Physics 307

Mathematical Physics

Luis Anchordoqui

ORDINARY DIFFERENTIAL EQUATIONS II

3.1 Setting the Stage ✓

3.2 Initial Value Problem

Picard's existence and uniqueness theorem

Systems of first-order linear differential equations

Green matrix as a generalized function

3.3 Boundary Value Problem

Self-adjointness of Sturm-Liouville operator

Green function of Sturm-Liouville operator

Series solutions to homogeneous linear equations

3.4 Fourier Analysis

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SYSTEM OF FIRST-ORDER LINEAR DIFFERENTIAL EQUATIONS

Picard's theorem can be generalized

to a system of first order ordinary differential equations

$$\frac{du_i}{dt} = f_i(t, u_1, \ldots, u_n), \quad i = 1, \ldots, n \qquad (3.2.70.)$$

$$\frac{d\mathbf{u}}{dt} = \mathbf{f}(t, \mathbf{u}) \qquad (3.2.71.)$$

$$\mathbf{u}(t) = \begin{pmatrix} u_1(t) \\ \vdots \\ u_n(t) \end{pmatrix} \quad \text{and} \quad \mathbf{f}(t, \mathbf{u}) = \begin{pmatrix} f_1(t, \mathbf{u}) \\ \vdots \\ f_n(t, \mathbf{u}) \end{pmatrix} \qquad (3.2.72.)$$

Proof is exactly same as proof of Picard's theorem

with substitution of f, u, v by $\mathbf{f}, \mathbf{u}, \mathbf{v}$

Proposition 3.2.1.

Any ordinary differential equation of order n

$$\frac{d^{(n)}u}{dt^n} = f\left(t, u, \frac{du}{dt}, \ldots, \frac{d^{(n-1)}u}{dt^{n-1}}\right) \quad (3.2.75.)$$

can can be written as a system of n first-order d.e.'s by defining a new family of unknown functions

$$u_1 = u,\ u_2 = \frac{du}{dt},\ \ldots,\ u_n = \frac{d^{(n-1)}u}{dt^{n-1}} \quad (3.2.76.)$$

with

$$\frac{du_1}{dt} = u_2, \frac{du_2}{dt} = u_3, \ldots, \frac{du_n}{dt} = f(t, u_1, \ldots, u_n) \quad (3.2.77.)$$

that is $f_1(t, \mathbf{u}) = u_2,\ f_2(t, \mathbf{u}) = u_3,\ \ldots, f_n(t, \mathbf{u}) = f(t, \mathbf{u})$

In addition ☛ if f satisfies hypotheses of Picard's theorem it is guaranteed existence and uniqueness of a solution to (3.2.75) for initial condition $\mathbf{u}(0) = \mathbf{u_0}$

$$\mathbf{u_0} = \left(u(0),\ \left.\frac{du}{dt}\right|_{t=0}, \ldots, \left.\frac{d^{(n-1)}u}{dt^{n-1}}\right|_{t=0}\right) \quad (3.2.78.)$$

Similarly ☛ a system of m coupled differential equations of order n can be reduced to a system of $n \times m$ first order equations

Definition 3.2.2.

A system of first-order ordinary differential equations of form (3.2.70.) is called linear if it can be written as

$$\frac{d\mathbf{u}}{dt} = \mathbb{A}(t)\,\mathbf{u} + \mathbf{f}(t) \qquad (3.2.79.)$$

where

$$\begin{pmatrix} a_{11}(t) & \dots & a_{1n}(t) \\ \vdots & \dots & \vdots \\ a_{n1}(t) & \dots & a_{nn}(t) \end{pmatrix} \qquad (3.2.80.)$$

are matrix-valued functions

The initial condition is given by $\mathbf{u}(t_0) = \mathbf{u_0}$

More explicitly

$$\frac{du_i(t)}{dt} = \sum_{j=1}^{n} a_{ij}(t)\,u_j(t) + f_i(t) \quad i = 1, \dots n \qquad (3.2.81.)$$

with initial values $u_i(t_0)$ for $i = 1, \dots n$

Theorem 3.2.3. [Superposition principle]

The solutions of a linear homogeneous n-vector system

$$\frac{d\mathbf{u}}{dt} = \mathbb{A}\mathbf{u} \qquad (3.2.82.)$$

form a linear space V of dimension n

(3.2.82) can be rewritten as $L[\mathbf{u}] = \mathbf{0}$ with $L \equiv d/dt - \mathbb{A}(t)$

Proof.

If $\mathbf{u}_1$ and $\mathbf{u}_2$ are solutions ☛ so is linear combination $c_1\mathbf{u}_1 + c_2\mathbf{u}_2$

(as can be verified directly by substitution)

This shows solutions form a vector space

We next demonstrate that

there exist exactly n linearly independent solution vectors

Since $\mathbf{u}$ has n components
we can find n linearly independent vectors $\mathbf{u}_j^0$

e.g. $$\mathbf{u_1}^0 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \mathbf{u_2}^0 = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \ldots, \mathbf{u_n}^0 = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix} \quad (3.2.83.)$$

Let
$$\mathbf{u}_j(t) = \begin{pmatrix} u_{1j}(t) \\ u_{2j}(t) \\ \vdots \\ u_{nj}(t) \end{pmatrix} \quad j = 1, \ldots n \quad (3.2.84.)$$

be solution of (3.2.82.) with initial condition $\mathbf{u}_j(t_0) = \mathbf{u}_j^0$

Invoking Picard's theorem we know there exists a unique solution for $|t - t_0| \leq r$ (i.e. for $t \in I_0$)

Consider now a given solution $\mathbf{u}(t)$ with i.c. $\mathbf{u}(t_0) = \mathbf{u_0}$

For $t = t_0$ ☛ the vectors $\mathbf{u}_j(t_0) = \mathbf{u}_j^0$ form a basis

and so we can write ☛ $$\mathbf{u}(t_0) = \sum_{j=1}^{n} c_j \mathbf{u_j}(t_0) \quad (3.2.85.)$$

However ☛ we know that for a given initial condition the solution must be unique ☛ therefore

$$\mathbf{u}(t) = \sum_{j=1}^{n} c_j \mathbf{u_j}(t) \qquad (3.2.86.)$$

must hold $\forall t \in I_0$

This shows that dimension of space is n

Finally we show ☛ n solutions remain linearly independent $\forall t \in I_0$

If the solutions were linearly dependent then there would exist a solution of form (3.2.86.) with $c_1, c_2, \ldots c_n$ not all zero that could be zero vector

e.g. ☛ for $t = t_1$ ➡ $\mathbf{u}(t_1) = \sum_{j=1}^{n} c_j \mathbf{u}_j(t_1) = \mathbf{0}$ (3.2.87.)

since there $\exists$ trivial solution $\mathbf{u}(t) = \mathbf{0} \forall t \in I_0$

because of uniqueness (3.2.87.) must coincide with trivial solution $\forall t \in I_0$

☛ $c_1 = c_2 = \cdots = c_n = 0$ in contradiction with our assumption

Definition 3.2.3. [Fundamental matrix]

A square matrix whose columns are
linearly independent solutions of homogeneous system (3.2.82.)

$$\mathbb{U}(t) = \begin{pmatrix} u_{11}(t) & u_{12}(t) & \dots & u_{1n}(t) \\ u_{21}(t) & u_{22}(t) & \dots & u_{2n}(t) \\ \vdots & \vdots & \dots & \vdots \\ u_{n1}(t) & u_{n2}(t) & \dots & u_{nn}(t) \end{pmatrix} \qquad (3.2.88.)$$

is called fundamental matrix

Since $d\mathbf{u_j}/dt = \mathbb{A}(t)\mathbf{u_j}$ with $\mathbf{u_j}(0) = \mathbf{u_j}^0$

$$\frac{d\mathbb{U}}{dt} = \mathbb{A}(t)\mathbb{U}(t), \quad \text{with} \quad \mathbb{U}(t_0) = \mathbb{U}_0 \qquad (3.2.89.)$$

$\mathbb{U}_0$ ☛ matrix containing n linearly independent i.c. $\mathbf{u_j}^0$

In particular ☛ for (3.2.83.) $\mathbb{U}_0 = \mathbb{I}$

Since determinant of n linearly independent vectors is non-zero

$$\det\left[\mathbb{U}(t_0)\right] \neq 0$$

then ☛ from Picard's theorem it follows that $\det\left[\mathbb{U}(t)\right] \neq 0$

The general solution of (3.2.82.) reads

$$\mathbf{u}(t) = \mathbb{U}(t)\,\mathbf{c} \qquad (3.2.90.)$$

with $\mathbf{c}$ a constant vector

For $\mathbf{c} = \mathbb{U}^{-1}(t_0)\mathbf{u}_0$

a particular solution with initial condition $\mathbf{u}(t_0) = \mathbf{u}_0$ reads

$$\mathbf{u}(t) = \mathbb{U}(t)\mathbb{U}^{-1}(t_0)\mathbf{u}_0 \qquad (3.2.91.)$$

For i.c. given in (3.2.83.) ☛ $\mathbb{U}(t_0) = \mathbb{I}$ and $\mathbf{u}(t) = \mathbb{U}(t)\mathbf{u}_0$

For $t \in I_0$ the general form of $\mathbb{U}(t)$ follows from Picard's theorem

$$\mathbb{U}(t) = \left[\mathbb{I} + \int_{t_0}^{t} \mathbb{A}(t')dt' + \int_{t_0}^{t} \mathbb{A}(t')dt' \int_{t_0}^{t'} \mathbb{A}(t'')dt'' + \ldots\right]\mathbb{U}_0$$

(3.2.92.)

Definition 3.2.4. [Green matrix]

If $\mathbb{U}(t)$ is fundamental matrix of homogeneous system (3.2.82.) we can write a particular solution of original system (3.2.79.) as

$$\mathbf{u}(t) = \mathbb{U}(t)\ \mathbf{c}(t) \qquad (3.2.97.)$$

From (3.2.89.) it follows that

$$\frac{d\mathbf{u}}{dt} = \frac{d\mathbb{U}}{dt}\mathbf{c} + \mathbb{U}\frac{d\mathbf{c}}{dt} = \mathbb{A}(t)\mathbb{U}\mathbf{c} + \mathbb{U}\frac{d\mathbf{c}}{dt} \qquad (3.2.98.)$$

thus ☛ the inhomogeneous system (3.2.79) can be rewritten as

$$\frac{d\mathbf{u}}{dt} - \mathbb{A}(t)\mathbf{u} = \mathbb{U}(t)\frac{d\mathbf{c}}{dt} = \mathbf{f}(t) \qquad (3.2.99.)$$

from which we obtain the relation

$$\frac{d\mathbf{c}}{dt} = \mathbb{U}^{-1}(t)\ \mathbf{f}(t) \qquad (3.2.100.)$$

and we have

$$\mathbf{c} = \int \mathbb{U}^{-1}(t)\ \mathbf{f}(t)\ dt \qquad (3.2.101.)$$

The general solution is of the form

$$\mathbf{u}(t) = \mathbb{U}(t)\left[\mathbf{c} + \int \mathbb{U}^{-1}(t)\ \mathbf{f}(t)\ dt\right] \qquad (3.2.102.)$$

Particular solution for $\mathbf{u}(t_0) = \mathbf{u}_0$ reads

$$\begin{aligned}\mathbf{u}(t) &= \mathbb{U}(t)\left[\mathbb{U}^{-1}(t_0)\mathbf{u}_0 + \int_{t_0}^{t} \mathbb{U}^{-1}(t')\ \mathbf{f}(t')\ dt'\right] \\ &= \mathbb{K}(t,t_0)\mathbf{u}_0 + \int_{t_0}^{t} \mathbb{K}(t,t')\ \mathbf{f}(t')\ dt' \qquad (3.2.102.)\end{aligned}$$

with

$$\mathbb{K}(t,t') = \mathbb{U}(t)\ \mathbb{U}^{-1}(t') \qquad (3.2.103.)$$

It is important to stress that $\mathbb{K}(t,t')$ satisfies

$$\frac{d\mathbb{K}(t,t')}{dt} = \mathbb{A}(t)\ \mathbb{K}(t,t') \qquad (3.2.104.)$$

with

$$\mathbb{K}(t',t') = \mathbb{I} \qquad (3.2.105.)$$

(extension of superposition principle)

If $\mathbf{u_1}(t)$ and $\mathbf{u_2}(t)$ are particular solutions for $\mathbf{f_1}(t)$ and $\mathbf{f_2}(t)$

then $\mathbf{u}(t) = c_1\mathbf{u_1}(t) + c_2\mathbf{u_2}(t)$ is also a particular solution

for $\mathbf{f}(t) = c_1\mathbf{f_1}(t) + c_2\mathbf{f_2}(t)$

We can decompose force in several terms or components and then add solutions for each of them

Consider solution with initial condition $\mathbf{u}_0 = 0$

From (3.2.103.) it follows that

$$\mathbf{u}(t) = \int_{t_0}^{t} \mathbb{K}(t,t')\ \mathbf{f}(t')\ dt' \qquad (3.2.106.)$$

If in addition $\mathbf{f}(t) = 0\ \forall t < 0$ and system is in equilibrium for $t < 0$

(with $\mathbf{u}(t) = 0\ \forall t < 0$)

we can rewrite (3.2.106.) as

$$\mathbf{u}(t) = \int_{-\infty}^{t} \mathbb{K}(t,t')\ \mathbf{f}(t')\ dt' \qquad (3.2.107.)$$

or equivalently

$$\mathbf{u}(t) = \int_{-\infty}^{\infty} \mathbb{G}(t,t')\ \mathbf{f}(t')\ dt' \qquad (3.2.108.)$$

with $$\mathbb{G}(t,t') = \begin{cases} 0 & 0 \le t \le t' \\ \mathbb{K}(t,t') & 0 \le t' \le t \end{cases} \qquad (3.2.109.)$$

The matrix $\mathbb{G}$ is called Green matrix of system

it allows one to express at a given value of t

the effect from a source that acts in any $t' < t$

For problems of initial values the variable t is in general time

and so $\mathbb{G}(t, t')$ is commonly called causal Green matrix

Note that $\mathbb{G}$ is discontinuous at $t = t'$ because $\lim_{t \to t'^+} = \mathbb{I}$ and $\lim_{t \to t'^-} = 0$

Example 3.2.3.

Let $\mathbb{A}$ in (3.2.79.) be independent of time ☛ that is $a_{ij}(t) = a_{ij} \ \forall i, j$

The homogeneous system

$$\frac{d\mathbf{u}}{dt} = \mathbb{A}\mathbf{u} \qquad (3.2.110.)$$

is now invariant under temporal translations

Without loss of generality we take $t_0 = 0$

indeed if $\mathbf{u}(t)$ is a solution of (3.2.82.) with $\mathbf{u}(0) = \mathbf{u}_0$

$\Rightarrow \mathbf{u}(t - t_0)$ is also a solution of (3.2.82.) for $\mathbf{u}(t_0) = \mathbf{u}_0$

From (3.2.92.) it follows that

$$\mathbb{U}(t) = \left[\mathbb{I} + \mathbb{A}t + \mathbb{A}^2\frac{t^2}{2!} + \dots\right]\mathbb{U}_0 = \left[\sum_{n=0}^{\infty} \frac{\mathbb{A}^n\ t^n}{n!}\right]\mathbb{U}_0 = e^{\mathbb{A}t}\mathbb{U}_0 \quad (3.2.111.)$$

series converges for any given square matrix of finite dimension m

if $|a_{ij}| \leq K \forall i,j \Rightarrow |(a^2)_{ij}| \leq mK^2$

and in general $|(a^n)_{ij}| \leq (mK)^n/m$ so that $|(e^a)_{ij}| \leq e^{mK}/m$

Next ☛ we verify that (3.2.111.) is a solution of (3.2.105.) $\forall t$

$$\frac{d}{dt}e^{\mathbb{A}t} = \frac{d}{dt}\sum_{n=0}^{\infty}\frac{\mathbb{A}^n t^n}{n!} = \sum_{n=1}^{\infty}\frac{\mathbb{A}^n t^{n-1}}{(n-1)!} = \mathbb{A}e^{\mathbb{A}t} \quad (3.2.112.)$$

General solution of homogeneous equation is given by

$$\mathbf{u}(t) = e^{\mathbb{A}t}\mathbf{c} \quad (3.2.113.)$$

Particular solution with $\mathbf{u}(t_0) = \mathbf{u}_0$ reads

$$\mathbf{u}(t) = e^{\mathbb{A}(t-t_0)}\mathbf{u}_0 \quad (3.2.114.)$$

Solution of inhomogeneous equation with $\mathbf{u}(t_0) = \mathbf{u}_0$ is given by

$$\mathbf{u}(t) = e^{\mathbb{A}(t-t_0)}\mathbf{u}_0 + \int_{t_0}^{t} e^{\mathbb{A}(t-t')}\ \mathbf{f}(t')\ dt' \quad (3.2.115.)$$

which corresponds to $\mathbb{K}(t,t') = e^{\mathbb{A}(t-t')}$ in (3.2.103)

Green matrix as a generalized function

Definition 3.2.5. [Dirac delta function as a limit]

Consider inhomogeneous linear differential equation

$$\frac{du}{dt} - au = f(t) \qquad (3.2.116.)$$

From an intuitive point view ☛ it seems reasonable to represent the inhomogeneity f as a sum of impulsive terms concentrated in very small time intervals and then obtain the solution as sum of individual solutions for each of these terms

Formalization of this idea requires concept of distribution (or generalized function)

Consider function $$g_\epsilon(x) = \begin{cases} 1/\epsilon & |x| \le \epsilon/2 \\ 0 & |x| > \epsilon/2 \end{cases}$$ with $\epsilon > 0$ (3.2.117.)

it follows that $$\int_{-\infty}^{\infty} g_\epsilon(x)dx = 1 \;\forall \epsilon > 0$$

In addition ☛ if f is an arbitrary continuous function

$$\int_{-\infty}^{\infty} g_\epsilon(x)\ f(x)dx = \epsilon^{-1}\int_{-\epsilon/2}^{\epsilon/2} f(x)\ dx = \frac{F(\epsilon/2) - F(-\epsilon/2)}{\epsilon} \qquad (3.2.118.)$$

F ☛ is primitive of f

For $\epsilon \to 0^+$ ☛ $g_\epsilon(x)$ is concentrated near the origin yielding

$$\lim_{\epsilon\to 0^+}\int_{-\infty}^{\infty} g_\epsilon(x) f(x)\ dx \ = \ \lim_{\epsilon\to 0^+}\frac{F(\epsilon/2) - F(-\epsilon/2)}{\epsilon} \qquad (3.2.119.)$$

$$= \ F'(0) = f(0)$$

We can define distribution $\delta(x)$ as the limit

$$\delta(x) = \lim_{\epsilon\to 0^+} g_\epsilon(x) \qquad (3.2.120.)$$

satisfying $\displaystyle\int_{-\infty}^{\infty} dx\ \delta(x)\ f(x) = f(0)$ (3.2.121.)

Although limit (3.2.120.) does not strictly exist

(it is 0 if $x \neq 0$ and ∞ if $x = 0$)

limit of integral (3.2.119.)

$\exists\ \forall f$ continuous in an interval centered at $x = 0$

➡ this is meaning of (3.2.120) and (3.2.121)

Note that if $a \neq b$ and $a < b$

$$\int_a^b \delta(x)\ f(x)\ dx = \lim_{\epsilon \to 0^+} \int_a^b g_\epsilon(x)\ f(x)\ dx = \begin{cases} f(0) & a < 0 < b \\ 0 & a < b < 0 \text{ or } 0 < a < b \end{cases}$$

We will consider from now on test functions f

which are bounded and differentiable functions to any order

and which vanish outside a finite range I

Remember first and foremost that such functions exist:

If $f(x) = 0$ for $x \leq 0$ and $x \geq 1$

and $f(x) = e^{-1/x^2} e^{-1/(1-x)^2}$ for $|x| < 1$

Function f has derivatives of any order at $x = 0$ and $x = 1$

In this case there are many other functions $g_\epsilon(x)$

that converge to $\delta(x)$ with derivatives of all orders

A well-known example is

$$\delta(x) = \lim_{\epsilon\to 0^+} \frac{e^{-x^2/2\epsilon^2}}{\sqrt{2\pi}\epsilon} \qquad (3.2.122.)$$

indeed

$$\frac{1}{\sqrt{2\pi}\epsilon}\int_{-\infty}^{\infty} e^{-x^2/2\epsilon^2} dx = 1 \ \forall \epsilon > 0 \qquad (3.2.123.)$$

and

$$\lim_{\epsilon\to 0^+} \frac{1}{\sqrt{2\pi}\epsilon}\int_{-\infty}^{\infty} e^{-x^2/2\epsilon^2} f(x) dx = f(0) \qquad (3.2.124.)$$

Here

$$g_\epsilon(x) = \frac{1}{\sqrt{2\pi}\ \epsilon} e^{-x^2/2\epsilon^2} \qquad (3.2.125.)$$

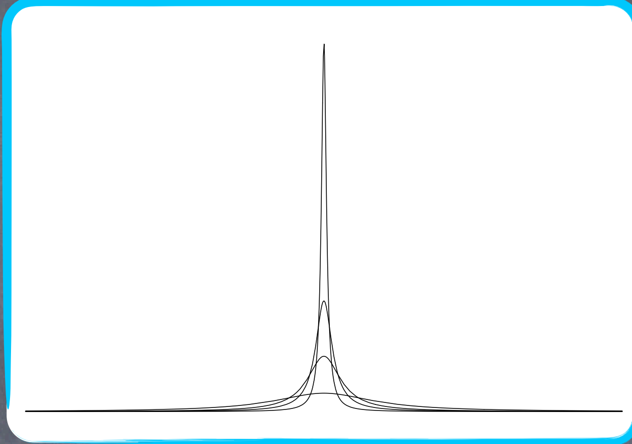
is normal (or Gaussian) distribution with area 1 and variance

$$\int_{-\infty}^{\infty} g_\epsilon\, x^2\, dx = \epsilon^2 \qquad (3.2.126.)$$

When $\epsilon \to 0^+$

$g_\epsilon(x)$ concentrates around $x = 0$

keeping its area constant ☛



In general ☛ if $g_\epsilon(x)$ is defined $\forall x \in \Re\mathrm{e}$ and $\epsilon > 0$ we have

$$\lim_{\epsilon \to 0^+} g_\epsilon(x) = \delta(x) \Leftrightarrow \lim_{\epsilon \to 0^+} \int_{-\infty}^{\infty} g_\epsilon(x) f(x) dx = f(0) \qquad (3.2.127.)$$

$\forall$ test function f

For example ☛ if $g(x) \geq 0 \forall x$

and $\displaystyle\int_{-\infty}^{\infty} g_\epsilon(x)\, dx = 1 \Rightarrow \lim_{\epsilon \to 0^+} \epsilon^{-1} g(x/\epsilon) = \delta(x)$

Indeed ☛ if $\epsilon > 0$

$$\frac{1}{\epsilon}\int_{-\infty}^{\infty} g(x/\epsilon)dx = \int_{-\infty}^{\infty} g(u)du = 1$$

and

$$\lim_{\epsilon\to 0^+}\frac{1}{\epsilon}\int_a^b g(x/\epsilon)dx = \lim_{\epsilon\to 0^+}\int_{a/\epsilon}^{b/\epsilon} g(u)du = \begin{cases} 1 & a < 0 < b \\ 0 & a < b < 0 \text{ or } 0 < a < b \end{cases}$$

Therefore ☛ if $|f(x)| \leq M \ \forall x$ and $ab > 0$

$$\lim_{\epsilon\to 0^+}\frac{1}{\epsilon}\left|\int_a^b g(x/\epsilon)\ f(x)\ dx\right| \leq M \lim_{\epsilon\to 0^+}\frac{1}{\epsilon}\int_a^b g(x/\epsilon)dx = 0 \quad (3.2.129.)$$

It follows that ☛ if $t > 0$ and f is continuous and bounded

$$I_f \equiv \lim_{\epsilon\to 0^+}\frac{1}{\epsilon}\int_{-\infty}^{\infty} f(x)\ g(x/\epsilon)\ dx = \lim_{\epsilon\to 0^+}\frac{1}{\epsilon}\int_{-t}^{t} f(x)\ g(x/\epsilon)\ dx \quad (3.2.130.)$$

If $m_t \leq f(x) \leq M_t$ with $x \in [-t, t] \Rightarrow m_t \leq I_f \leq M_t \ \forall t > 0$

and since f is continuous $\lim_{t\to 0^+} M_t = \lim_{t\to 0^+} m_t = f(0)$

we obtain $I_f = f(0)$

Other widely used examples are

$$\delta(x) = -\frac{1}{\pi} \lim_{\epsilon \to 0^+} \Im\text{m}\left[\frac{1}{x+i\epsilon}\right] = \frac{1}{\pi} \lim_{\epsilon \to 0} \frac{\epsilon}{x^2+\epsilon^2} \quad (3.2.131.)$$

and

$$\delta(x) = \frac{1}{\pi} \lim_{\epsilon \to 0^+} \epsilon \frac{\sin^2(x/\epsilon)}{x^2} \quad (3.2.132.)$$

with $g(x) = 1/[\pi(1+x^2)]$ and $g(x) = \sin^2(x)/(\pi x^2)$ respectively

Definition 3.2.6.

Convolution of $\delta(x)$ with other functions

is defined in such a way that the integration rules still hold

For example $$\int_{-\infty}^{\infty} \delta(x-x_0) f(x) dx = \int_{-\infty}^{\infty} \delta(u)\ f(u+x_0) du = f(x_0) \quad (3.2.133.)$$

Similarly ☛ if $a \neq 0$

$$\int_{-\infty}^{\infty} \delta(ax) f(x) dx = \frac{1}{|a|} \int_{-\infty}^{\infty} \delta(u)\ f(u/a) du = \frac{1}{|a|} f(0) \quad (3.2.134.)$$

and so $$\delta(ax) = \frac{1}{|a|}\delta(x) \qquad a \neq 0 \quad (3.2.135.)$$

in particular $\delta(-x) = \delta(x)$

Definition 3.2.7.

If we want that it keeps on fulfilling integration by parts we must define the derivative

$$\int_{-\infty}^{\infty} \delta'(x)\ f(x)\ dx = -\int_{-\infty}^{\infty} \delta(x)\ f'(x)\ dx = -f'(0) \quad (3.2.136.)$$

recall that $f = 0$ outside a finite interval

In general

$$\int_{-\infty}^{\infty} \delta^{(n)}(x)\ f(x)\ dx = (-1)^n f^{(n)}(0) \quad (3.2.137.)$$

therefore

$$f'(x_0) = -\int_{-\infty}^{\infty} \delta'(x - x_0)\ f(x)dx \quad (3.2.138.)$$

$$f^{(n)}(x_0) = (-1)^n \int_{-\infty}^{\infty} \delta^{(n)}(x - x_0)\ f(x)\ dx \quad (3.2.139.)$$

Note that ☛ if $a \neq 0$

$$\delta^{(n)}(ax) = \frac{1}{a^n |a|}\delta^{(n)}(x) \quad (3.2.140.)$$

In particular ☛ $\delta^{(n)}(-x) = (-1)^n \delta^{(n)}(x)$

Corollary 3.2.2. [Heaveside function]

Step (Heaveside) function

$$\Theta(x) = \begin{cases} 1 & x \geq 0 \\ 0 & x < 0 \end{cases} \quad (3.2.141.)$$

is primitive (at least in symbolic form) of $\delta(x)$

Equivalently $\Theta'(x)$ have symbolic limit $\delta(x)$

Proof.

For any given test function $f(x)$ ☛ integration by parts leads to

$$\int_{-\infty}^{\infty} dx\, \Theta'(x)\ f(x) = -\int_{-\infty}^{\infty} \Theta(x) f'(x)\ dx = -\int_{0}^{\infty} f'(x)\ dx = f(0)$$

(3.2.142.)

therefore $\Theta'(x) = \delta(x)$

Proposition 3.2.2.

Using $\Theta(x)$ function we write any integral over a finite interval $[a, b]$ as an integral where domain of integration is unbounded

$$\int_{-\infty}^{b} f(x)\ dx = \int_{-\infty}^{\infty} \Theta(b-x)\ f(x)\ dx \qquad (3.2.143.)$$

$$\int_{a}^{b} f(x)\ dx = \int_{-\infty}^{\infty} [\Theta(b-x) - \Theta(a-x)] f(x)\ dx \qquad (3.2.144.)$$

so that at most integrand is non-zero when $a < x < b$

Definition 3.2.9.

We can now return to our definition of Green matrix and rewrite (3.2.109.) as a distribution

$$\mathbb{G}(t, t') = \mathbb{K}(t, t')\Theta(t - t') \qquad (3.2.159.)$$

with $\mathbb{K}(t, t')$ as given in (3.2.104)

The system of first-order differential equations (3.2.79.)

can be rewritten as

$$L[\mathbf{u}(t)] = \mathbf{f}(t)$$

with

$$L = \mathbb{I}\frac{d}{dt} - \mathbb{A}(t)$$

$\mathbf{u}$ and $\mathbf{f}$ ☛ n-dimensional vectors

$\mathbb{A}$ ☛ $n \times n$ matrix

Since $\mathbb{K}(t, t')$ is a solution of homogeneous equation

$\mathbb{G}(t, t')$ satisfies $L[\mathbb{G}(t, t')] = \mathbb{I}\,\delta(t - t')$ (3.2.161.)

with $\mathbb{G}(t, t') = 0 \,\mathrm{for}\, t \to t'^{-}$

(where $\mathbb{I}$ is $n \times n$ identity matrix)

For $\mathbf{u}(t_0) = \mathbf{0}$ and $t_0 \to -\infty$ the solution of (3.2.79)

can be written as ☛ $$\mathbf{u}(t) = \int_{-\infty}^{\infty} \mathbb{G}(t,t')\ \mathbf{f}(t')\ dt' \quad (3.2.162.)$$

In particular ☛ if $\mathbf{f}(t) = \mathbf{f}_0\ \delta(t-t')$ then $\mathbf{u}(t) = \mathbb{G}(t,t')\mathbf{f}_0$

or equivalently $$u_i(t) = \sum_j G_{ij}(t,t')\ f_{0,j} \quad (3.2.163.)$$

with $\mathbf{f}_0$ a constant

Matrix element $G_{ij}(t,t')$ represents effect at time t in component i of a point source acting at time t' in component of j

Since $\lim_{t \to t^+} \mathbb{G}(t,t') = \mathbb{I}$ for $t > t'$

column j of $\mathbb{G}(t,t')$ is solution of homogeneous system

with initial condition $u_i(t') = \delta_{ij}$

This relation can be used to obtain $\mathbb{G}(t,t')$

Example 3.2.4.

If $\mathbb{A}(t) = \mathbb{A} \equiv$ constant ☛ then

$$\mathbb{U}(t) = e^{\mathbb{A}t}\, \mathbb{U}_0 \qquad (3.2.164.)$$

$$\mathbb{K}(t,t') = e^{\mathbb{A}(t-t')} \qquad (3.2.165.)$$

and

$$\mathbb{G}(t,t') = e^{\mathbb{A}(t-t')}\Theta(t-t') = \mathbb{G}(t-t') \qquad (3.2.166.)$$

In this case Green matrix is a function of $t - t'$

because of invariance of the homogeneous equation

with respect to temporal translations

Definition 3.2.8. [Theory of distributions]

Let V be finite-dimensional vector space ☛ such as $\mathbb{R}^n$

We can define linear functional (or linear form) $L : V \to \mathbb{R}$

which assigns to each vector $\mathbf{u} \in V$ a real number

and satisfies ☛ $L(c_1\mathbf{u_1} + c_2\mathbf{u_2}) = c_1 L(\mathbf{u_1}) + c_2 L(\mathbf{u_2})$

Exists unique vector $\mathbf{l}$ such that $L(\mathbf{u}) = \langle \mathbf{l}, \mathbf{u} \rangle \quad \forall \mathbf{u} \in V$

where $\langle \mathbf{l}, \mathbf{u} \rangle$ denotes inner product of two vectors

Expanding $\mathbf{u}$ in orthonormal basis

($\mathbf{v_i}, i = 1, \ldots, n$ such that $\langle \mathbf{v_i}, \mathbf{v_j} \rangle = \delta_{ij}$)

we obtain $\mathbf{u} = \sum_{i=1}^{n} c_i \mathbf{v_i}$ and $L(\mathbf{u}) = \sum_i c_i \, L(\mathbf{v_i}) = \sum_i c_i l_i = \langle \mathbf{l}, \mathbf{u} \rangle$

where $l_i = L(\mathbf{v_i})$ and $\mathbf{l} = \sum_i l_i^* \mathbf{v_i}$

Any linear form L on a finite-dimensional inner product space can be identified with a vector $\mathbf{l} \in V$

Consider space of test functions D made up of real functions $f(x)$ which have derivatives of any order

and cancel out beyond the bounds of a finite interval

We can define inner product $\langle g, f \rangle = \int_{-\infty}^{\infty} g(x)\ f(x)\ dx$

Consider now linear functional L

which assigns to each function $f(x)$ a real number

and satisfies ☛ $L[c_1 f_1 + c_2 f_2] = c_1 L[f_1] + c_2 L[f_2]$

where c_1 and c_2 are constants

$\forall g(x) \in D$ we can associate linear functional L_g

$$L_g[f] = \int_{-\infty}^{\infty} g(x)\ f(x)\ dx \qquad (3.2.150.)$$

Even though $\nexists g \in D$ such that $\int_{-\infty}^{\infty} g(x)\ f(x)\ dx = f(0)$

$\forall f \in D$ we now define functional δ such that $\delta[f] = f(0)$

Space of linear forms is greater than space of real functions f

We introduce symbol $\delta(x)$ such that

$$\delta[f] = \int_{-\infty}^{\infty} \delta(x)\ f(x)\ dx = f(0) \qquad (3.2.153.)$$

To continue to fulfill integration by parts

we define derivative of L as $L'[f] = -L[f']$ (3.2.154.)

it follows that

$$L_{g'}[f] = \int_{-\infty}^{\infty} g'(x)\ f(x)\ dx = -\int_{-\infty}^{\infty} g(x)\ f'(x)\ dx = L'_g[f] \qquad (3.2.155.)$$

In particular

$$\delta'[f] = -\delta[f'] = -f'(0) \qquad (3.2.156.)$$

Heaveside functional is defined by

$$\Theta[f] = \int_{0}^{\infty} f(x)\ dx \qquad (3.2.157.)$$

or equivalently $g(x) = \Theta(x)$ with

$$\Theta'[f] = -\Theta[f'] = -\int_{0}^{\infty} f'(x)\ dx = f(0) \qquad (3.2.158.)$$

Therefore ☛ $\Theta' = \delta$


TO BE
CONTINUED....