Physics 307

Mathematical Physics

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ORDINARY DIFFERENTIAL EQUATIONS V

3.1 Setting the Stage ✓

3.2 Initial Value Problem ✓

Picard's existence and uniqueness theorem

Systems of first-order linear differential equations

Green matrix as a generalized function

3.3 Boundary Value Problem ✓

Self-adjointness of Sturm-Liouville operator

Green function of Sturm-Liouville operator

Series solutions to homogeneous linear equations

3.4 Fourier Analysis

Fourier series

Fourier transform

Fourier Analysis

Consider Sturm-Liouville eigenvalue problem $L = -d^2/dx^2\,, \rho = 1$

$$f(-\pi) = 0 \quad \text{and} \quad f(\pi) = 0 \qquad (3.4.316.)$$

eigenfunctions are ☛ $b_n \sin(nx)$ with $n \in \mathbb{Z}$

Sturm-Liouville theorem states

if $f(x)$ is of class $\mathcal{C}^2[-\pi, \pi]$ and satisfies (3.4.316.)

it can be expanded in a convergent series

$$f(x) = \sum_{n=1}^{\infty} b_n \ \sin(nx)$$

if we replace boundary conditions ☛ $f'(-\pi) \quad \text{and} \quad f'(\pi) = 0$

We are now speaking of quite a different Sturm-Liouville system

in this space eigenfunctions are ☛ $a_n \cos(nx)$ with $n \in \mathbb{Z}$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx)$$

Fourier series

Central idea of todays class

investigate to what extent it is possible

to expand $f(x) \notin \mathcal{C}^2[-\pi, \pi]$ in an infinite series

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}[a_n \cos(nx) + b_n \sin(nx)] \qquad (3.4.319.)$$

1st order of business is to determine coefficients a_n and b_n

only then will we deal with convergence issues

Definition 3.4.1.

Assuming expansion (3.4.319.) holds we can determine a_n and b_n

using orthogonality (and norm) of $\sin(nx)$ and $\cos(nx)$

integrating on both sides of (3.4.319.) over $[-\pi, \pi]$ we obtain

$$\int_{-\pi}^{\pi} f(x)\, dx = \pi a_0 \qquad (3.4.320.)$$

note that $$\frac{a_0}{2} = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\, dx \qquad (3.4.321.)$$

is average value of $f(x)$ in the interval $[-\pi, \pi]$

To calculate a_n with $n \neq 0$

multiply both sides of (3.4.319) by $\cos(kx), k \neq 0$ and integrate

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\, \cos(kx)\, dx, \quad \text{with } k = 0, 1, 2, 3, \ldots, \infty \qquad (3.4.322.)$$

Similarly ☛ multiplying by $\sin(kx)$ and integrating

$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\, \sin(kx)\, dx \quad \text{with } k = 1, 2, 3, \ldots, \infty \qquad (3.4.323.)$$

Coefficients so determined are called Fourier coefficients

Associated Fourier series is given by

$$f(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx + \frac{1}{\pi}\sum_{n=1}^{\infty}\int_{-\pi}^{\pi} dx' f(x')\,[\cos(nx)\cos(nx') + \sin(nx)\sin(nx')] \quad (3.4.324.)$$

which can be written in a more compact form as

$$f(x) = \langle f\rangle + \frac{1}{\pi}\sum_{n=1}^{\infty}\int_{-\pi}^{\pi} dx'\; f(x')\;\cos[n(x-x')] \quad (3.4.325.)$$

The interval $[-\pi,\pi]$ was chosen rather arbitrarily

later on ☛ we will consider other intervals

Corollary 3.4.1.

Since $\begin{matrix}\cos(nx)\\ i\sin(nx)\end{matrix} = \frac{1}{2}\left(e^{inx} \pm e^{-inx}\right)$ we can rewrite (3.4.319.) as a power series in e^{ix}

$$f(x) = c_0 + \sum_{n=1}^{\infty} c_n e^{inx} + c_n^* e^{-inx} = \lim_{m\to\infty}\sum_{n=-m}^{m} c_n e^{inx} \quad (3.4.326.)$$

with $c_n = c_{-n}^* = \frac{1}{2}(a_n - ib_n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}\,dx$ (3.4.327.)

Definition 3.4.2.

$f(x)$ is said to be piecewise continuous on an interval $[a, b]$

if it is defined and continuous

except possibly at a finite number of points x_k

such that at each point of discontinuity left- and right-hand limits

$$f(x_k^-) = \lim_{x \to x_k^-} f(x), \qquad f(x_k^+) = \lim_{x \to x_k^+} f(x) \quad (3.4.328.)$$

exist

At endpoints a, b only require limits $f(a^+)$ and $f(b^-)$ to exist

Note that we do not require that $f(x)$ be defined at x_k

Even if $f(x_k)$ is defined

it does not necessarily equal either left- or right-hand limit

A function $f(x)$ defined for all $x \in \mathbb{R}$ is piecewise continuous provided it is piecewise continuous on every bounded interval

Points x_k are known as jump discontinuities of $f(x)$

difference between + and - limits is magnitude of jump

Definition 3.4.3.

A function $f(x)$ is called piecewise C^1 on an interval $[a,b]$ if it is defined continuous and continuously differentiable except possibly at a finite number of points x_k

such that at each exceptional point left- and right-hand limits

$$f(x_k^-) = \lim_{x \to x_k^-} f(x)\,, \qquad f(x_k^+) = \lim_{x \to x_k^+} f(x)\,,$$

$$f'(x_k^-) = \lim_{x \to x_k^-} f'(x)\,, \qquad f'(x_k^+) = \lim_{x \to x_k^+} f'(x)$$

exist

at endpoints we only require appropriate one-sided limits to exist

$$f(a^+), f(b^-), f'(a^+), f'(b^-)$$

For a piecewise continuous C^1 function an exceptional point x_k is either:

(i) a jump discontinuity of f but where left- and right-hand derivatives exist

(ii) a corner

meaning a point where f is continuous ☛ so $f(x_k^-) = f(x_k^+)$

but has different left- and right-hand derivatives $f'(x_k^-) \neq f'(x_k^+)$

Lemma 3.4.1. [Riemann-Lebesgue lemma]

If $g(x)$ is piecewise continuous in interval $[a, b]$

then $$\lim_{s\to\infty} \int_a^b g(x)\ \sin(sx + \alpha)\ dx = 0 \qquad (3.4.329.)$$

Proof.

If $g(x) \in \mathcal{C}^1([a, b])$ ☛ then integration by parts leads to

$$\int_a^b g(x)\ \sin(sx+\alpha)\ dx = -\int_a^b g(x) \frac{d}{dx}\left[\frac{\cos(sx+\alpha)}{s}\right]\ dx$$

$$= -g(x)\ \frac{\cos(sx+\alpha)}{s}\bigg|_a^b + \int_a^b g'(x) \frac{\cos(sx+\alpha)}{s}\ dx$$

which goes to zero as $s \to \infty$

same reasoning is valid if $g(x)$ is differentiable

except perhaps at a finite number of points in interval

where one-sided directional derivatives must exist and be finite

Next ☛ we discuss case in which $g(x)$ is continuous in $[a,b]$

Consider a partition of closed interval $[a,b], x_0 = a, x_1, \ldots, x_n = b$ with $x_i - x_{i-1} = (b-a)/n$

It follows that

$$
\begin{aligned}
\left|\int_a^b g(x)\sin(sx+\alpha)\ dx\right| &= \left|\sum_{i=1}^{n}\int_{x_{i-1}}^{x_i} g(x)\ \sin(sx+\alpha)\ dx\right| \\
&= \left|\sum_{i=1}^{n}\left\{g(x_i)\int_{x_{i-1}}^{x_i} \sin(sx+\alpha)\ dx\right.\right. \\
&+ \left.\left.\int_{x_{i-1}}^{x_i}[g(x)-g(x_i)]\ \sin(sx+\alpha)\ dx\right\}\right| \\
&\leq \sum_{i=1}^{n}\left[|g(x_i)|\frac{|\cos(sx_i+\alpha)-\cos(sx_{i-1}+\alpha)|}{s}\right. \\
&+ \left.\frac{m_i(b-a)}{n}\right] \\
&\leq \frac{2Mn}{s} + M_n\ (b-a)
\end{aligned}
$$

(3.4.330.)

where $\begin{cases} M \text{ is maximum value of } g \text{ in } [a,b] \\ m_i \text{ is maximum value of } |g(x) - g(x_i)| \text{ in } [x_{i-1}, x_i] \\ M_n \text{ is maximum value of } m_i \end{cases}$

With this in mind,

$$\lim_{s\to\infty} \left| \int_a^b g(x) \ \sin(sx) \ dx \right| \leq M_n(b-a) \qquad (3.4.329.)$$

Note that $\lim_{n\to\infty} M_n = 0$ ☛ because g is continuous and so M_n can be as small as desired by increasing n

Same applies if we replace $\sin(sx+\alpha)$ by $\cos(sx+\alpha)$

This proves lemma even if $g(x)$ is not differentiable at any point

If g is piecewise continuous we can separate jump discontinuity points x_c through integrals of form $\int_{x_c-\epsilon}^{x_c+\epsilon} g(x)\sin(sx+\alpha)dx$ which go to zero for $\epsilon \to 0$ ☛ because g is bounded

We repeat previous reasoning in remaining intervals where g is continuous

Definition 3.4.3. ++

A function is said to have period $2L$ if $f(x+2L) = f(x)\ \forall x$

For notational simplicity

we shall restrict our discussion to functions of period 2π

There is no loss of generality in doing so

since we can always use a simple change of scale $x = Ly/\pi$

to convert a function of period $2L$ into one of period 2π

the rescaled function $F(y) = f(Ly/\pi)$ lives on $[-\pi, \pi]$

Theorem 3.4.1. [Fourier convergence theorem]

If $f(x)$ is any 2π periodic piecewise C^1 function

then ☛ for any $x \in \mathbb{R}$ its Fourier series converges to:

$$\begin{cases} f(x) & \text{if } f \text{ is continuous at } x \\ \frac{1}{2}[f(x^+) + f(x^-)] & \text{if } x \text{ is a jump discontinuity} \end{cases} \qquad (3.4.330.)$$

Fourier series converges to $f(x)$ at all points of continuity

At discontinuities ☛ Fourier series cannot decide

whether to converge to right- or left-hand limit

and so ends up splitting difference by converging to their average

Proof.

Part I Let $f(x)$ be a differentiable function (therefore continuous) in interval $[a, b]$

We will show that identity (3.4.325.) is valid $\forall x \in (-\pi, \pi)$

If in addition ☛ $f(-\pi) = f(\pi)$

we will show that series converges to $f(x)$ in $[-\pi, \pi]$

n -th partial sum of (3.4.325.) is given by

$$S_n(x) = \frac{1}{2}a_0 + \sum_{m=1}^{n} a_m \cos(mx) + b_m \sin(mx)$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\,dt \left\{\frac{1}{2} + \sum_{m=1}^{\infty} \cos(mx)\cos(mt) + \sin(mx)\sin(mt)\right\}$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t) \left\{\frac{1}{2} + \sum_{m=1}^{n} \cos[(m(t-x))]\right\}\, dt$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\; K_n(t-x)\; dt \qquad (3.4.331.)$$

with ☛ $$K_n(s) = \frac{1}{2} + \sum_{m=1}^{n} \cos(ms) \qquad (3.4.332.)$$

We multiply both sides of (3.4.332.) by $\sin(s/2)$

and use relation $2\sin\alpha\ \cos\beta = \sin(\alpha+\beta) + \sin(\alpha-\beta)$ to obtain

$$\sin\left(\frac{s}{2}\right)\ K_n(s) = \frac{1}{2}\left\{\sin\left(\tfrac{s}{2}\right) + \sum_{k=1}^{n}\left[\sin\left(k+\tfrac{1}{2}\right)s\right] - \sin\left[\left(k-\tfrac{1}{2}\right)s\right]\right\}$$

one recognizes a telescopic sum

☛ all terms except $\sin\left[\left(n+\frac{1}{2}\right)s\right]$ cancel

Therefore ☛ $$K_n(s) = \frac{\sin\left[\left(n+\frac{1}{2}\right)s\right]}{2\sin\left(s/2\right)} \qquad (3.4.333.)$$

Note that $$K_n(2k\pi) = \lim_{s\to 2k\pi} K_n(s) = n + \tfrac{1}{2} \qquad k\in\mathbb{Z} \qquad (3.4.334.)$$

it is easily seen from sum of cosines that defines $K_n(s)$ that ↩

$$\int_{-\pi}^{\pi} K_n(t-x)\ dt = \pi \qquad (3.4.335.)$$

Therefore ☛ by adding and subtracting $f(x)$ to (3.4.331.) we have

$$S_n(x) = f(x) + \frac{1}{\pi}\int_{-\pi}^{\pi} [f(t) - f(x)]\; K_n(t-x)\, dt \qquad (3.4.336.)$$

$$= f(x) + \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{f(t)-f(x)}{2\sin[(t-x)/2]} \sin\left[\left(n+\tfrac{1}{2}\right)(t-x)\right] dt$$

For $t \neq x$ ☛ the function

$$g(t) = \frac{f(t)-f(x)}{2\sin[(t-x)/2]} \qquad (3.4.337.)$$

is continuous $\forall t \in [-\pi, \pi]$

Since $f(x) \in \mathcal{C}^1[-\pi,\pi]$ we have

$$g(x) = \lim_{t\to x} \frac{f(t)-f(x)}{2\sin[(t-x)/2]} = f'(x) \qquad (3.4.338.)$$

and so integral in (3.4.336.) vanishes for $n \to \infty$

yielding ☛ $\lim_{n\to\infty} S_n(x) = f(x)\ \forall x \in (-\pi, \pi)$ (3.4.339.)

If $x = \pm\pi$ ☛ the denominator of $g(t)$ is canceled for both $t \to \pi$ and $t \to -\pi$

However ☛ if $x = \pm\pi, K_n(t-x)$ is an even function of t

$$
\begin{aligned}
K_n(t \mp \pi) &= \frac{\sin\left[\left(n+\frac{1}{2}\right)(t \mp \pi)\right]}{2\sin[(t \mp \pi)/2]} \\
&= \frac{\sin\left[\left(n+\frac{1}{2}\right)(-t \pm \pi)\right]}{2\sin[(-t \pm \pi)/2]} \\
&= \frac{\sin\left[\left(n+\frac{1}{2}\right)(-t \mp \pi)\right]}{2\sin[(-t \mp \pi)/2]} \qquad (3.4.340.)
\end{aligned}
$$

Thus ☛ $\displaystyle\int_{-\pi}^{0} K_n(t-x)\,dt = \int_{0}^{\pi} K_n(t-x)\,dt = \frac{1}{2}\,\pi$ (3.4.341.)

For $x = \pm\pi$ we can then write

$$S_n(x) = \frac{f(\pi)+f(-\pi)}{2} + \frac{1}{\pi}\int_0^{\pi} \frac{f(t)-f(\pi)}{2\sin[(t-x)/2]} \sin\left[(n+\tfrac{1}{2})(t-x)\right]\, dt$$
$$+ \frac{1}{\pi}\int_{-\pi}^{0} \frac{f(t)-f(-\pi)}{2\sin[(t-x)/2]} \sin\left[(n+\tfrac{1}{2})(t-x)\right]\, dt \qquad (3.4.342.)$$

with ☛ $$\lim_{t\to\pm\pi} \frac{f(t)-f(\pm\pi)}{2\sin[(t-x)/2]} = f'(\pm\pi) \qquad (3.4.343.)$$

Applying Riemann-Lebesgue lemma to (3.4.342.) we obtain

$$\lim_{n\to\infty} S_n(\pm\pi) = \frac{f(\pi)+f(-\pi)}{2} \qquad (3.4.344.)$$

which gives desired result ☛ if $f(\pi) = f(-\pi)$

Part II Let $f(x)$ be a 2π periodic piecewise C^1 function

We will show that (3.4.330.) holds for any $x \in \mathbb{R}$

If x is a point of differentiability of function f

we repeat previous reasoning

If x_c is corner ☛ function $g(t)$ as given in (3.4.331.)

is continuous for $t \neq x$ ☛ and remains bounded for $t \to x$

satisfying hypotheses of Riemann-Lebesgue lemma

Therefore ☛ Fourier series also converges to f

If function f has a jump at point x

we can use (3.3.341.) and 2π periodicity of kernel K_n to write

$$S_n(x) - \frac{f(x^+) + f(x^-)}{2} = \frac{1}{2}\int_{x-\pi}^{x} K_n(x-t)[f(t) - f(x^-)]\, dt + \frac{1}{2}\int_{x}^{x+\pi} K_n(x-t)[f(t) - f(x^+)]\, dt$$

By a similar argument to one used in case

for which x is a point of differentiability of function f

we obtain desired outcome ☛ that is $\lim_{n\to\infty} S_n(x) = \frac{1}{2}\left[f(x^+) + f(x^-)\right]$

Definition 3.4.5.

Most familiar convergence mechanism for sequence of functions $S_n(x)$ is pointwise convergence.

This requires that functions' values at each individual point converge in usual sense:

$$\lim_{n\to\infty} S_n(x) = f(x) \qquad \forall x \in I \in \Re e$$

Pointwise convergence requires that for every $\epsilon > 0$ and every $x \in I$ there exists an integer N depending on ϵ and x such that

$$|S_n(x) - f(x)| < \epsilon \; \forall n \geq N$$

Pointwise convergence can be viewed as the function space version of the convergence of the components of a vector

Definition 3.4.6.

A stronger mode of convergence
is defined by demanding all points
to approach at more or less same rate to limit function

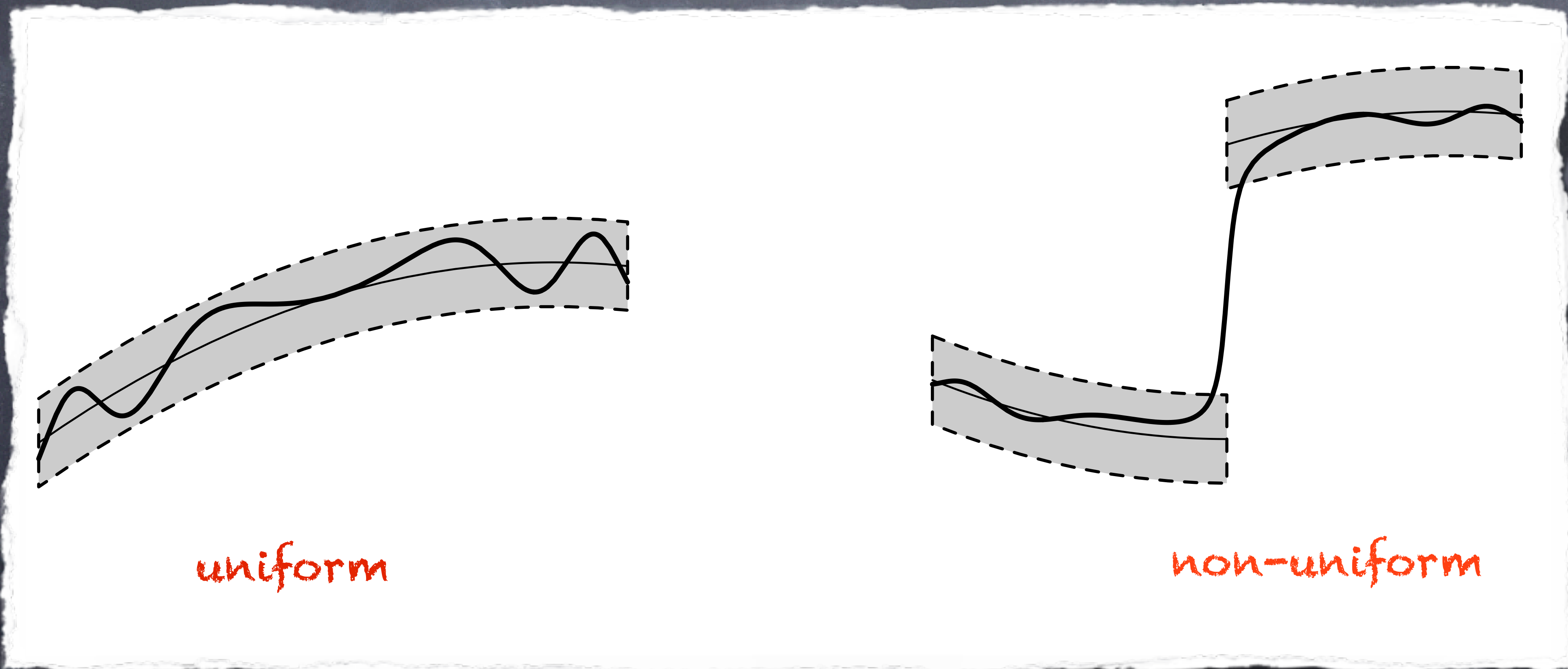
More precisely ☛ a sequence of functions $S_n(x)$
is said to converge uniformly to a function $f(x)$ on a subset $I \subset \mathbb{R}$
if for any $\epsilon > 0$
there exists an integer N -- depending solely on ϵ -- such that

$$|S_n(x) - f(x)| < \epsilon \quad \forall\, x \in I \text{ and } \forall n \geq N \qquad (3.4.348.)$$

Uniformly convergent sequence of functions converges pointwise
but converse does not hold

Key difference and reason for term uniform convergence
is that integer N depends only upon ϵ and not on point $x \in I$

According to (3.4.348.) the sequence converges uniformly if and only if for any small ϵ graphs of functions eventually lie inside a band of width 2ϵ centered around graph of limiting function



A key feature of uniform convergence is that it preserves continuity

Theorem 3.4.2.

If each $S_n(x)$ is continuous and $S_n(x) \to f(x)$ converges uniformly

then $f(x)$ is also a continuous function

Proof.

The proof is by reductio ad absurdum

Intuitively ☛ if $f(x)$ were to have a discontinuity

then a sufficiently small band around its graph

would not connect together

and this prevents the connected graph of any continuous function

such as $S_n(x)$

from remaining entirely within the band

Uniform convergence demands

all points to behave similarly in their approaching limit

Therefore ☛ it respects continuity and integration

but that mode of convergence is -- as expected -- not easy to get

For a normed vector space

one might instead consider an error of form

$$\lim_{n\to\infty} \|S_n(x) - f(x)\| \to 0 \qquad (3.4.349.)$$

This last one seems to be one of the best ways

of measuring error in case of Fourier series

For finite-dimensional vector spaces such as $\mathbb{R}^n$

convergence in norm is equivalent to ordinary convergence

On infinite-dimensional function spaces

convergence in norm differs from pointwise convergence

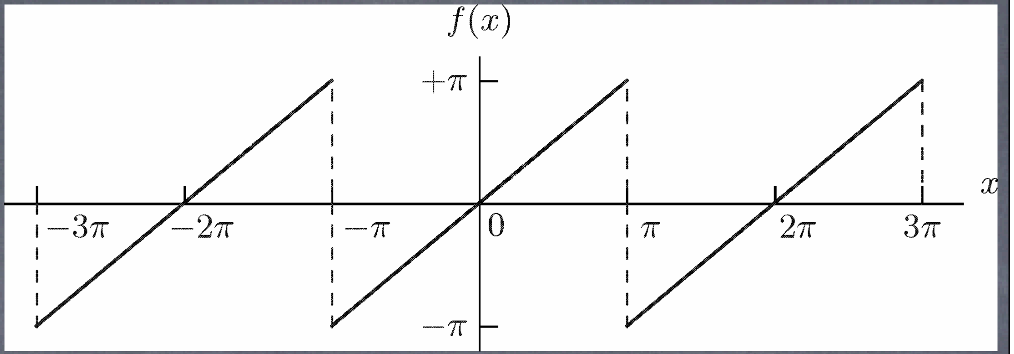
For instance ☛ it is possible to construct a sequence of functions

that converges in norm but does not converge pointwise anywhere

(We'll see this is the case in Example 3.4.3.)

Example 3.4.1.

For $f(x) = x$ with $x \in [-\pi, \pi]$



we obtain

$$a_n = 0 \forall n \quad \text{and} \quad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin(nx)\ dx = \frac{2(-1)^{n+1}}{n}$$

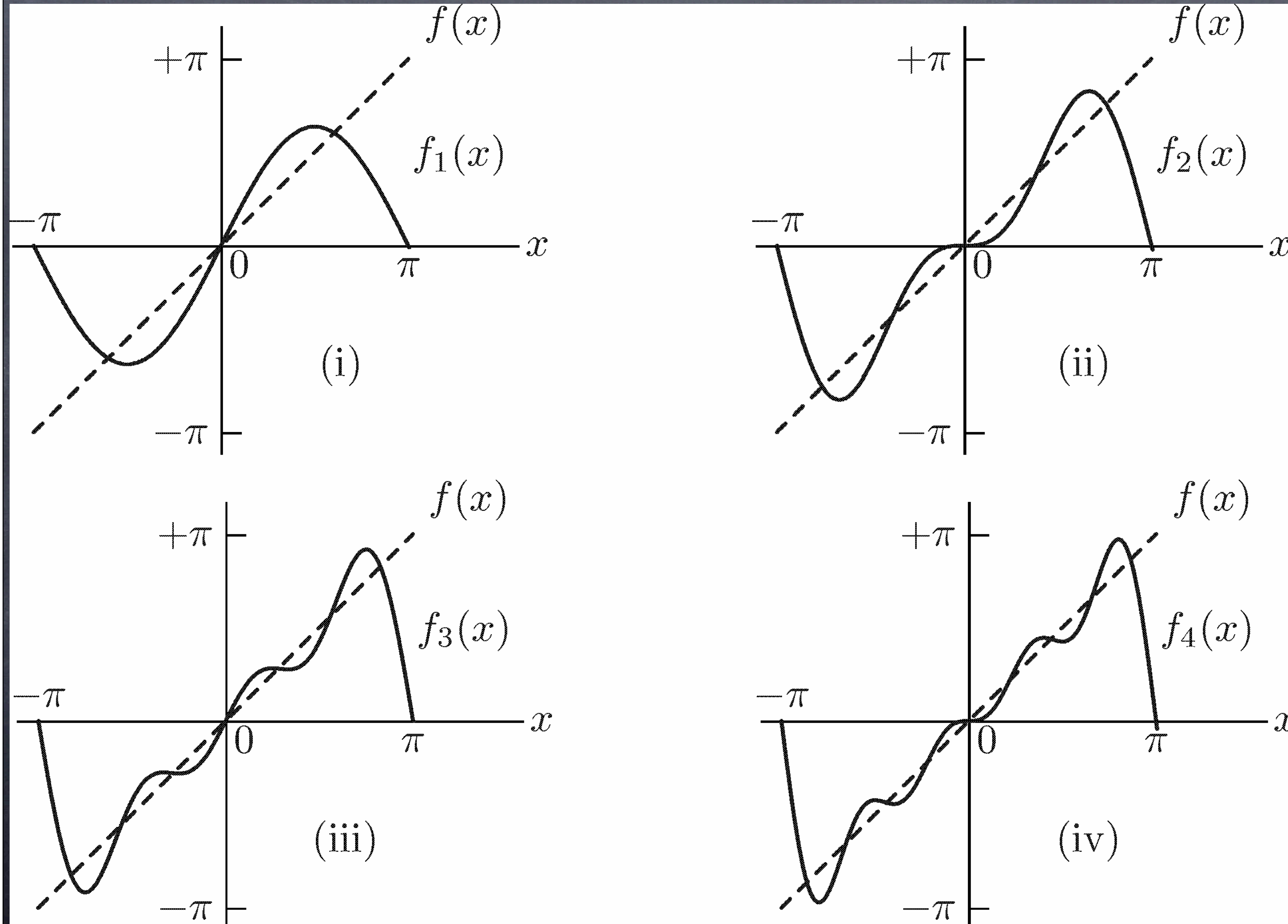
Therefore $x = 2\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\sin(nx)}{n}, \quad |x| < \pi$

As $x = \pm\pi$ series converges to $\frac{1}{2}[f(\pi) + f(-\pi)] = 0$

Series converges to 2π periodic extension

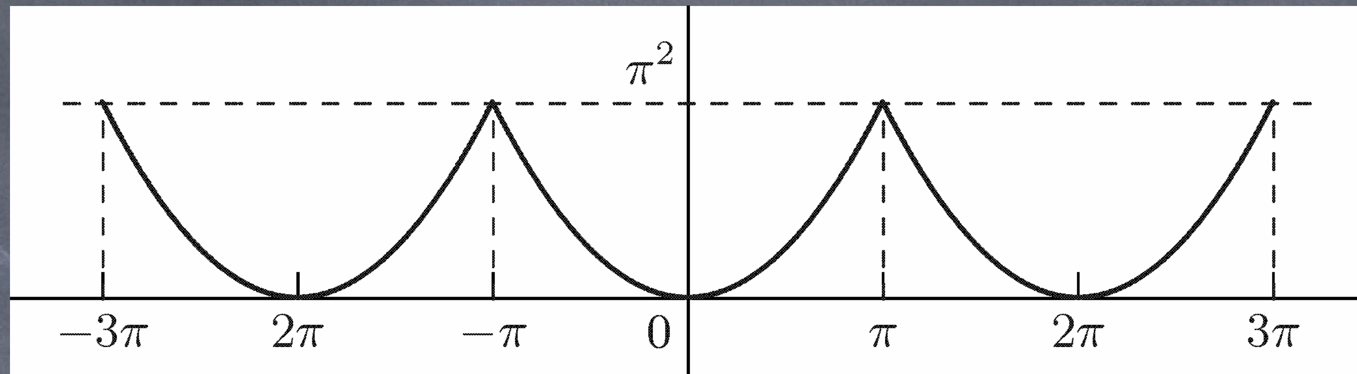
$$f(x) = x - 2n\pi \quad \text{if} \quad -\pi + 2n\pi < x < \pi + 2n\pi$$

and is discontinuous at $x = \pm\pi + 2n\pi$


$f(x)$
$+\pi$
$f_1(x)$
$-\pi$
x
0
π
(i)
$-\pi$
$f(x)$
$+\pi$
$f_2(x)$
$-\pi$
x
0
π
(ii)
$-\pi$
$f(x)$
$+\pi$
$f_3(x)$
$-\pi$
x
0
π
(iii)
$-\pi$
$f(x)$
$+\pi$
$f_4(x)$
$-\pi$
x
0
π
(iv)
$-\pi$

Example 3.4.2.

For $f(x) = x^2$



$$b_n = 0$$

$$a_0 = \frac{2}{3}\,\pi^2$$

$$a_n = \frac{4}{n^2}\,\cos(n\pi) \quad \text{if} \quad n \geq 1$$

Therefore ☛ $x^2 = \dfrac{\pi^2}{3} + 4\displaystyle\sum_{n=1}^{\infty}(-1)^n\,\frac{\cos(nx)}{n^2} \qquad |x| \leq \pi$

Fourier expansion also converges at $x = \pm\pi$ because $f(\pi) = f(-\pi)$

For $x = 0$ and $x = \pi$ Fourier expansion leads to

$$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12} \quad \text{and} \quad \sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}$$

Example 3.4.3.

For
$$f(x) = \begin{cases} (2a)^{-1} & \text{if } |x| < a < \pi \\ 0 & \text{if } |x| > a \end{cases}$$

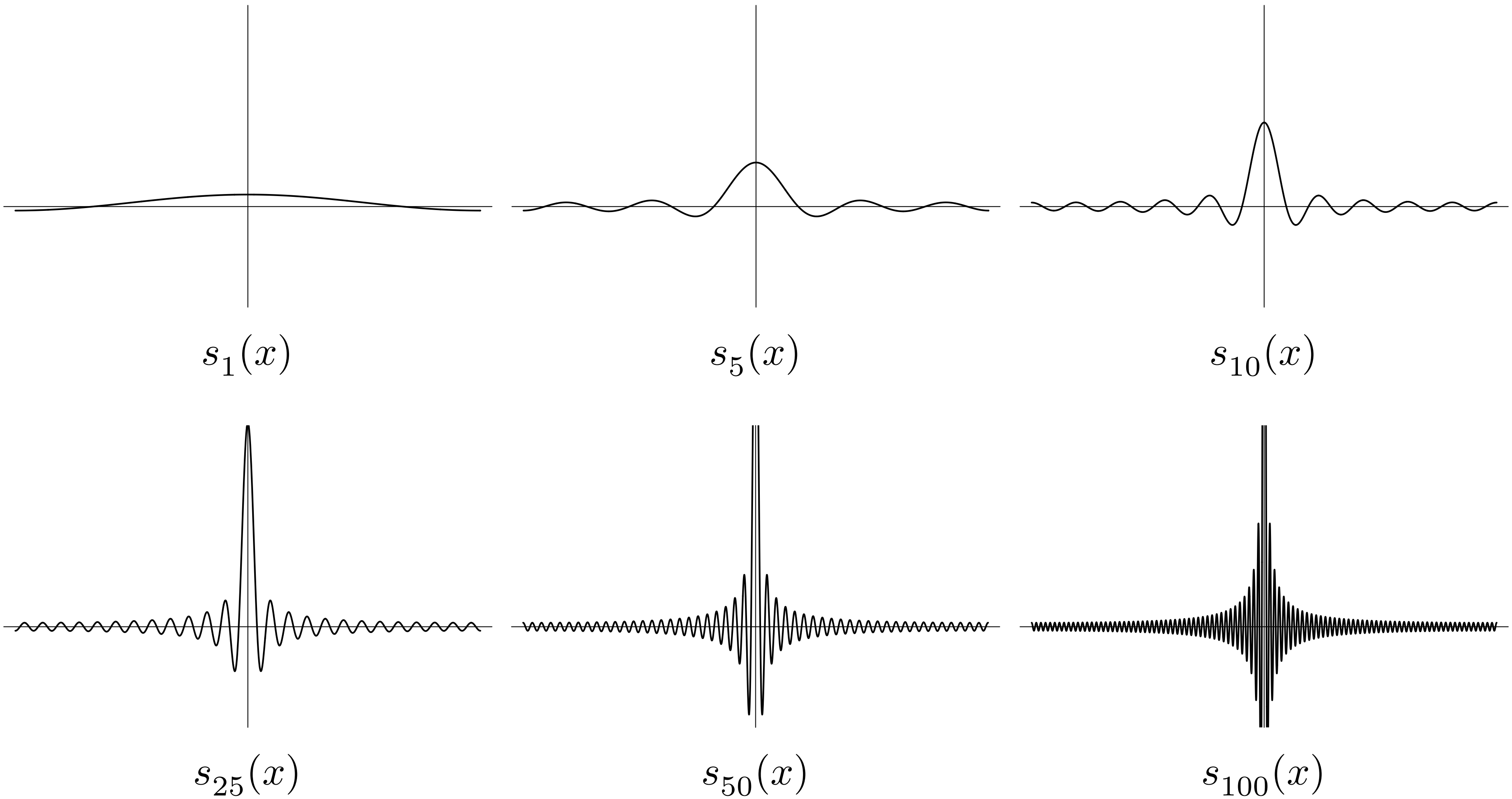
we obtain $b_n = 0, a_n = \sin(na)/(n\pi a)$ if $n \geq 1$ and $a_0 = 1/\pi$

Therefore ☛ $$f(x) = \frac{1}{\pi}\left[\frac{1}{2} + \sum_{n=1}^{\infty} \frac{\sin(na)}{na} \cos(nx)\right], \quad |x| \leq \pi$$

For $x = \pm a$ series converges to $(4a)^{-1}$

If $a \to 0$ the $f(x) \to \delta(x)$ with

$$\delta(x) = \frac{1}{\pi}\left[\frac{1}{2} + \sum_{n=1}^{\infty} \cos(nx)\right] = \frac{1}{2\pi} \lim_{n\to\infty} \sum_{n=-m}^{m} e^{inx}, \quad |x| \leq \pi$$



We had to truncate last two graphs; spikes extend beyond the top

Series does not converge pointwise

but it converges as a distribution to $\delta(x)$ for $|x| \leq \pi$

Indeed ☛ n-th partial sum

$$S_n(x) = \frac{1}{\pi}\left[\frac{1}{2} + \sum_{n=1}^{\infty}\cos(nx)\right] = \frac{\sin\left[\left(n+\frac{1}{2}\right)x\right]}{2\pi\sin(x/2)}$$

satisfies $\displaystyle\int_{-\pi}^{\pi} S_n(x)\, dx = 1, \quad \forall n \geq 0$

and

$$\begin{aligned}\lim_{n\to\infty}\int_{-\pi}^{\pi} S_n(x)\; f(x)\; dx &= f(0) + \lim_{n\to\infty}\int_{-\pi}^{\pi}\frac{f(x)-f(0)}{\sin(x/2)}\;\sin\left[\left(n+\tfrac{1}{2}\right)x\right]\,dx \\ &= f(0) \qquad \forall\, f \text{ test functions}\end{aligned}$$

Actually ☛ Fourier series converges to

2π periodic extension of original function

$$\lim_{m\to\infty}\sum_{n=-m}^{m}\delta(x-2n\pi) = \frac{1}{\pi}\left[\frac{1}{2} + \sum_{n=1}^{\infty}\cos(nx)\right] = \lim_{m\to\infty}\frac{1}{2\pi}\sum_{n=-m}^{m} e^{inx}$$

consisting of a periodic array of delta spikes

concentrated at all integer multiples of 2π

Example 3.4.4.

For

$$f(x) = \begin{cases} 1 & \text{if } |x| < 1/2 \\ 0 & \text{if } |x| > 1/2 \end{cases} \quad x \in [-1, 1] \qquad (3.4.357.)$$

we obtain $b_n = \int_{-1}^{1} f(x)\ \sin(n\pi x)\ dx = 0$ $\qquad a_0 = 1$

and $a_n = \int_{-1}^{1} f(x)\ \cos(n\pi x)\ dx = \begin{cases} (-1)^{(n+1)/2}\ 2/(n\pi) & n \text{ odd} \\ 0 & n \text{ even} \end{cases}$

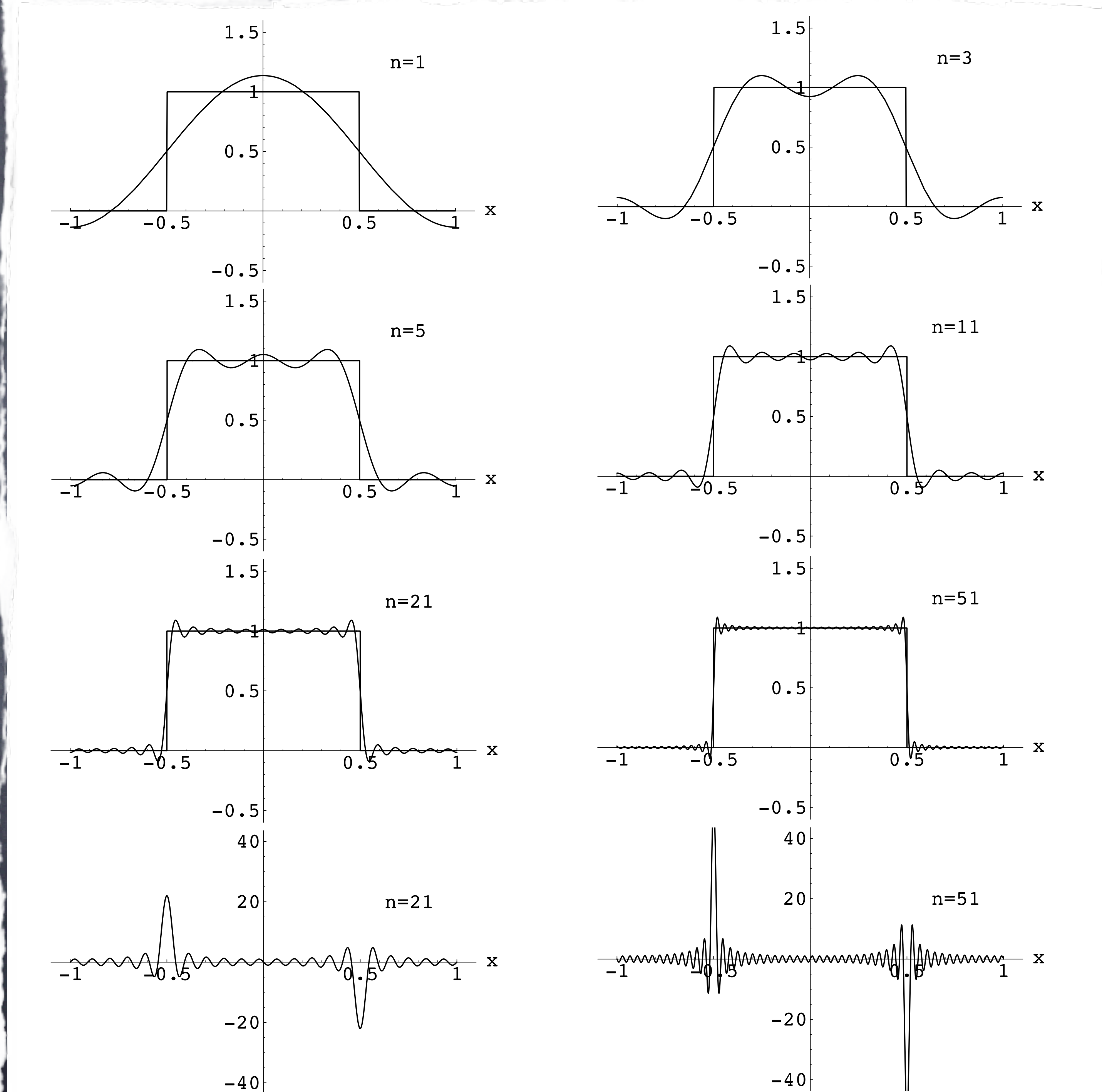
n-th partial sum reads $S_n(x) = \frac{1}{2}a_0 + \sum_{m=1}^{n} a_m \cos(m\pi x)$

$dS_n(x)/dx$ ☛ $\delta\left(x + \frac{1}{2}\right) - \delta\left(x + \frac{1}{2}\right)$

Near jumps there should be a consistent overshoot of about 9%

So-called Gibbs overshoot is a manifestation

of subtle non-uniform convergence of Fourier series


n=1
n=3
n=5
n=11
n=21
n=51
n=21
n=51
x
1.5
1
0.5
-0.5
-1
0.5
40
20
-20
-40

Definition 3.4.9.

Consider complex form of Fourier series of $f(x) : [-L, L] \to \mathbb{R}$

$$f(x) = \sum_{n=-\infty}^{\infty} c_n \, e^{in\pi x/L}, \quad c_n = \frac{1}{2L}\int_{-L}^{L} f(x) e^{-in\pi x/L} \, dx$$

We can rewrite this series as

$$f(x) = \frac{1}{\sqrt{2\pi}} \sum_k \widehat{f}(k) \, e^{ikx} \, \Delta k \qquad (3.4.370.)$$

where $k = n\pi/L, \Delta k = \pi/L$

and $$\widehat{f}(k) = \sqrt{2\pi} \, c_n \frac{L}{\pi} = \frac{1}{\sqrt{2\pi}} \int_{-L}^{L} f(x) \, e^{ikx} \, dx \qquad (3.4.373.)$$

Let us now consider limit $L \to \infty$

In such a case $\Delta k \to 0$ while (3.4.370.) and (3.4.373.)

$$\widehat{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) \, e^{-ikx} \, dx \qquad (3.4.375.)$$

approach ☛

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \widehat{f}(k) \, e^{ikx} \, dk \qquad (3.4.379.)$$

assuming that both integrals converge

$\widehat{f}:\mathbb{R}\to\mathbb{R}$ **Fourier transform** of function $f(f:\mathbb{R}\to\mathbb{R})$

Fourier transform operator $\widehat{f}(k)=\mathcal{F}[f(x)]$

maps each (sufficiently nice) function of spatial variable x

to a function of frequency variable k

Expression which retrieves f from $\widehat{f}$

is inverse of Fourier transform operator $f(x)=\mathcal{F}^{-1}\left[\widehat{f}(k)\right]$

$\mathcal{F}$ & $\mathcal{F}^{-1}$ are generalization of Fourier series

for functions f defined on $(-\infty,\infty)$

i.e. $f(x)\in\mathcal{L}^2$ is square-integrable

Before proceeding we show validity of (3.4.375.) and (3.4.379.)

for functions f which are piecewise $\mathcal{C}^1$ and satisfy $\int_{-\infty}^{\infty} |f(x)| dx < \infty$

Proof is very similar to one carried out for Fourier series

(3.4.375) and (3.4.379.) entail

$$f(x) = \int_{-\infty}^{\infty} \frac{1}{2\pi} \left[\int_{-\infty}^{\infty} e^{ik(x-x')}\, dk \right] f(x')\, dx'$$

so what must be shown is that

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ik(x-x')}\, dk = \delta(x - x') \quad ☛ \int_{-\infty}^{\infty} = \lim_{r\to\infty} \int_{-r}^{r} \quad (3.4.382.)$$

Indeed ☛ $$\frac{1}{2\pi} \int_{-r}^{r} e^{ikt}\, dk = \frac{1}{2\pi} \frac{e^{irt} - e^{-irt}}{it} = \frac{1}{\pi} \frac{\sin(rt)}{t}$$

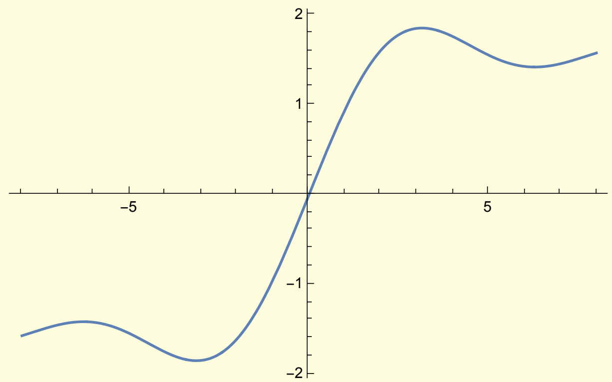
with $$\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sin(rt)}{t}\, dt = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sin(u)}{u}\, du = 1$$

In[6]:= `Integrate[(Sin[x]) / x, x]`

Out[6]= `SinIntegral[x]`

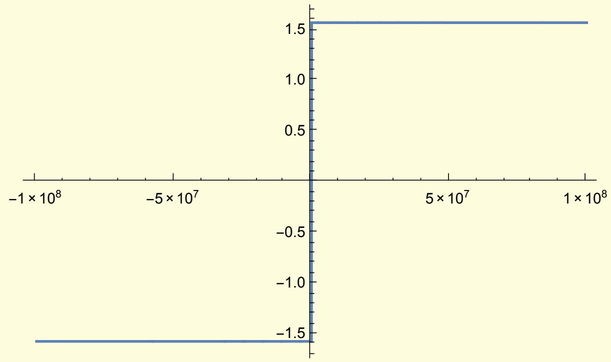
In[11]:= `Plot[SinIntegral[x], {x, -8, 8}]`

Out[11]=



In[12]:= `Plot[SinIntegral[x], {x, -100 000 000, 100 000 000}]`

Out[12]=



Therefore

$$f(x) = \lim_{r\to\infty} \int_{-\infty}^{\infty} \left[\frac{1}{2\pi} \int_{-r}^{r} e^{ik(x-x')} dk \right] f(x')\, dx'$$

$$= \lim_{r\to\infty} \int_{-\infty}^{\infty} \frac{\sin\left[r(x-x')\right]}{\pi(x-x')} f(x')\, dx'$$

$$= \lim_{r\to\infty} \int_{-\infty}^{\infty} \frac{\sin(rt)}{\pi t} \left[f(x+t) - f(x) + f(x)\right] dt$$

$$= f(x) + \lim_{r\to\infty} \int_{-\infty}^{\infty} \sin(rt)\, \frac{f(x+t) - f(x)}{\pi t}\, dt$$

For $r \to \infty$ second term cancels

because $[f(x+t) - f(x)]/t$ remains bounded for $t \to 0$

Equation (3.4.382.) also implies

$$\int_{-\infty}^{\infty} \varphi_{k'}^*(x)\ \varphi_k(x)\ dx = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ix(k-k')}\ dx = \delta(k-k') \qquad (3.4.386.)$$

indicating that functions $\varphi_k(x) = e^{ikx}/\sqrt{2\pi}$ are orthogonal with

with respect to inner product $\langle u, v\rangle = \int_{-\infty}^{\infty} u^*(x)\ v(x)\ dx$

and are normalized with respect to variable

Note that convergence of integrals (3.4.382.) and (3.4.386.)

should be understood as distributions

Example 3.4.5.

Fourier transform of rectangular pulse,

$$f(x) = \Theta(x+a) - \Theta(x-a) = \begin{cases} 1 & -a < x < a \\ 0 & |x| > a \end{cases} \quad (3.4.389.)$$

(a.k.a. box function of width $2a$) is easily computed

$$\widehat{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} e^{-ikx}\, dx = \frac{e^{ika} - e^{-ika}}{\sqrt{2\pi} ik} = \sqrt{\frac{2}{\pi}} \frac{\sin(ak)}{k} \quad (3.4.390.)$$

Reconstruction of pulse via inverse transform (3.4.379.) tells us that

$$\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{e^{ikx} \sin(ak)}{k}\, dk = \begin{cases} 1 & -a < x < a \\ 1/2 & x = \pm a \\ 0 & |x| > a \end{cases} \quad (3.4.391.)$$

Note convergence to middle of jump discontinuities at $x = \pm a$

Real part of this complex integral
produces a striking trigonometric integral identity

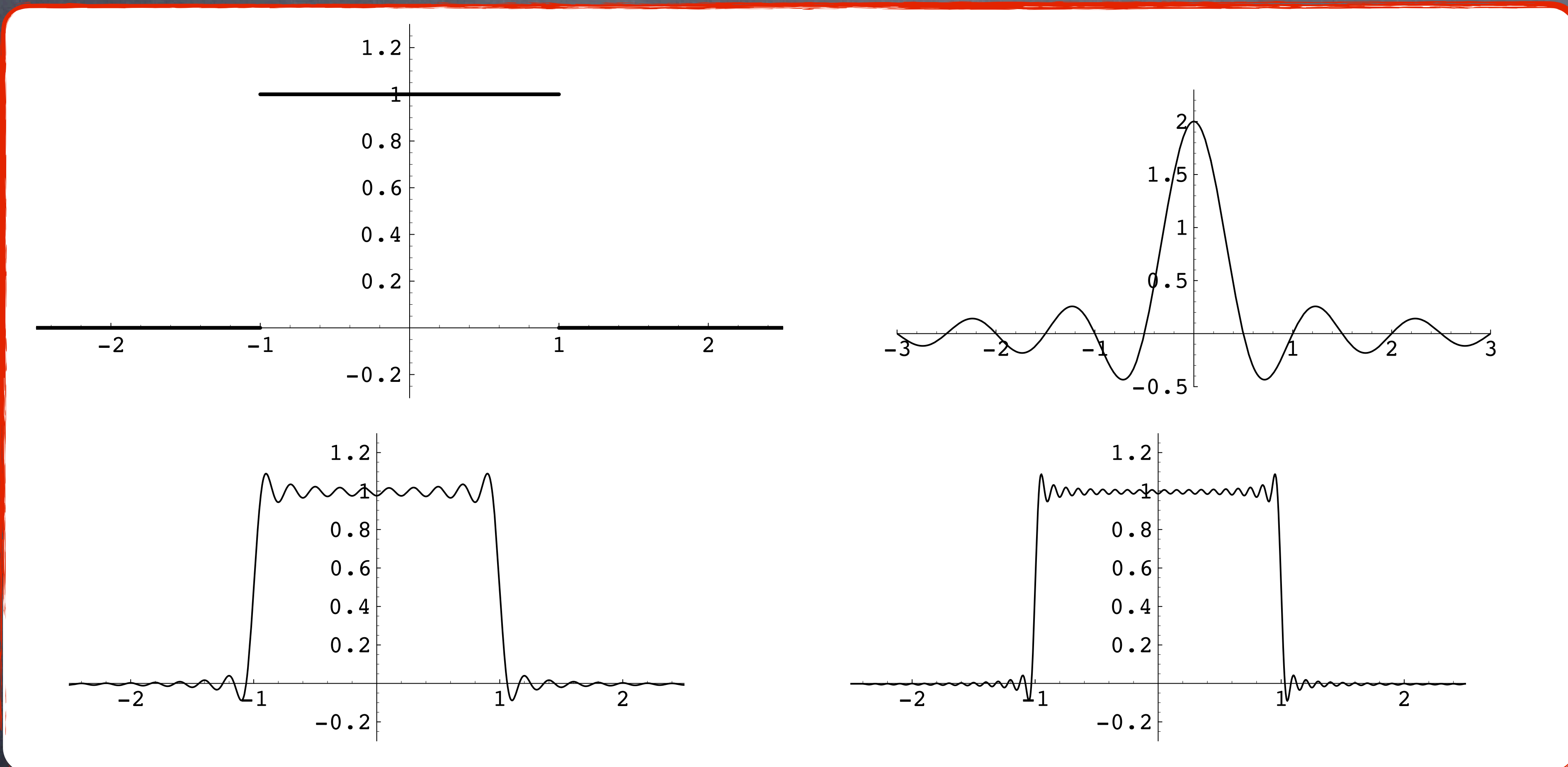
$$\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\cos(kx)\ \sin(ak)}{k}\ dk = \begin{cases} 1 & -a < x < a \\ 1/2 & x = \pm a \\ 0 & |x| > a \end{cases}$$

identity resulting from imaginary part

$$\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\sin(kx)\ \sin(ak)}{k}\ dk = 0$$

is not surprising because integrand is odd

Non-uniform convergence of the integral
leads to appearance of a Gibbs phenomenon at two discontinuities
similar to the non-uniform convergence of a Fourier series



Since we are dealing with an infinite integral
must break off numerical integrator restricting it to a finite interval

Left graph is obtained by integrating from $-5 \leq k \leq 5$

Right graph is obtained by integrating from $-10 \leq k \leq 10$


TO BE
CONTINUED....