with solution

$$\frac{\rho}{\rho_0} = \left(\frac{a}{a_0}\right)^{-3(1+\omega)} \tag{35}$$

- For non-relativistic matter (a.k.a. dust)

$$w = \frac{P}{\rho} \sim \frac{m v_{\rm th}^2}{mc^2} \sim \frac{v_{\rm th}}{c^2} \ll 1 \tag{36}$$

$v_{\rm th}$ ☞ thermal velocity of particles of mass m

- To near-perfect approximation $w = 0$ ☞ $\rho_m \propto a^{-3}$

- Light (or more generally any highly relativistic particle) has associated pressure ☞ radiation pressure

- For radition $w = 1/3$ ☞ $\rho_{\rm rad} \propto a^{-4}$

- This behavior also follows from a simple argument: number density of photons falls as $n \propto a^{-3}$ and energy per photon falls as $h\nu \propto a^{-1}$ because of cosmological redshift

Acceleration equation

- Multiply Friedmann equation by a^2

$$\dot{a}^2 = \frac{8\pi G}{3c^2}\rho a^2 - \frac{kc^2}{R_0^2} \tag{37}$$

- Take time derivative

$$2\dot{a}\ddot{a} = \frac{8\pi G}{3c^2}\left(\dot{\rho}a^2 + 2\rho a\dot{a}\right) \tag{38}$$

- Divide by $2\dot{a}a$ ☞

$$\frac{\ddot{a}}{a} = \frac{4\pi G}{3c^2}\left(\dot{\rho}\frac{a}{\dot{a}} + 2\rho\right) \tag{39}$$

- Substitutte from fluid equation

$$\dot{\rho}\frac{a}{\dot{a}} = -3(\rho + P) \Rightarrow \frac{\ddot{a}}{a} = -\frac{4\pi G}{3c^2}(\rho + 3P) \tag{40}$$

- If ρ and P are positive ☞ expansion of the universe decelerates

Example

- Flat universe ☞ $k = 0$
- For non-relativistic matter ☞ solution to Friedmann equation is

$$a(t) = \left(\frac{t}{t_0}\right)^{2/3} \quad \text{and} \quad \rho(t) = \frac{\rho_0}{a^3} = \frac{\rho_0 t_0^2}{t^2} \quad \text{with} \quad t_0 = \frac{2}{3}\frac{1}{H_0} \tag{41}$$

- Bizarre universe dominated today by radiation pressure ☞ yields

$$a(t) = \left(\frac{t}{t_0}\right)^{1/2} \quad \text{and} \quad \rho(t) = \frac{\rho_0}{a^4} = \frac{\rho_0 t_0^2}{t^2} \tag{42}$$

- We can picture time evolution of universe as follows
  - Early universe ☞ relativistic matter (radiation pressure dominates) $a(t) \propto t^{1/2}$, $\rho_{\rm rad} \propto t^{-2}$, $\rho_m \propto a^{-3} \propto t^{-3/2}$
  - Density of radiation falls more quickly than that of dust
  - When dust dominates ☞ $a(t) \propto t^{2/3}$, $\rho_m \propto t^{-2}$, $\rho_{\rm rad} \propto a^{-4} \propto t^{8/3}$ dust domination increases

- how to measure distances in FRW spacetime?
- Consider galaxy which is far away from us ☞ sufficiently far away that we may ignore small scale perturbations of spacetime and adopt FRW line element
- In expanding universe distance between two objects is increasing with time
- If we want to assign spatial distance between two objects we must specify t at which distance is the correct one
- Suppose that you are at the origin and galaxy which you are observing is at comoving coordinate position (r, θ, ϕ)
- Proper distance $d_{\rm p}(t)$ between two points equals length of spatial geodesic between them when scale factor is fixed at value $a(t)$

- Proper distance between the observer and galaxy can be found using FRW metric at fixed time t

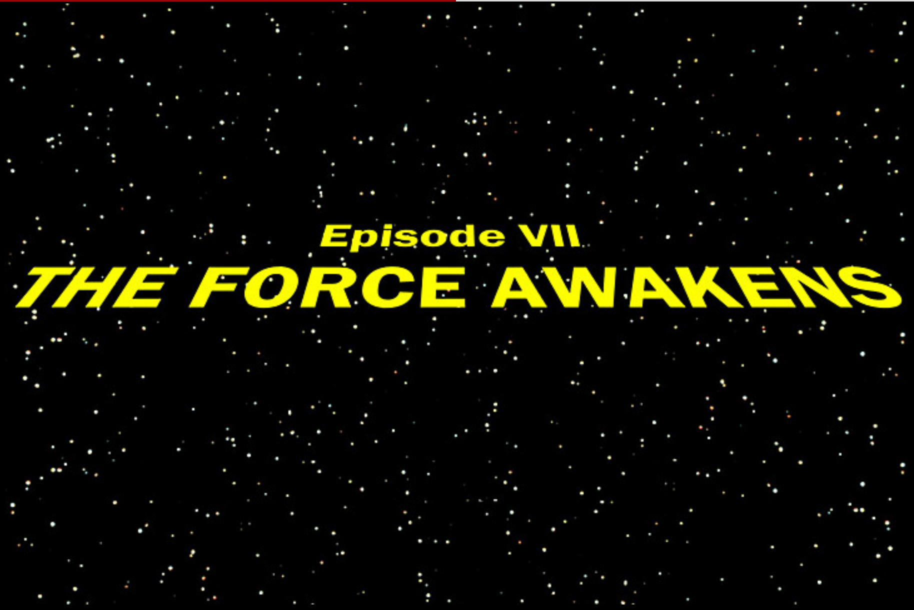
$$ds^2 = a^2(t)\ \left[dr^2 + S_k^2(r)\ \left(d\theta^2 + \sin^2\theta d\phi^2\right)\right] \tag{43}$$

- Along spatial geodesic between the observer and galaxy angle (θ, ϕ) is constant

$$ds = a(t)\ dr \tag{44}$$

- Proper distance $d_{\rm p}$ is found by integrating over r

$$d_{\rm p} = a(t) \int_0^r dr = a(t)\ r \tag{45}$$



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