Physics 307

Mathematical Physics

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ORDINARY DIFFERENTIAL EQUATIONS III

3.1 Setting the Stage ✓

3.2 Initial Value Problem ✓

Picard's existence and uniqueness theorem

Systems of first-order linear differential equations

Green matrix as a generalized function

3.3 Boundary Value Problem

Self-adjointness of Sturm-Liouville operator

Green function of Sturm-Liouville operator

Series solutions to homogeneous linear equations

3.4 Fourier Analysis

Fourier series

Fourier transform

BOUNDARY VALUE PROBLEM

Self-adjointness of Sturm-Liouville operator

So far we have seen differential equations with initial conditions

For a linear equation of second order

we have seen that integration constants are determined

from values of unknown function and its first derivative at $t = t_0$

We will begin to study problems of second order

in which constants of integration are determined

not by initial conditions ☛ but by boundary conditions

In these problems

range of variation of variable is restricted to a certain range
and integration constants are determined

from values of unknown function

or its derivative in extreme points of interval

Definition 3.3.1

General form of a second order linear equation reads

$$\frac{d^2u}{dx^2} + A(x)\frac{du}{dx} + B(x)u = F(x) \qquad (3.3.167.)$$

$A(x),\ B(x)$ and $F(x)$ are continuous functions

Preceding equation can be rewritten as

$$-\frac{d}{dx}\left[p(x)\frac{du}{dx}\right] + q(x)u = f(x) \qquad (3.3.168.)$$

$f(x) = -p(x)F(x),\ q(x) = -p(x)B(x)$ and $p(x) = e^{\int A(x)dx}$

Note that $p'(x) = A(x)p(x)$

and $(pu')' = pu'' + p'u' = p[u'' + A(x)u']$ (3.3.169.)

so (3.3.168.) reduces to (3.3.167.) multiplied by $-p(x)$

Eq. (3.3.168.) is generally written as $L[u(x)] = f(x)$ (3.3.170.)

where $L = -\frac{d}{dx}\left[p(x)\frac{d}{dx}\right] + q(x)$ ☛ is Sturm-Liouville (SL) operator (3.3.171.)

This operator acts on functions $u(x)$

defined in a given real interval $a \le x \le b$

and is only completely defined

after specifying values of unknown function or its first derivative

or linear combinations of them at boundaries (a, b)

These conditions are known as boundary conditions

and the functions that satisfy them

constitute domain of Sturm-Liouville operator

In this class ☛ we study the case in which $p(x)$ is non-zero in $[a, b]$

Next class we will study the case ⤸

in which $p(x)$ vanishes at one or both ends of the interval

that leads to study of so-called special functions

Throughout we let $[a, b]$ be a bounded interval in $\mathbb{R}$

$\mathcal{C}^{(n)}([a, b])$ denotes the space of functions

with derivatives of n-th order continuous up to endpoints

$p(x) \in \mathcal{C}^1([a, b])$ and $q(x) \in \mathcal{C}^0([a, b])$

$\mathcal{L}^2([a, b])$ is subspace of functions that vanish near endpoints

Theorem 3.3.1.

$\forall u, v \in \mathcal{L}^2([a,b])$ ☛ L is most general second order real operator

which can be formally self-adjoint $\langle v, L[u]\rangle = \langle L[v], u\rangle$ (3.3.172.)

$\langle v, u\rangle$ denotes usual inner product $\langle \mathbf{v}, \mathbf{u}\rangle = \int_a^b v^*(x)\, u(x)\, dx$

Proof.

Integration by parts leads to

$$\begin{aligned}\langle v, L[u]\rangle - \langle L[v], u\rangle &= \int_a^b [v(pu')' - u(pv')']dx \\ &= \int_a^b [(vpu')' - (upv')']\, dx \qquad (3.3.173.) \\ &= p\,[vu' - uv']_b^a\end{aligned}$$

For L to be self-adjoint ☛ we need to impose conditions on u, v at the endpoints to make right hand side of (3.3.173.) vanish

Definition 3.3.2.

A boundary condition B is an expression of the form

$$Bu = c_a u(a) + c_b u(b) + d_a u'(a) + d_b u'(b) \quad (3.3.174.)$$

for real constants ☛ $c_a,\ c_b,\ d_a,\ d_b$

Definition 3.3.3.

Boundary conditions B_1, B_2 are self-adjoint for L

if $\forall u, v \in \mathcal{C}^2([a,b])$ satisfying $B_1 u = B_2 u = B_1 v = B_2 v = 0$ ⤵

$$\int_a^b L[u]\ v\ dx = \int_a^b u\ L[v]\ dx \qquad (3.3.175.)$$

⬇

vanishing of $B_j u$ and $B_j v$

implies right-hand side of (3.3.173.) vanishes

Definition 3.3.4.

Local boundary conditions are those establishing a relationship between unknown function and its derivative in each edge separately

We say that $B_1 u = 0$ and $B_2 u = 0$ are local or separated b.c if B_1 and B_2 are independently chosen to guarantee that right hand side of (3.3.173.) vanish

- Dirichlet conditions $B_1 u = u(a)$ & $B_2 u = u(b)$
- Neumann conditions $B_1 u = u'(a)$ & $B_2 u = u'(b)$
- Robin-Cauchy conditions $c_a u(a) + d_a u'(a) = 0$ & $c_b u(b) + d_b u'(b) = 0$

These are separated boundary conditions

B_1 is a condition at a and B_2 is a condition at b

Any pair of separated conditions is self-adjoint for general L

Definition 3.3.5.

Non-local boundary conditions establish a relationship between value of unknown function and its derivative in one and other edge

Most common examples of non-separated boundary conditions are

➤Periodic conditions $B_1 u = u(b) - u(a)$ & $B_2 u = u'(b) - u'(a)$

➤Anti-Periodic conditions $B_1 u = u(b) + u(a)$ & $B_2 u = u'(b) + u'(a)$

These are self-adjoint for L if $p(b) = p(a)$

Next ☛ we discuss whether homogeneous equation $L[u] = 0$ with Dirichlet boundary conditions has non-trivial solutions $u(x) \neq 0$

Later ☛ we will see that

the non-existence of such solutions (a.k.a. zero modes) is a necessary and sufficient condition for inhomogeneous equation to have a unique solution via Green's function

Example 3.3.1.

Consider case with $p(x) = 1$ and $q(x) = 0$

$$L = -\frac{d^2}{dx^2} \quad (3.3.177.)$$

General solution of homogeneous equation is $u(x) = cx + d$ (3.3.178.)

Therefore ☛ if $u(a) = u(b) = 0 \Rightarrow c = d = 0$

and homogeneous equation only has trivial solution

Example 3.3.2.

For $L = -\frac{d^2}{dx^2} - k^2$

general solution of homogeneous equation can be written as

$$u(x) = ce^{ikx} + de^{-ikx} \quad (3.3.179.)$$

or equivalently

$$u(x) = c' \sin[k(x-a)] + d' \cos[k(x-a)] \quad (3.3.180.)$$

If $u(a) = 0 \Rightarrow d' = 0$

The condition $u(b) = 0$ leads to $c' \sin[k(b-a)] = 0$

which has a non-trivial solution $(c' \neq 0) \Leftrightarrow k(b-a) = n\pi, n \in \mathbb{Z}$

Otherwise ☛ $c' = 0$ and only solution is $u(x) = 0$

Definition 3.3.6.

Fix a positive weight function $\rho(x) \in \mathcal{C}^2([a,b])$

so that $\rho(x) \geq c > 0$ for $x \in [a,b]$

and consider the Sturm-Liouville eigenvalue problem

$$L\ u = \lambda\ \rho\ u, \quad \text{with} \quad B_1 u = B_2 u = 0 \qquad (3.3.181.)$$

We say that λ is an eigenvalue of L

if there is a non-zero solution $u \in \mathcal{C}^2([a,b])$ of (3.3.181.)

and we call u an eigenfunction

Lemma 3.3.1.

Let (L, B_1, B_2, ρ) be a self-adjoint Sturm-Liouville system

Eigenfunctions associated to different eigenvalues are orthogonal

in the inner product ☛ $\langle u, v\rangle_\rho = \int_a^b u^*(x)\ v(x)\ \rho(x)\ dx$ (3.3.182.)

(i) eigenfunction u is an eigenvector for operator $\rho^{-1}L$

i.e. $\rho^{-1}Lu = \lambda u$

(ii) $\langle \rho^{-1}Lu, v\rangle_\rho = \langle u, \rho^{-1}Lv\rangle_\rho$

so that $\rho^{-1}L$ is self-adjoint in domain $\mathcal{C}^2([a,b])$

with respect to inner product $\langle\ ,\ \rangle$

Proof.

If $L[u_i(x)] = \lambda_i u_i(x)$ and $L[u_j(x)] = \lambda_j u_j(x)$ we have

$$0 = \langle u_j, L[u_i]\rangle_\rho - \langle L[u_j], u_i\rangle_\rho = (\lambda_i - \lambda_j)\int_a^b u_i(x)\ u_j(x)\ \rho(x)dx$$

and so $$\int_a^b u_i(x)\ u_j(x)\ \rho(x)\ dx = 0 \text{ if } \lambda_i \neq \lambda_j \qquad (3.3.184.)$$

Theorem 3.3.2.

(i) For any self-adjoint SL system (L, B_1, B_2, ρ) there exists:

a countable set of eigenvalues $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \leq \ldots$

with associated eigenfunctions $u_1(x), u_2(x), \ldots, u_n(x) \ldots$

which satisfy $L[u_n(x)] = \lambda\rho(x)u_n(x)$

and are orthogonal with respect to inner product (3.3.182.)

For local boundary conditions ☛ $\lambda_i < \lambda_j$ if $i \neq j$

because there cannot be two l.i. solutions of $L[u] = \lambda\rho u$

for same λ satisfying (3.3.181.)

(as the Wronskian would be null)

(ii) Any function $f(x) \in \mathcal{C}^2([a,b])$ that satisfies boundary conditions

can be written in terms of eigenfunctions $u_n(x)$

as absolutely uniformly convergent series

$$f(x) = \sum_{n=1}^{\infty} c_n u_n(x) \qquad (3.3.185.)$$

Set of all eigenfunctions forms a basis

of vector space of differentiable functions to second order

that satisfy boundary conditions on interval $[a,b]$

In other words ☛ eigenfunctions of L form a complete set

Proof.

Note that space $\mathcal{L}^2_\rho([a,b])$

consisting of eigenfunctions for Sturm-Liouville problem

is space of measurable u on $[a,b]$ such that

$$\|u\|^2_{\mathcal{L}^2_\rho} = \int_a^b |u(x)|^2 \ \rho(x) \ dx < \infty \qquad (3.3.186.)$$

Since $\rho(x)$ is bounded from above and below

this is the same space of functions as $\mathcal{L}^2([a,b])$

but norm and inner product $\langle \, , \rangle_\rho$ are different

$u \to \sqrt{\rho}\ u$ is unitary map of $\mathcal{L}_\rho$ onto $\mathcal{L}^2 : \|\sqrt{\rho}\ u\|_{\mathcal{L}^2} = \|u\|_{\mathcal{L}^2_\rho}$

In particular

$\{u_j\}_{j=1}^\infty$ is an orthonormal basis for $\mathcal{L}^2_\rho \Leftrightarrow \{\sqrt{\rho}\ u_j\}_{j=1}^\infty$

is an orthonormal basis for $\mathcal{L}^2$

Coefficients c_n are given by

$$c_n = \frac{\int_a^b \rho(x)\ u_n(x)\ f(x)dx}{\int_a^b \rho(x)\ u_n^2(x)dx} = \frac{\langle u_n, f\rangle_\rho}{\langle u_n, u_n\rangle_\rho} \qquad (3.3.187.)$$

Indeed ☛ if we multiply (3.3.185.) by $\rho(x)u_n(x)$ and integrate

$$\int_a^b \rho(x)\ u_n(x)\ f(x)\ dx = \sum_{m=1}^{\infty} c_m \int_a^b \rho(x)u_m(x)\ u_n(x)$$

$$= c_n \int_a^b \rho(x)\ u_n^2(x)dx \qquad (3.3.188.)$$

where we have used orthogonality of eigenfunctions

For a given f ☛ coefficients of expansion are unique

If $\sum_{n=1}^{\infty} c_n u_n(x) = 0 \Rightarrow c_n = 0 \forall n$

Corollary 3.3.2.

Let $\rho : [a,b] \to \mathbb{R}$ be an arbitrary function such that $\rho(x) \geq c > 0$

The space ☛ $\mathcal{L}^2_\rho([a,b];\mathbb{C}) = \{f : [a,b] \to \mathbb{C} \text{ such that } \sqrt{\rho} f \in \mathcal{L}^2\}$

equipped with inner product

$$\langle f, g\rangle_\rho = \langle \sqrt{\rho} f, \sqrt{\rho} g\rangle_{\mathcal{L}^2} = \int_a^b \rho(x)\ f(x)\ g^*(x)dx$$

is a Hilbert space ☛ which we denote by $(\mathcal{L}^2_\rho([a,b];\mathbb{C}), \langle , \rangle_\rho)$

Green function of Sturm-Liouville operator

Definition 3.3.7.

The Green function is defined as as solution of the equation

$$L_x[G(x,x')] = \delta(x-x') \qquad (3.3.191.)$$

which satisfies (in first variable) Robin-Cauchy boundary conditions

i.e
$$c_a G(a,x') + d_a \frac{dG}{dx}|_{(x=a,x')} = 0\,,$$
$$c_b G(b,x') + d_b \frac{dG}{dx}|_{(x=b,x')} = 0 \qquad (3.3.192.)$$

with $a < x$ and $x' < b$

subscript in operator indicates that it acts

on the first variable of the argument of the Green's function

Theorem 3.3.3.

(i) Solution of (3.3.191.) exists and is unique

if and only if trivial solution $u(x) = 0 \forall x \in [a,b]$

is only solution of $L[u(x)] = 0$ subject to RC boundary condition

i.e. there are no zero modes

(ii) Solution of inhomogeneous equation with RC boundary conditions

is given by ☛ $$u(x) = \int_a^b G(x,x')\ f(x')\ dx' \qquad (3.3.193.)$$

Proof.

We first show that (ii) holds

if $G(x,x')$ exists then (3.3.193.) is a solution of (3.3.170.)

$$L[u(x)] = \int_a^b L_x[G(x,x')]\ f(x')\ dx' = \int_a^b \delta(x-x')\ f(x')\ dx' = f(x)$$

Note that L acts on first variable ☛ which is unaffected by integral

In addition ☛ u satisfies boundary condition

because G satisfies this condition

in first variable which is unaffected by integral

$$c_a u(a) + d_a u'(a) = \int_a^b \left[c_a G(a,x') + d_a \left.\frac{dG}{dx}\right|_{(x=a,x')} \right] f(x')dx' = 0$$

and

$$c_b u(b) + d_b u'(b) = \int_a^b \left[c_b G(b,x') + d_b \left.\frac{dG}{dx}\right|_{(x=b,x')} \right] f(x')dx' = 0$$

To show (i) we now construct Green function

Let $u_1(x)$ and $u_2(x)$ be two solutions of homogeneous equation

$$L[u(x)] = 0$$

each satisfying one of boundary conditions

$$c_a u_1(a) + d_a u_1'(a) = 0 \quad \text{and} \quad c_b u_2(b) + d_b u_2'(b) = 0 \qquad (3.3.194.)$$

For $x < x'$ we have $L[G(x,x')] = 0$ with $G(x,x') = c_1(x')u_1(x)$

satisfying boundary condition at $x = a$

In an analogous manner ☛ if $x > x'$ then $G(x,x') = c_2(x')u_2(x)$

satisfying boundary condition at $x = b$

Therefore ☛ $G(x,x') = \begin{cases} c_1(x')u_1(x) & x < x' \\ c_2(x')u_2(x) & x > x' \end{cases}$ (3.3.195.)

Integration of (3.3.191.) over $[x' - \epsilon, x' + \epsilon]$ (with $\epsilon > 0$)

$$-\int_{x'-\epsilon}^{x'+\epsilon} \frac{d}{dx}[p\,G'(x,x')]dx + \int_{x'-\epsilon}^{x'+\epsilon} q(x)\,G(x,x')dx = \int_{x'-\epsilon}^{x'+\epsilon} \delta(x - x')\,dx$$

leads to

$$-[p\,G'(x,x')]_{x=x'-\epsilon}^{x=x'+\epsilon} + \int_{x'-\epsilon}^{x'+\epsilon} q(x)\,G(x,x')dx = 1 \quad \text{(3.3.196.)}$$

Due to continuity of p and q this equation can only be satisfied if:

$G(x,x')$ is continuous

its derivative has a discontinuity of magnitude $-1/p(x')$ at $x = x'$

Indeed

$$\lim_{\epsilon\to 0}\int_{x'-\epsilon}^{x'+\epsilon} q(x)\,G(x,x')\,dx \to 0 \qquad (3.3.197.)$$

because q and G are continuous functions

and thus ☛ $$-p(x')\left\{\left.\frac{dG}{dx}\right|_{x'+\epsilon} - \left.\frac{dG}{dx}\right|_{x'-\epsilon}\right\} = 1 \qquad (3.3.198.)$$

or equivalently $$\left.\frac{dG}{dx}\right|_{x\to x'+} - \left.\frac{dG}{dx}\right|_{x\to x'-} = -\frac{1}{p(x')} \qquad (3.3.199.)$$

in summary

$-p(x)G'(x,x')$ must be of the form

$$\Theta(x-x') + \phi(x)$$

with ϕ a continuous function at $x = x'$

so that $(-pG')'$ contains a $\delta(x-x')$ term

We impose G requirements on (3.3.195.) to obtain

$$\begin{aligned} c_1(x')u_1(x') - c_2(x')u_2(x') &= 0 \\ c_1(x')u_1'(x') - c_2(x')u_2'(x') &= \frac{1}{p(x')} \end{aligned} \qquad (3.3.200.)$$

which determines

$$c_1(x') = -u_2(x')/C \quad \text{and} \quad c_2(x') = -u_1(x')/C \qquad (3.3.201.)$$

where

$$\begin{aligned} C &= p(x')[u_1(x')u_2'(x') - u_2(x')u_1'(x')] \\ &= [pW(u_1, u_2)]_{x=x'} \end{aligned} \qquad (3.3.202.)$$

with

$$W(u_1, u_2) = \begin{vmatrix} u_1 & u_2 \\ u_1' & u_2' \end{vmatrix} \qquad (3.3.203.)$$

and $u'(x) = du/dx$

The solution exists only if $C \neq 0$

that is ☛ only if Wronskian $W(u_1, u_2)$ is non-zero

Note that this is satisfied if $u_1(x)$ and $u_2(x)$

are two linearly independent solutions of $L[u] = 0$

In such a case ☛ C is a constant

$$\begin{aligned}
[p(u_1 u_2' - u_2 u_1')]' &= p'(u_1 u_2' - u_2 u_1') + p(u_1 u_2'' - u_2 u_1'') \\
&= u_1 (p u_2')' - u_2 (p u_1')' \\
&= q(u_1 u_2 - u_2 u_1) = 0 \qquad (3.3.204.)
\end{aligned}$$

Therefore ☛ if $C \neq 0$ we have

$$G(x, x') = \begin{cases} -u_1(x) u_2(x')/C & x \leq x' \\ -u_1(x') u_2(x)/C & x \geq x' \end{cases} \qquad (3.3.205.)$$

If $C = 0$ Green function does not exist

In this case solutions u_1 and u_2 are linearly dependent

i.e. $u_2(x) = c u_1(x)$ ☛ $u_1(x)$ satisfies boundary conditions at both ends

This implies that if $C = 0$ ☛ there is a non-trivial solution $u_1 \neq 0$ satisfying $L[u_1] = 0$ and RC boundary conditions

Green function exists ⇔ the only solution of homogeneous equation

$$L[u] = 0$$

that satisfies RC conditions is $u = 0$

This conclude proof of (i)

Additionally ☛ (3.3.205.) gives explicit expression for Green function

Linear operator $G[u(x)] = \int_a^b G(x, x')\; u(x')\; dx'$ (3.3.206.)

is then inverse of L operator and it is sometimes denoted also as L^{-1}

Let us note that:

(i) inverse of differential linear operator L is integral linear operator

($G(x, x')$ is known as kernel of the integral operator)

(ii) G depends not only on coefficients $p(x)$ and $q(x)$ of L

but also on the boundary condition

(iii) symmetry of (3.3.205.) yields

$$G(x, x') = G(x', x) \qquad (3.3.207.)$$

and it allows to state theorem that follows

Theorem 3.3.4. [Reciprocity of Green function]

Response of system in x to a point source in x'

is identical to response of system in x' to a point source in x

even if p and q depend on x

This is due to self-adjointness of L

$$\langle L_x G(x,x'), G(x,x'')\rangle = \langle G(x,x'), L_x G(x,x'')\rangle \quad (3.3.208.)$$

Using differential equation that satisfies Green function

and definition of Dirac distribution we have

$$\int_a^b \delta(x-x')\; G(x,x'')\; dx = \int_a^b G(x,x')\; \delta(x-x'')\; dx \quad (3.3.209.)$$

which leads to $G(x',x'') = G(x'',x')$ (3.3.210.)

Inverse operator G is also self-adjoint

$$\langle v, G[u]\rangle = \int_a^b \int_a^b v(x)\; G(x,x')\; u(x')\; dx\; dx' = \langle G[v], u\rangle$$

(3.3.211.)

Note that Green's function (3.3.205.)

is not invariant under space translations (due to boundary conditions)

Translational invariance is broken ☛ even if p and q are constants

Therefore ☛ $G(x, x') \neq G(x - x')$

Solution (3.3.193.) can be rewritten as

$$u(x) = -\frac{1}{C}\left[u_2(x)\int_a^x u_1(x')\ f(x')\ dx' + u_1(x)\int_x^b u_2(x')\ f(x')\ dx'\right]$$

and one can explicitly verify that $L[u] = f$

It is always possible to write solution in form or $u(x) = u_p(x) + u_h(x)$

u_p ☛ particular solution of inhomogeneous equation $(L[u_p] = f)$

u_h ☛ solution of homogeneous equation $(L[u_h] = 0)$

subject to boundary condition

Example 3.3.3.

Consider case $p(x) = 1$ and $q(x) = 0$

i.e. ☛ $$L = -\frac{d}{dx^2} \qquad (3.3.212.)$$

Let us set $a = 0, b > 0$ and $u(0) = u(b) = 0$

It follows that $u_1(x) = x$ and $u_2(x) = x - b$ with $C = x - (x - b) = b$

Then we obtain

$$G(x, x') = \begin{cases} x(b - x')/b & x \leq x' \\ x'(b - x)/b & x \geq x' \end{cases} \qquad (3.3.213.)$$

Solution of inhomogeneous equation

$$-\frac{d^2u}{dx^2} = f(x) \quad (0 \leq x \leq b) \qquad (3.3.214.)$$

with $u(a) = u(b) = 0$ is then

$$\begin{aligned} u(x) &= \int_a^b G(x, x')\ f(x')\ dx \\ &= \frac{1}{b}\left[\int_0^x x'(b - x)\ f(x')\ dx' + \int_x^b x\ (b - x')\ f(x')\ dx'\right] \end{aligned}$$

If $f(x) = x^2$ we have

$$u(x) = \frac{1}{12}x(b^3 - x^3) = -\frac{x^4}{12} + \frac{x\ b^3}{12} \qquad (3.3.215.)$$

that consists of particular solution $-x^4/12$

and solution of homogeneous equation $xb^3/12$

such that $u(0) = u(b) = 0$

Example 3.3.4.

Let

$$L = -\frac{d^2}{dx^2} - \omega^2 \qquad (3.3.216.)$$

with $a = 0, b > 0$

In this case ☛ for $u(a) = u(b) = 0$

we have ☛ $u_1(x) = \sin(\omega x)$ and $u_2(x) = \sin(\omega(x - b))$ (3.3.217.)

which lead to

$$C = \omega\left[\sin(\omega x)\cos(\omega(x - b)) - \cos(\omega x)\sin(\omega(x - b))\right] = \omega\sin(\omega b)$$

Green function exists only if $\sin(\omega b) \neq 0$ that is $\omega \neq n\pi/b$ with $n \in \mathbb{Z}$

in such a case we have

$$G(x,x') = \frac{1}{\omega \sin(\omega b)} \begin{cases} \sin(\omega x)\sin(\omega(b-x')) & x \leq x' \\ \sin(\omega x')\sin(\omega(b-x)) & x \geq x' \end{cases} \quad (3.3.218.)$$

Note that for $\omega \to 0$ we recover (3.3.213.)

If $\omega = ik$ with $k \in \mathbb{R}$ ☛ Green function exists $\forall k \neq 0$

$G(x,x')$ follows from (3.3.218.) with substitution $\omega \to k$ & $\sin \to \sinh$

Example 3.3.5.

Let us consider again $L = -\frac{d}{dx^2}$ operator

with boundary condition $u'(a) = u'(b) = 0$

Green function does not exist because

$$u_1(x) = c_1 \quad \text{and} \quad u_2(x) = c_2 \quad (3.3.219.)$$

yielding $C = 0$

This is due to constant solution $u(x) = c \neq 0$

non-zero solution of $L[u] = 0$ and satisfies $u'(a) = u'(b) = 0$

In this case ☛ solution of inhomogeneous problem

if it exists ➡ is not unique

For a given a solution one can always add up an arbitrary constant which satisfies homogeneous equation and the boundary condition

Example 3.3.6.

Finally ☛ consider $L = -\frac{d^2}{dx^2} - \omega^2$ operator

with boundary condition $u'(a) = u'(b) = 0$

It follows that

$$u_1(x) = \cos(\omega x) \quad \text{and} \quad u_2(x) = \cos\left(\omega(x-b)\right) \qquad (3.3.220.)$$

with $C = -\omega \sin(\omega b)$

Green function exists only if $\sin(\omega b) \neq 0$ that is $\omega \neq n\pi/b$

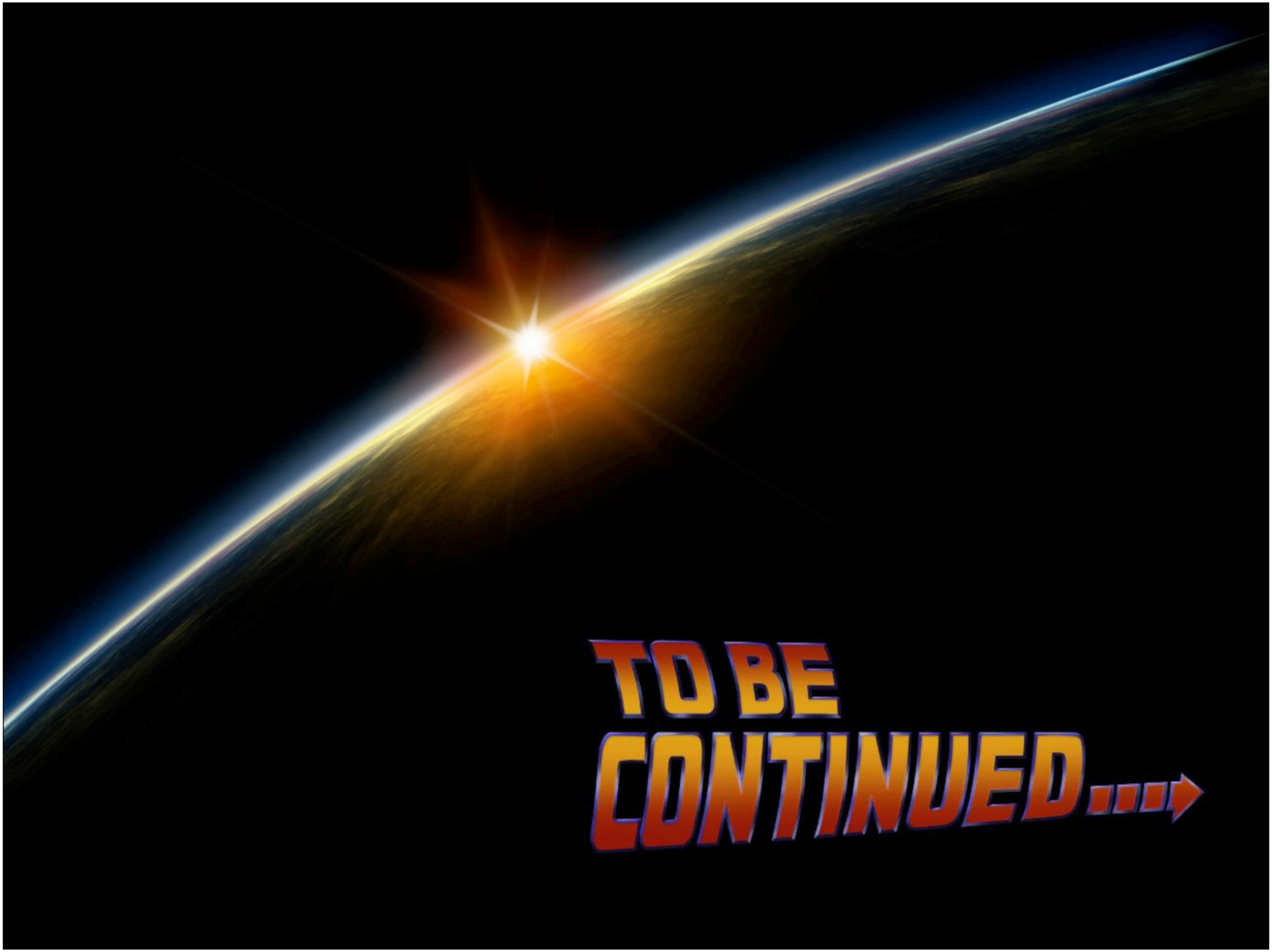
with $n \in \mathbb{Z}$

In such a case we have

$$G(x,x') = \frac{1}{\omega \sin(\omega b)} \begin{cases} \cos(\omega x) \cos\left(\omega(b-x')\right) & x \leq x' \\ \cos(\omega x') \cos\left(\omega(b-x)\right) & x \geq x' \end{cases} \qquad (3.3.221.)$$

If $\omega \to 0$ ☛ $|G(x,x')| \to \infty$

On other hand ☛ if $\omega = ik$ with $k \in \mathbb{R}, G(x,x')$ exists $\forall k \neq 0$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