Astronomy, Astrophysics, and Cosmology

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Lesson IV
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The hunter and the bear

- A hunter is tracking a bear
- Starting at his camp ☞ he walks one mile due south
- The bear changes direction and the hunter follows it due east
- After one mile ☞ the hunter loses the bear's track
- He turns north and walks for another mile
  at which point he arrives back at his camp
- What was the color of the bear?

✧ This certainly does not work everywhere on Earth

but it does if you start at North pole

✧ Therefore ☞ color of bear has to be white

✧ Surprisingly ☞ sum of all three angles is greater than 180°

✧ This implies space is curved

- To understand the idea of curved metric space
  we'll simplify discussion considering only 2-dimensional surfaces
- Parameterization of surface maps points (u, v) in domain
  to points $\vec{\sigma}(u, v)$ in space

$$\vec{\sigma}(u,v) = \begin{pmatrix} x(u,v) \\ y(u,v) \\ z(u,v) \end{pmatrix} \tag{1}$$

- First derivatives ☞ $\vec{\sigma}_u(u, v)$ and $\vec{\sigma}_v(u, v)$
  span tangent plane to surface at point $\vec{\sigma}(u, v)$
- Surface normal $\hat{n}$ at point $\vec{\sigma}$

$$\hat{n}(\vec{\sigma}) = \frac{\vec{\sigma}_u \times \vec{\sigma}_v}{||\vec{\sigma}_u \times \vec{\sigma}_v||} \tag{2}$$

- Tangent vectors and surface normal
  define orthogonal coordinate system at point $\vec{\sigma}(u, v)$
  which is framework for describing local shape of surface

- First fundamental form

$$\begin{aligned} \mathrm{I} &\equiv ds^2 = d\vec{\sigma}\cdot d\vec{\sigma} = (\vec{\sigma}_u du + \vec{\sigma}_v dv)\cdot(\vec{\sigma}_u du + \vec{\sigma}_v dv) \\ &= (\vec{\sigma}_u\cdot\vec{\sigma}_u)du^2 + 2(\vec{\sigma}_u\cdot\vec{\sigma}_v)du dv + (\vec{\sigma}_v\cdot\vec{\sigma}_v)dv^2 \\ &= E du^2 + 2F du dv + G dv^2 \end{aligned} \tag{3}$$

☞ distance of neighboring points on surface
with parameters (u, v) and $(u + du, v + dv)$

- Area bounded by vertices
$\vec{\sigma}(u, v), \vec{\sigma}(u + \delta u, v), \vec{\sigma}(u, v + \delta v), \vec{\sigma}(u + \delta u, v + \delta v)$

$$\delta A = |\vec{\sigma}_u\, \delta u \times \vec{\sigma}_v\, \delta v| = \sqrt{EG - F^2}\, \delta u\, \delta v \tag{4}$$

- In differential form

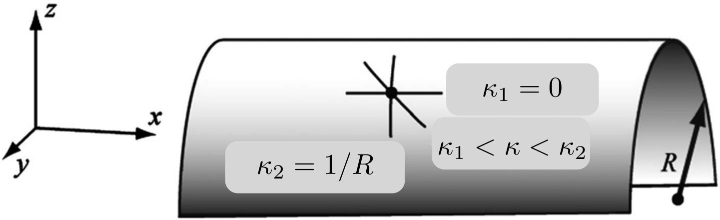
$$dA = \sqrt{EG - F^2}\, du\, dv \tag{5}$$

- Expression under square root ☞ $|\vec{\sigma}_u \times \vec{\sigma}_v|$
and so it is strictly positive at regular points

Gaussian curvature

- At any point on $\vec{\sigma}$ we can find $\hat{n}$
- Planes containing normal vector are called normal planes
- Intersection of normal plane and $\vec{\sigma}$ forms curve ☞ normal section
- Curvature of normal section ☞ normal curvature
- For most points on most surfaces
  different sections will have different curvatures
- Maximum (κ_2) and minimum (κ_1) values ☞ principal curvatures
- Gaussian curvature ☞ $K = \kappa_1 \kappa_2$

E.g. ☞



- Second fundamental form

$$\begin{aligned} \mathrm{II} &= (\vec{\sigma}_{uu} \cdot \hat{n})\, du^2 + 2(\vec{\sigma}_{uv} \cdot \hat{n})\, du dv + (\vec{\sigma}_{vv} \cdot \hat{n})\, dv^2 \\ &= e\, du^2 + 2f\, du\, dv + g\, dv^2 \end{aligned} \quad (6)$$

- K ☞ calculated using first and second fundamental coefficients
- At each grid point ☞ 2 matrices are defined
- Matrix of first fundamental form

$$\mathrm{I} = \begin{pmatrix} E & F \\ F & G \end{pmatrix} \quad (7)$$

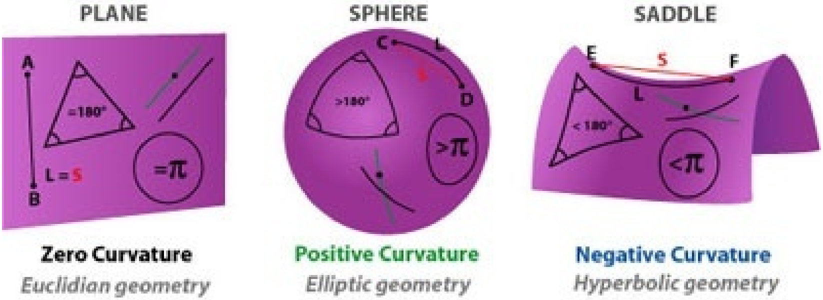
- Matrix of second fundamental form

$$\mathrm{II} = \begin{pmatrix} e & f \\ f & g \end{pmatrix} \quad (8)$$

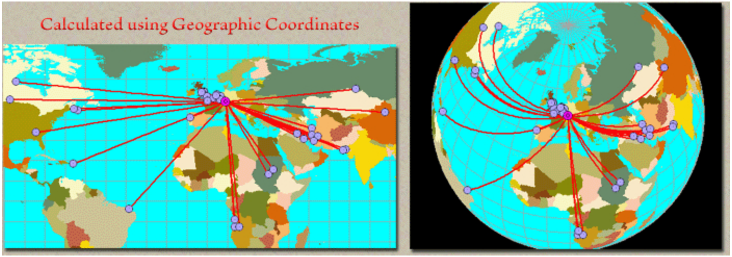
- Gaussian curvature

$$K = \frac{\det \mathrm{II}}{\det \mathrm{I}} \quad (9)$$

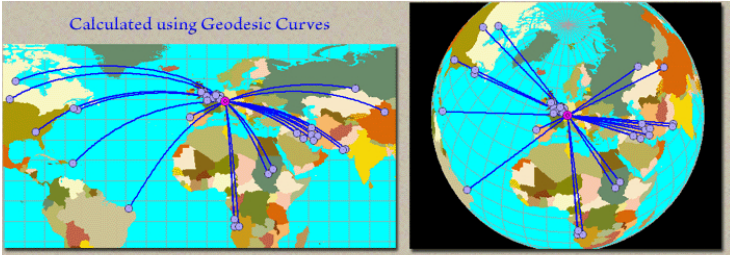
Geometry classification according to Gaussian curvature



Gedesic ☞ curve $\gamma(t)$ on surface $\vec{\sigma}(u,v)$
for which at every point $\ddot{\gamma}(t)$ is either 0 or $\parallel$ to $\hat{n}$



Straight line distances are not shortest route on Earth



Ricci sacalar

- Scalar curvature R
  simplest curvature invariant of n-dimensional hypersurface
- To each point on hypersurface
  assigns number determined by intrinsic geometry near that point
- R measures degree to which given metric
  might differ from that of ordinary Euclidean n-space
- In two dimensions ☞ $R = 2K$
  and completely characterizes curvature of surface
- In more than two dimensions ☞ curvature of hypersurfaces
  involves more than one functionally independent quantity

- Consider freely falling spacecraft in gravitational field
  of radially symmetric mass distribution with total mass M
- Because spacecraft is freely falling
  no effects of gravity are felt inside
- Spacetime coordinates from $r \to \infty$ ☞ valid inside spacecraft
  $\vec{\Sigma}_\infty(t_\infty, x_\infty, y_\infty, z_\infty)$ ☞ $x_\infty \parallel$ to movement & $y_\infty, z_\infty \perp$ to movement
- Spacecraft has velocity v at the distance r from mass M
  measured in system $\vec{\Sigma} = (r, \theta, \phi, t)$ with mass M at rest @ $r = 0$
- As long as the *gravitational field is weak*
  to first order approximation laws of special relativity hold
  Lorentz transformation relates $\vec{\Sigma}$ @ rest and $\vec{\Sigma}_\infty$ moving with β

- We'll define shortly what "weak" means in this context
- For the moment ☞ assume effects of gravity are small $\to v \ll c$
- If this were case

$$\begin{aligned} dt_\infty &= dt\sqrt{1-\beta^2} \\ dx_\infty &= \frac{dr}{\sqrt{1-\beta^2}} \\ dy_\infty &= r d\theta \\ dz_\infty &= r\sin\theta d\phi \end{aligned} \tag{10}$$

- Infinitesimal distance between two spacetime events

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = c^2 dt_\infty^2 - dx_\infty^2 - dy_\infty^2 - dz_\infty^2 \tag{11}$$

for case at hand

$$ds^2 = (1-\beta^2) c^2 dt^2 - \frac{dr^2}{1-\beta^2} + r^2 (d\theta^2 + \sin^2 d\phi^2) \tag{12}$$

- Consider energy of spacecraft with rest mass m

$$(\gamma - 1)mc^2 - \frac{G\gamma m M}{r} = 0 \tag{13}$$

- Dividing by γmc^2

$$\left(1 - \frac{1}{\gamma}\right) - \frac{GM}{rc^2} = 0 \tag{14}$$

- Introducing $\alpha = GM/c^2$

$$\sqrt{1 - \beta^2} = 1 - \frac{\alpha}{r} \tag{15}$$

yielding

$$1 - \beta^2 = 1 - \frac{2\alpha}{r} + \frac{\alpha^2}{r^2} \approx 1 - \frac{2\alpha}{r} \tag{16}$$

- Schwarzschild line element

$$ds^2 = \left(1 - \frac{2\alpha}{r}\right) c^2 dt^2 - \left(1 - \frac{2\alpha}{r}\right)^{-1} dr^2 - r^2 d\Omega^2 \tag{17}$$

- ds^2 ☞ time and spatial distance between two spacetime events
- t measured by observer in instantaneous rest frame ☞ $d\tau = ds/c$
- Δt between two events at same point ☞ $dx^i = 0$
- Consider two static observers at position r and r' in Sch. metric then ☞ $dr = d\phi = d\theta = 0$

$$\frac{d\tau(r)}{d\tau(r')} = \frac{\sqrt{g_{00}(r)}dt}{\sqrt{g_{00}(r')}dt} = \sqrt{\frac{g_{00}(r)}{g_{00}(r')}} \tag{18}$$

- Time intervals $d\tau(r')$ and $d\tau(r)$ are different
  time measured by clocks at different r from M will differ too
- In particular ☞ time τ_∞ measured by observer at infinity
  will pass faster than time experienced in gravitational field

$$\tau_\infty = \frac{\tau(r)}{\sqrt{1 - 2\alpha/r}} < \tau(r) \tag{19}$$

Gravitational redshift

- Since frequencies are inversely proportional to time
  frequency of photon traveling from r to r'
  will be affected by gravitational field

$$\frac{\nu(r')}{\nu(r)} = \sqrt{\frac{1-2\alpha/r}{1-2\alpha r'}} \qquad (20)$$

- Observer @ $r' \to \infty$ will receive photons emitted @ r with ν

$$\nu_\infty = \sqrt{1-\frac{2GM}{rc^2}}\,\nu(r) \qquad (21)$$

- Photon frequency is redshifted by gravitational field
- Size of effect is of order Φ/c^2
  $\Phi = -GM/r$ ☞ Newtonian gravitational potential
- Weak gravitational field ☞ As long as $|\Phi|/c^2 \ll 1$ deviation of

$$g_{00} = 1-\frac{2GM}{rc^2} \approx 1-2\frac{\Phi(r)}{c^2} \qquad (22)$$

  from $g_{00} = 1$ of Minkowski spacetime is small
  and Newtonian gravity is sufficient approximation

Event horizon

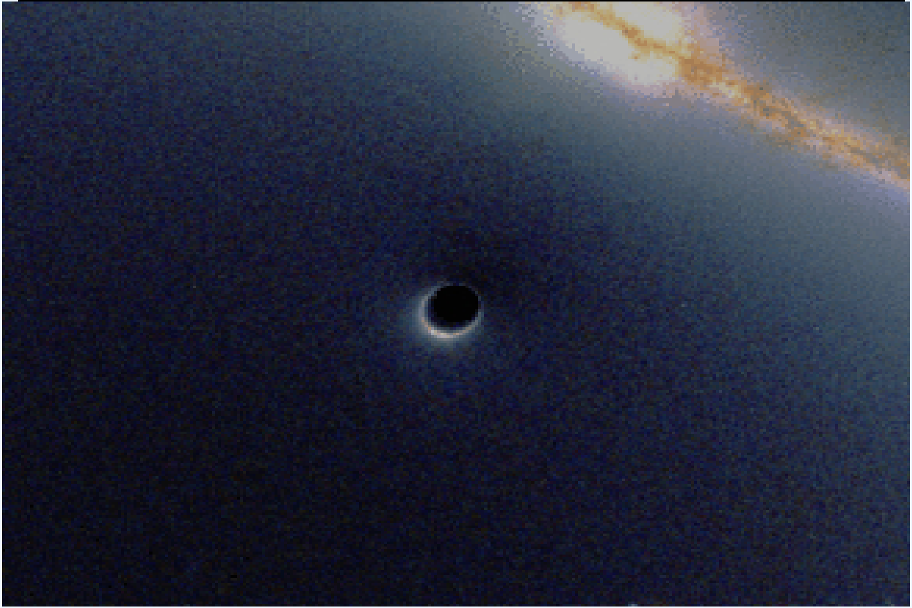
- What is the meaning of $r = R_{\rm Sch} = 2\alpha$?
- At
$$R_{\rm Sch} = \frac{2GM}{c^2} = 3\ {\rm km}\frac{M}{M_\odot} \qquad (23)$$
Schwarzschild oordinate system becomes ill-defined
- Apparent singularity
  physical quantities like tidal forces doesn't become infinite
- Whether or not singularity is moved to origin
  only depends on coordinate frame used
  and has no physical significance whatsoever
- If gravitating mass is concentrated inside radius smaller than $R_{\rm Sch}$ we can't obtain any information about what is going on inside $R_{\rm Sch}$
- $r = R_{\rm Sch}$ ☞ defines event horizon
- Black hole ☞ object smaller than its Schwarzschild radius
  object that has "cut itself off" from rest of universe

- Light rays are characterized by $ds^2 = 0$
- Consider light ray traveling in radial direction ☞ $d\phi = d\theta = 0$

$$\frac{dr}{dt} = \left(1 - \frac{2\alpha}{r}\right) c \tag{24}$$

- As seen from far away ☞ light ray approaching a massive star will travel slower and slower as it comes closer to $R_{\rm Sch}$
- For observer at infinity signal will reach $r = R_{\rm Sch}$ only asymptotically for $t \to \infty$
- Factors $(1 - 2\alpha/r)$ in line element control bending of light phenomenon known as gravitational lensing
- First observation of light deflection change in position of stars as they passed near the Sun
- Observations performed in May 1919 during total solar eclipse

Animated simulation of gravitational lensing caused by a black hole



- Binary X-ray sources ☞ places to find black hole candidates
- Companion star ☞ source of infalling material for black hole
- As matter falls or is pulled towards black hole
  it gains kinetic energy, heats up, and is squeezed by tidal forces
- Heating ionizes atoms ☞ atoms emit X-rays when $T > 10^6$ K
- X-rays are sent off into space before the matter crosses $R_{\rm Sch}$
- Then ☞ we can see this X-ray emission
- Cygnus X-1 first black hole candidate
- There is natural limit to luminosity L
  that can be radiated by compact object of mass M
- Limit arises because both:
  (i) attractive gravitational force acting on an electron-ion pair
  (ii) repulsive force due to radiation pressure
  decrease inversely with square of distance from black hole
- When luminosity exceeds Eddington limit

$$L_{\rm Edd} = 30,000 (M/M_\odot) L_\odot \tag{25}$$

gas will be blown away by radiation

Active galactic nuclei

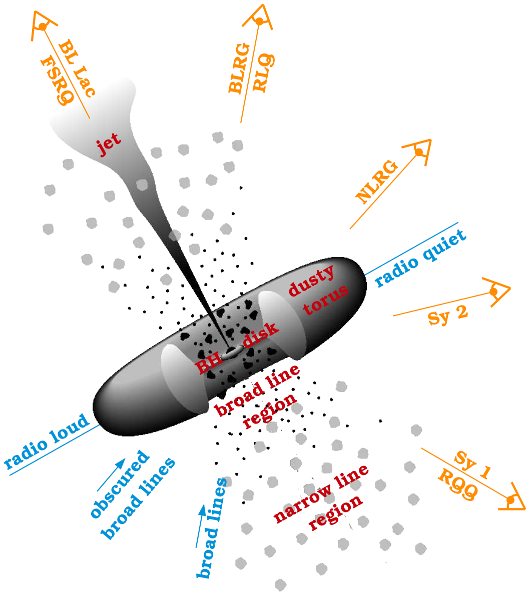
- AGNs are galaxies that harbor compact masses at center
  exhibiting intense non-thermal emission that is often variable
  which indicates small sizes (light months to light years)

- Under favorable conditions
  accretion leads to formation of highly relativistic collimated jet

- Formation of the jet is not well constrained ☞ overall:
  - magnetic-field-dominated near central engine
  - particle dominated beyond pc distances

- AGN taxonomy ☞ controlled by
  dichotomy between radio-quiet and radio-loud classes

AGN taxonomy



Kruskal coordinates

- Elegant coordinate substitution ☞ $xy = \left(\frac{r}{2M} - 1\right) e^{r/(2M)}$ and $x/y = e^{t/(2M)}$
- Schwarzschild line element becomes

$$ds^2 = -16M^2 \left(1 - \frac{2M}{r}\right) \frac{dx}{x}\frac{dy}{y} - r^2 d\Omega^2 = -\frac{32M^3}{r} e^{-r/(2M)}\, dx\, dy - r^2 d\Omega^2$$

- Zero and pole at $r = 2M$ have cancelled out