Physics 307

Mathematical Physics

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ORDINARY DIFFERENTIAL EQUATIONS IV

3.1 Setting the Stage ✓

3.2 Initial Value Problem ✓

Picard's existence and uniqueness theorem

Systems of first-order linear differential equations

Green matrix as a generalized function

3.3 Boundary Value Problem

Self-adjointness of Sturm-Liouville operator

Green function of Sturm-Liouville operator

Series solutions to homogeneous linear equations

3.4 Fourier Analysis

Fourier series

Fourier transform

Series solutions to homogeneous linear equations

Theorem 3.3.5.

If $A(x)$ and $B(x)$ are analytic functions in a neighborhood of $x = 0$

$$A(x) = \sum_{n=0}^{\infty} A_n x^n, \quad B(x) = \sum_{n=0}^{\infty} B_n x^n, \quad |x| < R \qquad (3.3.222.)$$

the solutions of

$$u'' + A(x)u' + B(x)u = 0 \qquad (3.3.223.)$$

are also analytic functions & can be represented as a power series

$$u(x) = \sum_{n=0}^{\infty} c_n \, x^n, \quad |x| < R \qquad (3.3.224.)$$

Proof.

We first assume that there exists a solution of form (3.3.224) and we will show that it converges

Since

$$B(x)\ u(x) = \sum_{n=0}^{\infty} x^n \sum_{m=0}^{n} B_{n-m} c_m \qquad (3.3.225.)$$

and

$$A(x)\ u'(x) = \sum_{n=0}^{\infty} x^n \sum_{m=0}^{n+1} A_{n-m+1}\ m\ c_m \qquad (3.3.226.)$$

we substitute (3.3.224) into (3.3.223) to obtain

$$\sum_{n=0}^{\infty} x^n \left[c_{n+2}(n+2)(n+1) + \sum_{m=0}^{n+1} c_m (m A_{n-m+1} + B_{n-m}) \right] = 0 \qquad (3.3.227.)$$

Since coefficient of x^n must cancel

$$c_{n+2} = -\frac{\sum_{m=0}^{n+1} c_m [m A_{n-m+1} + B_{n-m}]}{(n+2)(n+1)}, \quad n \geq 0 \qquad (3.3.228.)$$

it follows that c_{n+2} only depends on previous coefficients $c_0, \ldots, c_{n+1}$

For $n \geq 2$ ☛ this leads to a recursive relationship that determines every c_n in terms of c_0 and c_1

For example

$$\begin{aligned} c_2 &= -\frac{A_0 c_1 + B_0 c_0}{2}, \\ c_3 &= -\frac{A_1 c_1 + 2A_0 c_2 + B_1 c_0 + B_0 c_1}{6} \\ &= -\frac{c_0(B_1 - A_0 B_0) + c_1(A_1 + B_0 - A_0^2)}{6} \end{aligned} \quad (3.3.229.)$$

This series can be proven to converge for $|x| < R$

Let t be such that $0 \leq |x| < t < R$

A necessary condition for series (3.3.222) to converge is that

$$\lim_{n\to\infty} A_n t^n = \lim_{n\to\infty} B_n t^n = 0$$

This implies that $\exists M > 0$ such that

$$|B_n| \leq M/t^n \quad \text{and} \quad |A_n| \leq M/t^{n-1} \quad \forall n \quad (3.3.230.)$$

Therefore ☛ $$|c_{n+2}| \leq \frac{M \sum_{m=0}^{n+1} |c_m| \; t^m \; (m+1)}{t^n \; (n+2) \; (n+1)} \quad (3.3.231.)$$

By recursively defining non-negative coefficients

$$d_{n+2} = \frac{M \sum_{m=0}^{n+1} d_m \; t^m \; (m+1)}{t^n \; (n+2) \; (n+1)} \qquad (3.3.232.)$$

with $d_0 = |c_0|, d_1 = |c_1|$ we have $|c_n| \leq d_n \forall n$

In addition

$$d_{n+2} = d_{n+1} \left[\frac{n}{t(n+2)} + \frac{Mt(n+2)}{(n+2)(n+1)} \right] \qquad (3.3.233.)$$

and so using the ratio test

$$\lim_{n \to \infty} \frac{d_{n+2}|x|^{n+2}}{d_{n+1}|x|^{n+1}} = \frac{|x|}{t} < 1 \qquad (3.3.234.)$$

we show that series $\sum_{n=0}^{\infty} d_n x^n$ is absolutely convergent for $|x| < t$

i.e. $\forall x < R$

This in turn entails that $\sum_{n=0}^{\infty} c_n x_n$ is absolutely convergent for $|x| < R$

The general solution can be written as

$$u(x) = c_0 u_1(x) + c_1 u_2(x) \quad (3.3.235.)$$

u_1 is solution for $c_0 = 1$ and $c_1 = 0$

u_2 is solution for $c_0 = 0$ and $c_1 = 1$

i.e.

$$\begin{aligned} u_1(x) &= 1 - \frac{B_0}{2}x^2 - \frac{B_1 - A_0 B_0}{6}x^3 + \dots, \\ u_2(x) &= x - \frac{A_0}{2}x^2 - \frac{A_1 + B_0 - A_0^2}{6}x^3 + \dots \end{aligned}$$

Of course same considerations apply

if coefficients are analytical in neighborhood of x_0

in which case $A(x), B(x)$, and $u(x)$

can be expressed as power series of $(x - x_0)$

Example 3.3.8. [Legendre polynomials]

The second order differential equation

$$(1-x^2)u'' - 2xu' + l(l+1)u = 0 \qquad (3.3.238.)$$

can be written in Sturm-Liouville form

$$[(1-x^2)u']' + l(l+1)u = 0 \qquad (3.3.239.)$$

it corresponds to $A(x) = -\dfrac{2x}{1-x^2}$ and $B(x) = \dfrac{l(l+1)}{1-x^2}$

both functions are analytic if $|x| < 1$

Legendre equation arises when considering

the angular part of Laplacian in spherical coordinates

if u is only a function of θ

$$\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)\right]u(\theta) = -l(l+1)u(\theta) \qquad (3.3.240.)$$

Note that with substitution ☛ $x = \cos\theta \quad \Rightarrow \quad \dfrac{1}{\sin\theta}\dfrac{\partial}{\partial\theta} = -\dfrac{\partial}{\partial x}$

(3.3.240.) becomes (3.3.238.)

Since $l(l+1) = (l+1/2)^2 - 1/4$

it is sufficient to consider $\Re\mathrm{e}(l) \geq -\frac{1}{2}$

Series (3.3.224.) takes form

$$\sum_{n=0}^{\infty} x^n \{c_{n+2}(n+2)(n+1) - c_n[n(n-1) + 2n - l(l+1)]\} = 0 \quad (3.3.241.)$$

and thus

$$c_{n+2} = c_n \frac{n(n+1) - l(l+1)}{(n+2)(n+1)} = -c_n \frac{(n+l+1)(l-n)}{(n+2)(n+1)} \quad (3.3.242.)$$

for $c_1 = 0$ and $c_0 \neq 0$ we obtain even solution ☛ $c_{2n+1} = 0$

$$c_2 = -c_0 \frac{l(l+1)}{2!}, \quad c_4 = c_0 \frac{l(l-2)(l+1)(l+3)}{4!}, \ldots$$

for $c_0 = 0$ and $c_1 \neq 0$ we obtain odd solution ☛ $c_{2n} = 0$

$$c_3 = -c_0 \frac{(l-1)(l+2)}{3!}, \quad c_5 = c_0 \frac{(l-1)(l-3)(l+2)(l+4)}{5!}, \ldots$$

Since $\lim_{n\to\infty} |c_{n+2}/c_n| \to 1$ radius of convergence of series is $|x| < 1$

If $l \notin \mathbb{Z}$ ☛ it is easily seen that $|u(x)| \to \infty$ at $x = \pm 1$

If $l > 0 \in \mathbb{Z}$ ☛ $c_{l+2} = 0$

the solution becomes a polynomial of degree l

called Legendre polynomial

non-zero coefficients of Legendre polynomial are given by

$$c_{2n+i} = c_i \frac{(-1)^n \; (l_2!)^2 (l+2n)!}{l!(l_2-n)!(l_2+n)!(2n+i)!}, \; i = 0, 1, \quad n = 0, \ldots, l_2$$

$$l_2 = \lfloor l/2 \rfloor \quad (3.3.243.)$$

$i = 0$ corresponds to solution with even l

$i = 1$ corresponds to solution with odd l

Legendre polynomials are defined as solution to coefficients

$$c_i = (-1)^{l_2} l! / [(l_2!)^2 2^{l-i}], i = 0, 1$$

$$P_l(x) = \frac{1}{2^l} \sum_{k=0}^{l_2} \frac{(-1)^k (2l-2k)!}{k!(l-k)!(l-2k)!} \; x^{l-2k} \quad (3.3.244.)$$

$$k = l_2 - n$$

Since

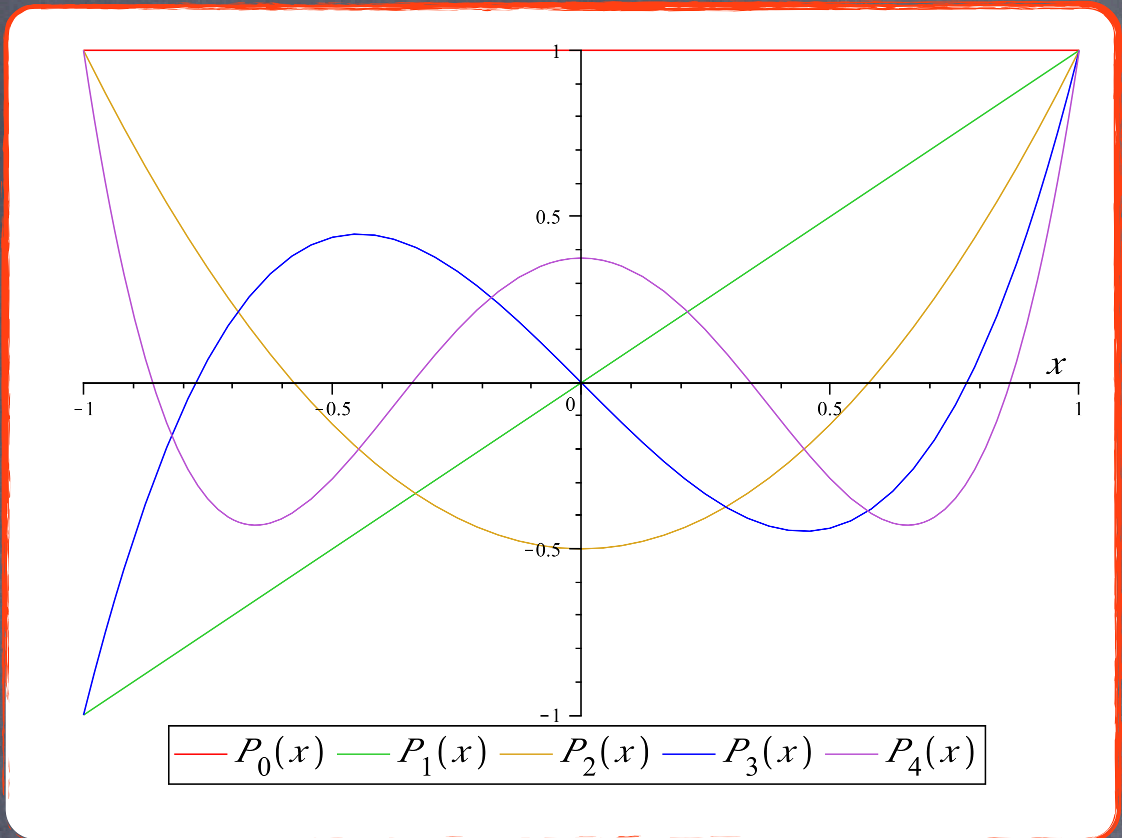
$$\frac{(2l-2k)!}{(l-2k)!}x^{l-2k} = \frac{d^{(l)}}{dx^l}x^{2l-2k} \quad \text{and} \quad k!(l-k)! = l!/\binom{l}{k} \qquad (3.3.245.)$$

polynomials can also be written using Rodrigues formula

$$\begin{aligned} P_l(x) &= \frac{1}{2^l l!}\frac{d^{(l)}}{dx^l}\sum_{k=0}^{l}(-1)^k \binom{l}{k} x^{2l-2k} \\ &= \frac{1}{2^l l!}\frac{d^{(l)}}{dx^l}(x^2-1)^l \end{aligned} \qquad (3.3.246.)$$

First five Legendre polynomials are:

$$\begin{aligned} P_0(x) &= 1 \\ P_1(x) &= x \\ P_2(x) &= \frac{1}{2}(3\ x^2-1) \\ P_3(x) &= \frac{1}{2}(5\ x^3-3\ x) \\ P_4(x) &= \frac{1}{8}(35\ x^4-30\ x^2+3); \end{aligned}$$



Legendre polynomials satisfy following relations:

$$P_l(1) = 1$$

$$\int_{-1}^{1} P_l(x)\ P_{l'}(x)\ dx = \delta_{ll'}\ \frac{2}{2l+1}$$

$$lP_l(x) = (2l-1)xP_{l-1}(x) - (l-1)P_{l-2}(x), \quad l \geq 2$$

(3.3.247.)

In summary ➡ bounded solutions of (3.3.238.)

for $x \in (-1, 1)$ and $l \geq -1/2$ are only obtained when $l \in \mathbb{Z}$

Solution is a polynomial proportional to polynomial of Legendre

Polynomial functions are odd or even (with respect to $x = 0$)

according to whether index l is odd or even

For example

for $l = 0$ linearly independent solutions of (3.3.238.) are

$$u_1(x) = P_0(x) = 1 \qquad \text{and} \qquad u_2(x) = \frac{1}{2}\ \ln[(1+x)/(1-x)]$$

where $u_2(x)$ is odd solution which diverges for $x \to \pm 1$

Example 3.3.9. [Associated Legendre polynomials]

Associated Legendre polynomials are canonical solutions of

$$(1-x^2)u'' - 2xu' + \left[l(l+1) - \frac{m^2}{1-x^2}\right]u = 0 \quad (3.3.248.)$$

For $m = 0$ (3.3.248.) reduces to Legendre equation

Associated Legendre equation arises when considering angular part of Laplacian in spherical coordinates

if $u(\theta, \phi) = u(\theta)e^{im\phi}$

$$\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) - \frac{m^2}{\sin^2\theta}\right]u(\theta) = -l(l+1)u(\theta) \quad (3.3.249.)$$

Note that with substitution $x = \cos\theta$ (3.3.249.) becomes (3.3.248.)

We can assume that $\Re\mathrm{e}\,(m) \geq 0$ and $\Re\mathrm{e}\,(l) \geq -1/2$

It is convenient to introduce following change of variable

$$u = (1-x^2)^{m/2}\, w \quad (3.3.250.)$$

so that (3.3.248.) can be rewritten as

$$(1-x^2)w'' - 2(m+1)xw' + [l(l+1) - m(m+1)]w = 0 \quad (3.3.251.)$$

Introducing a power series for w ☛ (3.3.228.) takes form

$$c_{n+2} = c_n \frac{n(n-1) + (2n+m)(m+1) - l(l+1)}{(n+2)(n+1)}$$

$$= -c_n \frac{(n+l+1+m)(l-m-n)}{(n+2)(n+1)} \quad (3.3.252.)$$

If $l - m = k$ with k a positive integer ☛ then $c_{k+2} = 0$

and solution (with same parity of k) is a polynomial of degree k

Although we can obtain such polynomials by means of (3.3.252.)

it is easily seen (with help of Leibniz formula) that

if $u(x)$ is a solution of Legendre equation (3.3.238.)

then ☛ m-th derivative $u^{(m)}(x)$ satisfies (3.3.251.)

Namely

$$(1-x^2)u^{(m+2)} - 2(m+1)xu^{(m+1)} + [l(l+1) - m(m+1)]u^{(m)} = 0 \quad (3.3.253.)$$

For $m > 0 \in \mathbb{Z}$ solutions of (3.3.248.) are of form $(1-x^2)^{m/2} u^{(m)}(x)$ with $u(x)$ a solution of Legendre equation (3.3.238)

For integers m and l we can write the so-called associated Legendre polynomials

$$P_l^m(x) = (1-x^2)^{m/2} \frac{d^{(m)}}{dx^m} P_l(x), \quad \text{with} \quad 0 \le m \le l \quad (3.3.254.)$$

which constitute only bounded solutions of (3.3.248.) in $[-1,1]$

Using Leibniz differentiation formula once again it is easily seen that $P_l^m(x)$ and P_l^{-m} are related by

$$P_l^{-m}(x) = (-1)^m \frac{(l-m)!}{(l+m)!} P_l^m(x) \quad (3.3.255.)$$

Orthogonality integral reads

$$\int_{-1}^{1} P_l^m(x)\, P_{l'}^m(x)\, dx = \delta_{ll'} \frac{2}{2l+1} \frac{(l-m)!}{(l+m)!} \quad (3.3.256.)$$

and hence

$$\int_{-1}^{1} P_l^m(x)\, P_{l'}^{-m}(x)\, dx = \delta_{ll'} \frac{2}{2l+1} \quad (3.3.257.)$$

Theorem 3.3.6. [Frobenius-Fuchs theorem]

If $A(x)$ has at least a simple pole at $x = 0$

and $B(x)$ has at least a pole of order two at $x = 0$

such that

$$A(x) = \sum_{n=-1}^{\infty} A_n x^n = \frac{A_{-1}}{x} + A_0 + \ldots$$

$$B(x) = \sum_{n=-2}^{\infty} B_n x^n = \frac{B_{-2}}{x^2} + \frac{B_{-1}}{x} + \ldots,$$

$$0 < |x| < R$$

one of linearly independent solutions of (3.3.223.)

can expanded as a generalized power series

$$u_1(x) = \sum_{n=0}^{\infty} a_n x^{n+s_+} = x^{s_+} \sum_{n=0}^{\infty} a_n x^n \qquad (3.3.258.)$$

where $a_0 \neq 0$ and s_+ is a root of indicial polynomial

$$s(s-1) + A_{-1}s + B_{-2} = 0 \qquad (3.3.259.)$$

that is ☛ $s_\pm = \dfrac{1 - A_{-1} \pm r}{2}$ with $r = \sqrt{(1 - A_{-1})^2 - 4B_{-2}}$ (3.3.260.)

If the difference between two roots of this equation $r \notin \mathbb{Z}$

2nd solution of (3.3.223.) is also a generalized power series on s_- ☛ other root of (3.3.259.)

On other hand ☛ if $r \in \mathbb{Z}$ second solution has form

$$u_2(x) = C u_1(x) \ln x + x^{s_-} \sum_{n=0}^{\infty} b_n x^n \qquad (3.3.261.)$$

if $r = 0$ (i.e. $s_+ = s_-$) then $C \neq 0$

Proof. Need for a generalized power series can be understood by analyzing behavior of solution for $x \to 0$

Retaining only higher-order terms (3.3.223.) becomes Euler eq.

$$u'' + \frac{A_{-1}}{x} u' + \frac{B_{-2}}{x^2} u = 0 \qquad (3.3.262.)$$

For single roots ☛ solutions are of the form $u(x) = cx^s$ (3.3.263.)

for multiple roots ☛ $x^s \ln s$ is also a solution

Substituting (3.3.263.) into (3.3.262.) we obtain

$cx^s \left[s(s-1) + A_{-1}s + B_{-2} \right] = 0$ which leads to indicial equation

Alternatively ☛ we can obtain indicial equation substituting (3.3.258.) in (3.3.223.)

$$\sum_{n=0}^{\infty} x^{s+n-2} \left\{ a_n(n+s)(n+s-1) + \sum_{m=0}^{n} a_m \left[A_{n-m-1}(m+s) + B_{n-m-2}\right] \right\}$$

Cancellation of coefficient of x^{s-2} (i.e. ☛ $n=0$) implies

$$a_0[s(s-1) + A_{-1}s + B_{-2}] = 0 \qquad (3.3.265.)$$

which leads to indicial equation because $a_0 \neq 0$

For $n \geq 1$ ☛ cancelation of coefficient of x^{n+s-2} leads to recursive relation

$$a_n = -\frac{\sum_{m=0}^{n-1} a_m(A_{n-m-1}(m+s) + B_{n-m-2})}{(n+s)(n+s-1) + A_{-1}(n+s) + B_{-2}}$$

$$= -\frac{\sum_{m=0}^{n-1} a_m(A_{n-m-1}(m+s) + B_{n-m-2})}{n(n+2s+A_{-1}-1)} \qquad (3.3.266.)$$

Since $n + 2s + A_{-1} - 1 = n \pm r$ ☛ this is valid if $n \pm r \neq 0$

Example 3.3.10. [Bessel functions]

Bessel equation

$$u'' + \frac{u'}{x} + \left(1 - \frac{\nu^2}{x^2}\right) u = 0 \qquad (3.3.267.)$$

is of the form contemplated in theorem 3.3.6.

It arises when finding separable solutions
to Laplace & Helmhotz eqs. in cylindrical or spherical coordinates

By employing method of generalized power series we obtain

$$\sum_{n=0}^{\infty} x^{s+n-2}\{a_n[(n+s)(n+s-1)+(n+s)-\nu^2]+a_{n-2}\}=0 \qquad (3.3.268.)$$

with $a_{-2} = a_{-1} = 0$

For $n = 0$ ☛ $a_0[s(s-1)+s-\nu^2]=0$ (3.3.269.)

which leads to indicial equation $s^2 - \nu^2 = 0$ ☛ $s = \pm\nu$

For $n \geq 2$ ☛ $a_n = -\dfrac{a_{n-2}}{(n+s)^2-\nu^2} = -\dfrac{a_{n-2}}{n(n+2s)}$ (3.3.270.)

Substituting $s = \nu$ into (3.3.268.) we determine $a_1 = 0$

and using (3.3.270.) we see that $a_n = 0$ for all $n =$ odd integers

The coefficients for even powers of n are found to be

$$\frac{a_{2n}}{a_{2n-2}} = -\frac{1}{4n(n+\nu)}$$

$$= \frac{(-1)^{n+1}\ 2^{2n}\ (n-1)!\ \Gamma(n+\nu)}{(-1)^n\ 2^{2n+2}\ n!\ \Gamma(n+\nu+1)} \quad (3.3.271.)$$

where in last line we have use properties of Gamma function

The recursive relation is satisfied if

$$a_{2n} = (-1)^n\ \frac{c}{2^{2n}\ n!\ \Gamma(n+\nu+1)} \quad (3.3.272.)$$

For $c = 2^{-\nu}$ ☛ we obtain Bessel function of first kind of order ν

$$J_\nu(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma(n+1)\Gamma(n+\nu+1)} \left(\frac{x}{2}\right)^{2n+\nu} \quad (3.3.273.)$$

which is one of solutions of (3.3.267.)

By applying ratio test ☛ series converges $\forall x \in \mathbb{R}$ (or $x \in \mathbb{C}$)

If $\nu \notin \mathbb{Z}$ ☛ other linear independent solution of (3.3.267.) is

$$J_{-\nu}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(n-\nu+1)} \left(\frac{x}{2}\right)^{2n-\nu} \quad (3.3.274.)$$

If $n \geq 0 \in \mathbb{Z}, J_{-n}(x) \equiv \lim_{\nu \to -n} J_\nu(x) = (-1)^n J_n(x)$

Thus ☛ we take particular linear combination of $J_\nu(x)$ and $J_{-\nu}(x)$

$$Y_\nu(x) = \frac{\cos(\nu\pi) J_\nu(x) - J_{-\nu}(x)}{\sin(\nu\pi)} \quad (3.3.275.)$$

known as Bessel function of second kind

For $\nu \notin \mathbb{Z}$ ☛ $Y_\nu(x)$ satisfies Bessel equation

for it is linear combination of known solutions $J_\nu(x)$ and $J_{-\nu}(x)$

However ☛ for $\nu \in \mathbb{Z}$ (3.3.275.) becomes indeterminate

In fact ☛ $Y_n(x)$ for integer n is defined as

$$Y_n(x) = \lim_{\nu \to n} Y_\nu(x) \quad (3.3.277.)$$

It is easily seen that explicit form of Y_ν for $\nu \in \mathbb{Z}$ is

$$Y_\nu(x) = \frac{2}{\pi}\left[J_\nu(x)\ln\left(\frac{x}{2}\right) - \left(\frac{x}{2}\right)^\nu \sum_{n=0}^{\infty}(-1)^n \frac{[\phi(n)+\phi(n+\nu)]}{2n!\,(n+\nu)!}\left(\frac{x^2}{4}\right)^n \right.$$
$$\left. - \left(\frac{x}{2}\right)^{-\nu}\sum_{n=0}^{\nu-1}\frac{(\nu-n-1)!}{2n!}\left(\frac{x^2}{4}\right)^{2n}\right] \qquad (3.3.278.)$$

$\phi(m) = \Gamma'(m+1)/\Gamma(m)$ and last sum is only present if $\nu \neq 0$

Asymptotic forms are

$$J_\nu(x) \approx \sqrt{\frac{2}{\pi x}}\,\cos\left[x - \left(\nu + \frac{1}{2}\right)\frac{\pi}{2}\right] \qquad (3.3.279.)$$

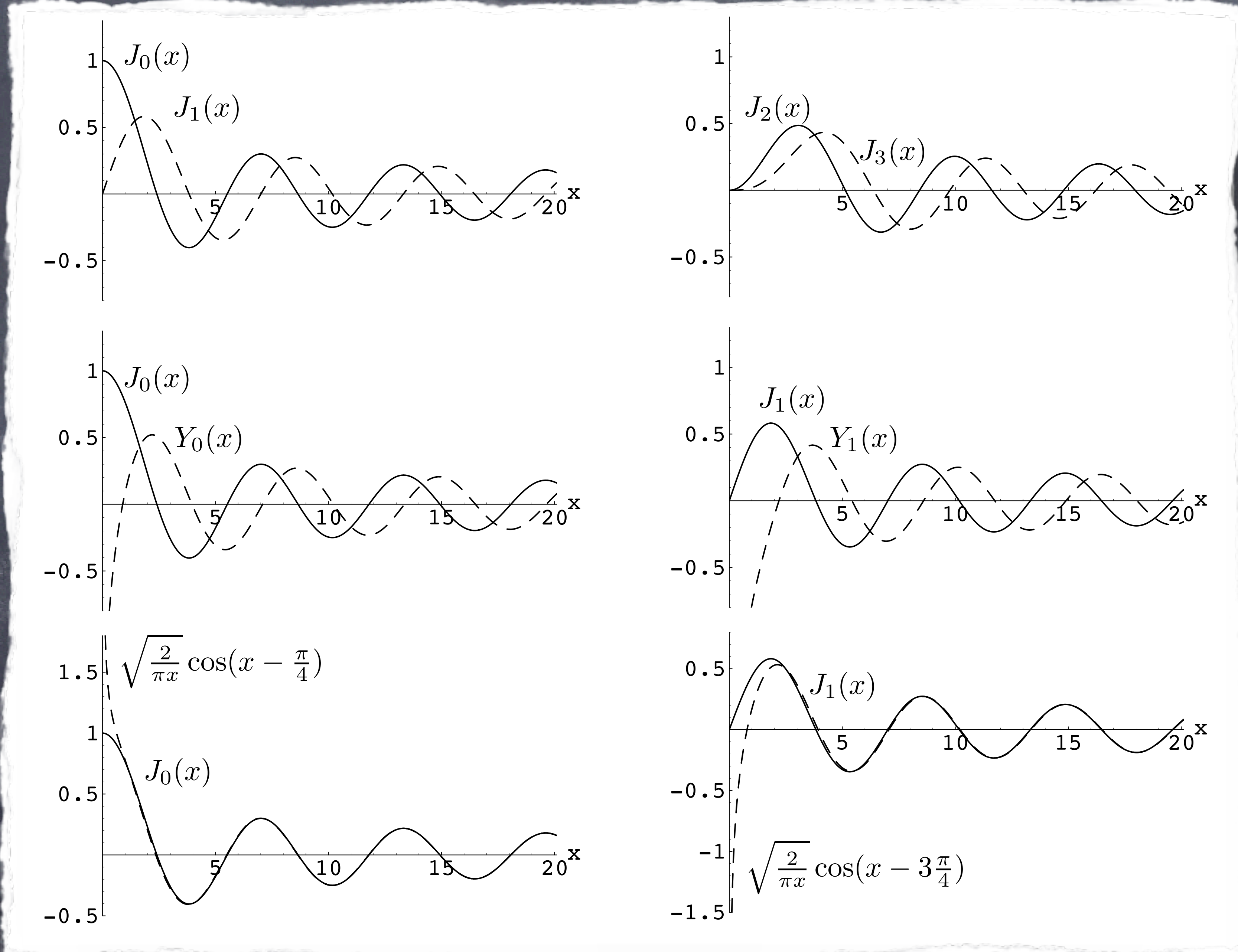
and

$$Y_\nu(x) \approx \sqrt{\frac{2}{\pi x}}\,\sin\left[x - \left(\nu + \frac{1}{2}\right)\frac{\pi}{2}\right] \qquad (3.3.280.)$$

Note that for $k \neq 0$ ☛ $J_\nu(kx)$ and $Y_\nu(kx)$ are solutions of

$$u'' + \frac{u'}{x} + \left(k^2 - \frac{\nu^2}{x^2}\right)u = 0 \qquad (3.3.281.)$$

Properties of various Bessel functions



Bessel functions of second kind are irregular at $x = 0$
but with increasing x ☛ become oscillatory

Theorem 3.3.8.

Last class we saw that for $p(x) > 0$ and $q(x)$ continuous in $[a, b]$

Sturm-Liouville eigenvalue problem (3.3.181.) can be written as

$$u'' + A(x)u' + B(x)u = 0 \qquad (3.3.288.)$$

with $$A(x) = \frac{p'(x)}{p(x)} \quad \text{and} \quad B(x) = \frac{\lambda\rho(x) - q(x)}{p(x)} \qquad (3.3.289.)$$

Consider the case in which $p(x)$ has a zero of order one at $x = a$

$$p(x) = c_1(x-a) + c_2(x-a)^2 + \ldots, \quad c_1 \neq 0 \qquad (3.3.290.)$$

and is positive and continuous in rest of interval $[a, r]$

In such a case $A(x)$ has a single pole with residue $A_{-1} = 1$

There is only one l.i. bounded solution of (3.3.288.) for $x \to a^+$

Then ☛ boundary condition to be imposed ↩

eigenfunction remains bounded for $x \to a$

Proof.

If $q(x)$ is continuous in $[a, b]$ then

$B(x)$ has at most a simple pole at $x = a$ and so $B_{-2} = 0$

roots of indicial equation at $x = a$ are:

$s = 0$ and $s = 1 - A_{-1} = 0$

one of solutions of (3.3.288.) is a power series

(which is bounded in $x = a$)

whereas other solution has a logarithmic divergence at $x = a$

Note that Sturm-Liouville operator remains self-adjoint if

$$\lim_{x \to a^+} p(x)u(x) = 0 \quad \text{and} \quad \lim_{x \to a^+} p(x)u'(x) = 0 \qquad (3.3.291.)$$

If $u(x)$ and $u'(x)$ are bounded for $x \to a^+$

then (3.3.291.) conditions are satisfied as $p(a) = 0$ by hypothesis

The above reasoning extends easily

to the case in which $q(x)$ has a simple pole at $x = a$

Corollary 3.3.3.

(i) Legendre polynomial series

Legendre equation (3.3.238.)

can be rewritten as a Sturm-Liouville eigenvalue problem

$$-\left[(1-x^2)u'\right]' = \lambda u, \quad x \in [-1,1] \qquad (3.3.292.)$$

$p(x) = 1 - x^2 = (1+x)(1-x)$ has a zero of order one at $x = \pm 1$

Boundary condition to be imposed is that $u(x)$ remains bounded

This condition determines eigenvalues $\lambda = l(l+1)$ with $l = 0, 1, \ldots$

$P_l(x)$ polynomials form a complete set in $[-1,1]$

☛ orthogonal basis of vector space of differentiable $f(x)$ to 2nd order

Any function $f(x) \in \mathcal{C}^2([-1,1])$ can be expanded as

$$f(x) = \sum_{l=0}^{\infty} c_l P_l(x)$$

Since $\int_{-1}^{1} P_l^2(x)dx = 2/(2l+1)$ we have ☛ $c_l = \frac{2l+1}{2} \int_{-1}^{1} f(x)\ P_l(x)\ dx$

(iii) Bessel series

(3.3.281.) also defines a Sturm-Lioville eigenvalue problem

$$-(xu')' + \frac{\nu^2}{x} u = \lambda x u \qquad (3.3.298.)$$

where $p(x) = x, q(x) = \nu/x$ and $\rho(x) = x$ with $\lambda = k^2$

As an illustration ☛ we consider here $x \in [0, a]$

Since $p(x)$ has a simple pole at $x = 0$

we impose the boundary condition $|u(0)| < \infty$

At $x = a$ we set a Dirichlet boundary condition: $u(a) = 0$

General solution of (3.3.298.) is

$$u(x) = A\, J_\nu(\sqrt{\lambda} x) + B\, Y_\nu(\sqrt{\lambda} x) \qquad (3.3.299.)$$

Boundary condition $|u(0)| < \infty$ sets $B = 0$

whereas $u(a) = 0$ leads to $J_\nu(\sqrt{\lambda} a) = 0$

that is $\sqrt{\lambda} a = k_n^\nu, \quad n = 1, 2, \ldots$ with $J_\nu(k_n^\nu) = 0$ (3.3.300.)

k_n^ν are roots of $J_\nu(x)$ that form a countable set in $\mathbb{R}$

Eigenfunctions associated with eigenvalues $\lambda = (k_n^\nu/a)^2$

are orthogonal in inner product $\langle u, v\rangle_x = \int_0^a u(x)\ v(x)\ x\ dx$

Eigenfunction set $\{J_\nu(k_n^\nu x/a),\ n = 1, 2, \ldots,\}$ is complete on $[0, a]$

because L is self-adjoint with present boundary conditions

We can then expand a function $f(x) \in [0, a]$ as

$$f(x) = \sum_{n=0}^{\infty} c_n\ J_\nu(k_n^\nu x/a) \tag{3.3.308.}$$

with $$c_n = \frac{2}{a^2 J_\nu'^{\,2}(k_n^\nu)} \int_0^a f(x)\ J_\nu(k_n^\nu x/a)\ x\ dx \tag{3.3.309.}$$

Note that summation in (3.3.308.) is over n and not over ν

First zeros of $J_0(x)$ are

$$k_1^0 \approx 2.405 = 0.765\pi,\ \ k_2^0 \approx 5.52 = 1.76\pi,\ \ k_3^0 \approx 8.65 = 2.75\pi$$

Asymptotic form ☛ $k_n^0 \approx (n - \frac{1}{4})\pi$


TO BE
CONTINUED....