Physics 307

Mathematical Physics

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ORDINARY DIFFERENTIAL EQUATIONS I

3.1 Setting the Stage

3.2 Initial Value Problem

Picard's existence and uniqueness theorem

Systems of first-order linear differential equations

Green matrix as a generalized function

3.3 Boundary Value Problem

Self-adjointness of Sturm-Liouville operator

Green function of Sturm-Liouville operator

Series solutions to homogeneous linear equations

3.4 Fourier Analysis

Fourier series

Fourier transform

SETTING THE STAGE

Definition 3.1.1.

A **differential equation** is an equation for an unknown function of one (or several) variable(s) that expresses a relationship between the function itself and its derivatives of various orders

Example 3.1.1.

Law of radioactive decay

$$\frac{dN(t)}{dt} = -kN(t) \qquad (3.1.1.)$$

Example 3.1.2.

Newton's second law for a particle of constant mass

$$m\frac{d^2\mathbf{r}}{dt^2} = \mathbf{F}\left(t, \mathbf{r}, \frac{d\mathbf{r}}{dt}\right) \qquad (3.1.2.)$$

Note that because of the vector nature of unknown function this is actually a system of three coupled equations

Example 3.1.3.

Laplace equation for electrostatic potential in absence of charges

$$\nabla^2\phi(x,y,z) = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} = 0 \qquad (3.1.3.)$$

Definition 3.1.2.

An ordinary differential equation is a d.e. in which all derivatives are with respect to a single independent variable

Examples include equations (3.1.1.) and (3.1.2)

Definition 3.1.3.

A partial differential equation is a d.e. that involves
two or more independent variables
an unknown function (dependent on those variables)
and partial derivatives of the unknown function
with respect to the independent variables

An example is given in equation (3.1.3)

Definition 3.1.4.

The order of a differential equation

is order of highest derivative involved

Definition 3.1.5.

A solution to a d.e. is a function that solves the equation or turns it into an identity when substituted into the equation

Example 3.1.4.

$N(t) = Ce^{-kt}$ ☛ is a solution of (3.1.1.)

$$\frac{dN(t)}{dt} = -C\,k\,e^{-kt} = -k\,N(t) \qquad (3.1.4.)$$

arbitrary constant C can be determined

if the value of N is given at some particular time

$$N(0) = N_0 \qquad (3.1.5.)$$

Then $C = N_0$ yielding ☛ $N(t) = N_0\,e^{-kt}$

Equation (3.1.1) together with initial condition (3.1.5.) define an initial value problem

Definition 3.1.6.

Process of finding solutions of a d.e. is known as **resolution or integration** of the equation

Such a process can be simple ☛ as in example above

but in general one has to rely on approximate methods

that end up with a numerical integration

Sometimes we only want to understand certain solutions properties

like system's behavior for small variation of initial conditions

or obtain a global idea of derivative fields & equipotential curves

Resolution of a d.e. of order n requires n integrations

and therefore integration constants must be determined

This leads to following definition ➡

Definition 3.1.7.

A solution in which one or more integration constants take a particular value is called a particular solution of the d.e.

A solution is called general

if it contains all particular solutions of the d.e.

that is ☛ n integration constants are left undetermined

Definition 3.1.8.

Any ordinary differential equation of order n

can be written in the general form:

$$F\left(t, u, \frac{du}{dt}, \ldots, \frac{d^n u}{dt^n}\right) = 0 \qquad (3.1.6.)$$

where u ☛ is the unknown function

Definition 3.1.9.

The degree of a d.e. is power of highest derivative term

However ☛ not every differential equation has a degree

If derivatives occur within radicals or fractions

d.e. may not have a degree

If equation can be rationalized and cleared of fractions with with regard to all derivatives present

then its degree is the degree of highest ordered derivative

Example 3.1.5.

$$\left[\frac{d^2 f(x)}{dx^2}\right]^{2/3} = 2 + 3\frac{df(x)}{dx} \qquad (3.1.7.)$$

can be rationalized by cubing both sides to obtain

$$\left[\frac{d^2 f(x)}{dx^2}\right]^{2} = \left(2 + 3\frac{df(x)}{dx}\right)^3 \qquad (3.1.8.)$$

hence it is of degree two

Definition 3.1.10.

A d.e. is called homogeneous of degree p
if multiplying $u(t)$ and all its derivatives by a parameter λ leads to

$$F\left(t, \lambda u, \lambda \frac{du}{dt} \ldots \lambda \frac{d^{(n)}u}{dt^n}\right) = \lambda^p F\left(t, u, \frac{du}{dt} \ldots \frac{d^{(n)}u}{dt^n}\right) \quad (3.1.9.)$$

for arbitrary p

That is ☛ F is a homogeneous function of degree p
on the unknown variable and all its derivatives

Definition 3.1.11.

A differential equation is said to be linear if F
can be written as a linear combination of u and its derivatives
together with a constant term (all possibly depending on t)

Example 3.1.6.

The d.e. given in (3.1.1.) is linear

Example 3.1.7.

The system of coupled equations (3.1.2.) would be linear
if and only if $\mathbf{F}$ is a linear function of $\mathbf{r}$ and $d\mathbf{r}/dt$

Definition 3.1.12.

For a scalar function u

the most general form of linear ordinary differential equation is

$$a_n(t)\,\frac{d^n u}{dt^n} + a_{n-1}(t)\,\frac{d^{n-1}u}{dt^{n-1}} + \cdots + a_0(t)u = f(t)\,, \qquad a_n(t) \neq 0$$

(3.1.10.)

Equation (3.1.10) can be rewritten as

$$L[u] = f(t)\,, \qquad L = \sum_{m=o}^{n} a_m(t)\frac{d^m}{dt^m}$$

(3.1.11.)

where L is a differential linear operator

That is ☛ if $c_1,\ c_2$ are constants

and $u_1(t),\ u_2(t)$ are n times differentiable functions

$$L[c_1u_1(t) + c_2u_2(t)] = c_1L[u_1(t)] + c_2L[u_2(t)]$$

(3.1.12.)

$\forall c_1,\ c_2,\ u_1(t),\ u_2(t)$

Example 3.1.8.

Using our definition 3.1.10 ☛ (3.1.10) will be homogeneous

if and only if $f(t) = 0$

If this were case ☛ equation will be homogeneous of degree n

If $f(t) \neq 0$ linear equation will be inhomogeneous

Definition 3.1.13. [superposition principle]

If u_1 and u_2 are solutions of homogeneous equation

(i.e. $L[u_1] = L[u_2] = 0$) then

$$u(t) = c_1 u_1(t) + c_2 u_2(t) \quad (3.1.13.)$$

is also a solution of homogeneous equation $\forall c_1, c_2$

because of (3.1.12.)

Solutions of the homogeneous d.e. comprise vector space over $\mathbb{R}$

(i) an ordinary linear d.e. has n particular solutions

which are linearly independent

(ii) general solution of the homogeneous equation

is a linear combination of n particular solutions

Corollary 3.1.1.

General solution of linear inhomogeneous equation (3.1.10.) is given by sum of general solution of homogeneous equation and a particular solution to inhomogeneous equation

Proof.

Let $u_p(t)$ be a particular solution of inhomogeneous equation

Using superposition principle

general solution of homogeneous equation is given by

$$u_h(t) = \sum_{i=1}^{n} c_i u_{h,i}(t) \qquad (3.1.14.)$$

where $u_{h,i}$ are n particular solutions of homogeneous equation

Consider $u(t) = u_p(t) + u_h(t)$ ☛ recalling that L is linear

$$\begin{aligned} L[u(t)] &= L[u_p(t) + u_h(t)] = L[u_p(t)] + L[u_h(t)] \\ &= L[u_p(t)] + \sum_{i=1}^{n} c_i L[u_{h,i}(t)] = f(t) + 0 = f(t) \qquad (3.1.15.) \end{aligned}$$

Therefore ☛ $u(t)$ is a solution of inhomogeneous equation

Since it has n undetermined constant

it is general solution of inhomogeneous equation

Definition 3.1.14.

A first-order ordinary differential equation is of the form

$$\frac{du}{dt} = f(t, u) \qquad (3.1.16.)$$

If $f(t, u)$ does not depend on u ☛ then (3.1.16.) becomes

$$\frac{du}{dt} = f(t) \qquad (3.1.17.)$$

and the general solution reads

$$u(t) = \int f(t')\, dt' + c \qquad (3.1.18.)$$

where c is the so-called integration constant

Constant c can be determined if we know the initial condition

i.e. if $u(t_0) = u_0$ then

$$u(t) = \int_{t_0}^{t} f(t')\, dt' + u_0 \qquad (3.1.19.)$$

Definition 3.1.15.

A first-order differential equation is called separable

if $f(t,u) = h(t)g(u)$ ☛ (3.1.16.) becomes

$$\frac{du}{dt} = h(t)g(u) \qquad (3.1.20.)$$

If $g(u) \neq 0$ you can separate variables as

$$\frac{du}{g(u)} = h(t)dt \qquad (3.1.21.)$$

and then integrate both sides to get

$$\int \frac{du}{g(u)} = \int h(t)dt + c \qquad (3.1.22.)$$

This equation (of form $\phi(t,u) = c$)

determines u as an implicitly-defined function t

The constant of integration is chosen from a particular solution

$$u(t_0) = u_0 \text{ with } g(u_0) \neq 0$$

$$\int_{u_0}^{u} \frac{du'}{g(u')} = \int_{t_0}^{t} h(t')dt' \quad (3.1.23.)$$

If in addition there are roots u_r such that $g(u_r) = 0$

one should add to (3.1.22.) the constant solutions

$$u(t) = u_r, \quad \text{with} \quad g(u_r) = 0 \quad (3.1.24.)$$

which do not necessarily follow from (3.1.23) and (3.1.22)

but are undoubtedly solutions of (3.1.20.)

Example 3.1.9.

If $N(t) \neq 0$ ☛ (3.1.1.) can be rewritten as $\frac{dN(t)}{N} = -k dt$ (3.1.25.)

integration leads to

$$\int \frac{dN}{N} = \ln|N| = -\int k\, dt + c = -kt + c \quad (3.1.26.)$$

or equivalently $N(t) = c'\, e^{-kt}$ (3.1.27.)

with $c' = \pm e^c$

If $N(t_0) = N_0 \Rightarrow c' = N_0 e^{kt_0}$ and therefore

$$N(t) = N_0\, e^{-k(t-t_0)} \quad (3.1.28.)$$

for $k > 0$ ☛ gives formula of radioactive decay

for $k < 0$ ☛ formula for exponential growth of bacteria colonies

Previous calculation is valid for $N_0 \neq 0$

for $N_0 = 0$ one recovers constant solution of (3.1.1.)

namely ☛ $N(t) = 0 \; \forall t$ which implies $c' = 0 (c \to -\infty)$

Definition 3.1.16.

A first order linear differential equation has form of (3.1.16) with $f(t,u)$ a linear function of u

$$\frac{du}{dt} = a(t) + b(t)u \qquad (3.1.29.)$$

Note that (3.1.29) can be rewritten as $L[u] = a(t)$

$$\text{where } L = \frac{d}{dt} - b(t) \qquad (3.1.30.)$$

is a linear operator

If $a(t) = 0$

then (3.1.29.) is an homogeneous equation of separated variables

$$\frac{du}{u} = b(t)\,dt \qquad (3.1.31.)$$

Integration of left-hand side leads to

$$\ln|u(t)| = \int b(t)dt + c' \qquad (3.1.32.)$$

or equivalently $$u(t) = ce^{\int b(t)\,dt} \qquad (3.1.33.)$$

Setting $u(t_0) = u_0$ we have

$$u(t) = u_0 e^{\int_{t_0}^{t} b(t')dt'} \qquad (3.1.34.)$$

If $a \neq 0$ we can use method of variation of parameters
(a.k.a. variation of constants)

We envisage a solution of of form (3.1.33.)
but with c being a function of t

$$u(t) = u_h(t)\ c(t) \qquad (3.1.35.)$$

where $u_h(t) = e^{\int b(t)dt}$

Since $L[u_h(t)] = 0$ it follows that

$$L[u] = L[u_h(t)]c(t) + u_h(t)\frac{dc}{dt} = u_h(t)\frac{dc}{dt} = a(t) \qquad (3.1.36.)$$

and so

$$c(t) = \int \frac{a(t)}{u_h(t)} dt + c' \qquad (3.1.37.)$$

Substituting in (3.1.35.)

$$u(t) = u_h(t)\left[c' + \int \frac{a(t)}{u_h(t)}\,dt\right]$$

$$= e^{\int b(t)dt}\left[c' + \int e^{-\int b(t)dt} a(t)dt\right] \qquad (3.1.38.)$$

General solution is then a solution of homogeneous equation plus particular solution of inhomogeneous equation

Particular solution for $u(t_0) = u_0$ reads

$$u(t) = e^{\int_{t_0}^{t} b(t')dt'}\left[u_0 + \int_{t_0}^{t} e^{-\int_{t_0}^{t'} b(t'')dt''} a(t')dt'\right]$$

$$= K(t,t_0)\; u_0 + \int_{t_0}^{t} K(t,t')\; a(t')\; dt' \qquad (3.1.39.)$$

where $K(t_2,t_1) = e^{\int_{t_1}^{t_2} b(t)dt} = u_h(t_2)/u_h(t_1)$

Note that ☛ $K(t,t) = 1$

Example 3.1.10.

Consider a series resistor-inductor $(R-L)$ circuit

driven by a voltage source $V(t)$

The complete response to input voltage is described by

$$L\frac{dI}{dt}+IR=V(t) \qquad (3.1.40.)$$

where I is current

Solution for $I(0)=I_0$ with L and R constants is

$$I(t)=I_0e^{-Rt/L}+\int_0^t e^{-R(t-t')/L}V(t')dt' \qquad (3.1.41.)$$

Initial Value Problem

3.2.1. Existence and uniqueness of solutions

Consider the first-order ordinary differential equation

$$\frac{du}{dt} = f(t, u) \qquad (3.2.42.)$$

with initial condition

$$u(t_0) = u_0 \qquad (3.2.43.)$$

Except for some special cases

it is not usually possible to obtain analytical solution of (3.2.42)

We then have to resort on approximate methods

that can be used to solve (3.2.42.) numerically

We must first be sure that for given $f(t, u)$ and initial condition

there is indeed a solution of (3.2.42.)

The following theorem shows that a solution exists

and is unique for a very wide class of functions

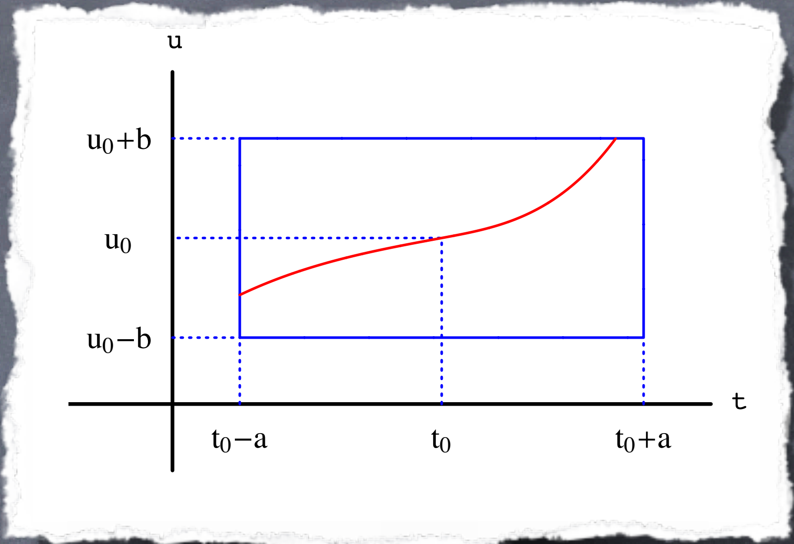
The theorem also provides an approximate solution of (3.2.42.)

which turns out to be useful both formally and numerically

Theorem 3.2.1. [Picard's theorem]

Let $f(t,u)$ be a continuous function in the rectangle

$$R = \{t, u / |t - t_0| \leq a, |u - u_0| \leq b\}$$



Let's further assume that f satisfies Lipschitz condition in R

$$|f(t,u_2) - f(t,u_1)| \leq N|u_2 - u_1| \qquad (3.2.44.)$$

Sufficient condition for Lipschitz inequality (3.2.44.) to hold is that $f_u = \partial f / \partial u$ exists and is bounded in R

Indeed ☛ if $|f_u| \leq N$ in R it follows that

$$|f(t,u_2) - f(t,u_1)| = |f_u(t,\xi)(u_2 - u_1)| \leq N|u_2 - u_1| \qquad (3.2.47.)$$

with $\xi \in [u_1, u_2]$

Then ☛ in the interval

$$|t - t_0| \leq r, \quad r = \min[a, b/M] \qquad (3.2.45.)$$

there exits a unique solution $u(t)$ satisfying d.e. (3.2.42.)

where M is maximum value of f in R

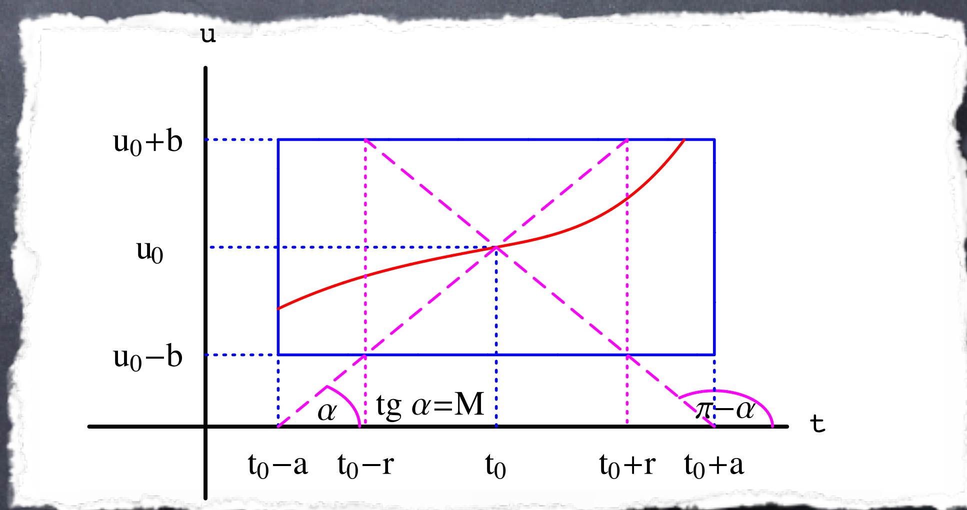
Condition $|t - t_0| \leq r$ entails solution will remain in R

Indeed ☛ if $|t - t_0| \leq r$ then $|f| \leq M$ in R

and so we can integrate(3.2.42.) and take absolute value to obtain

$$|u(t) - u_0| = \left| \int_{t_0}^{t} f(t', u(t')\, dt' \right| \leq \left| \int_{t_0}^{t} |f(t', u(t')|\, dt' \right|$$

$$\leq M|t - t_0| \leq Mr = b \qquad (3.2.46.)$$



Proof.

First we note that solution $u(t)$ will satisfy the integral equation

$$u(t) = u_0 + \int_{t_0}^{t} f\left(t', u(t')\right)\, dt' \quad (3.2.48.)$$

Conversely ☛ any solution of integral equation must satisfy both the d.e. and the initial condition

For example ☛ if we set $t = t_0$ in (3.2.48) we find that i.c. holds

A sequence $u_0, u_1(t), \ldots u_n(t), \ldots$ of successive approximations is now defined with $u_0(t) = u_0$

and

$$u_n(t) = u_0 + \int_{t_0}^{t} f\left(t', u_{n-1}(t')\right) dt' \,, \quad n \geq 1 \qquad (3.2.49.)$$

Restriction (3.2.45) ensures that $u_n(t)$ belongs to R for all n

that is ☛ $|u_n(t) - u_0| \leq b$ if $|t - t_0| \leq r$

Indeed ☛ for $n = 0$ the condition is trivially satisfied

Assuming that condition holds for u_{n-1}

since $|f| \leq M$ in R we obtain

$$|u_n(t) - u_0| \leq \int_{t_0}^{t} |f(t', u_{n-1}(t'))| \, dt' \leq M|t - t_0| \leq b$$

$\forall |t - t_0| \leq r$ (3.2.50.)

To establish convergence of the sequence

we calculate difference of two successive members of it

and find for $n \geq 1$ and $|t - t_0| \leq r$

$$\begin{aligned} |u_{n+1}(t) - u_n(t)| &= \left| \int_{t_0}^{t} [f(t', u_n(t')) - f(t', u_{n-1}(t'))] \, dt' \right| \\ &\leq \left| \int_{t_0}^{t} |[f(t', u_n(t')) - f(t', u_{n-1}(t'))]| \, dt' \right| \\ &\leq N \left| \int_{t_0}^{t} |u_n(t') - u_{n-1}(t')| \, dt' \right| \end{aligned}$$

(3.2.51.)

For $n = 1$ (3.2.50.) implies $|u_1(t) - u_0| \leq M|t - t_0|$

so (3.2.51.) leads to

$$|u_2(t) - u_1(t)| \leq NM \left| \int_{t_0}^{t} |t' - t_0| \, dt \right| = MN \frac{|t - t_0|^2}{2} \quad (3.2.52.)$$

For a general n we have

$$|u_n(t) - u_{n-1}(t)| \leq \frac{MN^{n-1}|t - t_0|^n}{n!} \quad (3.2.53.)$$

Now assuming (3.2.53.) holds we get

$$|u_{n+1}(t) - u_n(t)| \leq MN^n \left| \int_{t_0}^{t} \frac{|t' - t_0|^n}{n!} \, dt' \right| = MN^n \frac{|t - t_0|^{n+1}}{(n+1)!}$$

(3.2.54.)

Therefore ☛ $$\lim_{n \to \infty} |u_{n+1}(t) - u_n(t)| = 0 \quad (3.2.55.)$$

In summary ☛ since

$$u_n(t) = u_0 + (u_1(t) - u_0) + \cdots + (u_n(t) - u_{n-1}(t)) + \ldots \quad (3.2.56.)$$

the limit

$$u(t) \equiv \lim_{n\to\infty} u_n(t) = u_0 + \sum_{n=1}^{\infty} (u_n(t) - u_{n-1}(t)) \quad (3.2.57.)$$

exists because the series

$$\sum_{n=1}^{\infty} |u_n(t) - u_{n-1}(t)| \leq M \sum_{n=1}^{\infty} \frac{N^{n-1}|t-t_0|^n}{n!} = M \frac{e^{N|t-t_0|} - 1}{N}$$

(3.2.58.)

is absolutely convergent

The limit of integral in (3.2.49.) is equal to integral of limit and therefore $u(t)$ is a solution of (3.2.42.)

To study the question of uniqueness

we assume $v(t)$ is another solution of (3.2.42.) satisfying $v(t_0) = u_0$

Then ☛ if $|t - t_0| \leq r$

subtraction of one such equation from the other yields

$$\begin{aligned} |u(t) - v(t)| &\leq \int_{t_0}^{t} |f(t', u(t')) - f(t', v(t'))| \; dt' \\ &\leq N \int_{t_0}^{t} |u(t') - v(t')| \, dt' \\ &\leq KN|t - t_0| \end{aligned} \quad (3.2.60.)$$

where K is maximum of $|u(t) - v(t)|$ for $|t - t_0| < r$

using (3.2.60) for $|u(t') - v(t')|$

$$|u(t) - v(t)| \leq KN^2 \int_{t_0}^{t} |t' - t_0| dt' = KN^2 \frac{|t - t_0|^2}{2}$$

Replicating the procedure n times $|u(t) - v(t)| \leq KN^n \frac{|t - t_0|^n}{n!}$

it follows that ☛ $|u(t) - v(t)| \leq \lim_{n \to \infty} KN^n \frac{|t - t_0|^n}{n!} = 0$

Example 3.2.1.

Consider again linear equation

$$\frac{du}{dt} = -\lambda u \qquad (3.2.64.)$$

with $t_0 = 0$ and $u(t_0) = u_0$

We invoke Picard's theorem to obtain

$$\begin{aligned} u_1 &= u_0 - \lambda \int_0^t u_0\, dt' = u_0[1 - \lambda t] \\ u_2 &= u_0 - \lambda \int_0^t u_1(t')\, dt' = u_0[1 - \lambda t + \lambda^2 t^2/2] \end{aligned}$$

In general ☛ $u_n = u_0 \sum_{m=0}^{n} (-1)^m (\lambda t)^m / m!$

$$u(t) = \lim_{n\to\infty} u_n(t) = \sum_{n=0}^{\infty} (-1)^n \frac{(\lambda t)^n}{n!} = u_0 e^{-\lambda t} \qquad (3.2.65.)$$

Series converges $\forall t$ ☛ but condition (3.2.45.) provides overly conservative estimate of interval of convergence

For $u_0 > 0, \ M = |\lambda|(b + u_0)$ and

$$r = \min\left[a, \frac{b}{|\lambda|(u_0 + b)}\right] \quad (3.2.66.)$$

If $a > |\lambda|^{-1}$ then $r \leq |\lambda|^{-1}$ because $b/(u_0 + b) < 1 \qquad \forall b > 0$

In general condition (3.2.45.) is very restrictive and Picard's expansion converges in a larger interval

Definition 3.2.1.

Points (t_0, u_0) in which there is either no solution of (3.2.42.) or solution is not unique ☛ are called singular points

Hypotheses of Picard's theorem are sufficient for existence of a solution ☛ but not necessary

Indeed ☛ if f is continuous inside a ball centered at (t_0, u_0) there is always a solution of (3.2.42.) but this may not be unique if Lipschitz condition is not met

The curve formed by singular points is called singular curve

A solution made up entirely of singular points is called singular solution

Example 3.2.2.

Consider first-order ordinary differential equation

$$\frac{du}{dt} = q\frac{u}{t} \qquad (3.2.67.)$$

with $q > 0$

Function $f(t,u)$ has a discontinuity at $t = 0$

A possible solution is $u(t) = 0$

If $u(t) \neq 0$ ☛ integration of (3.2.67.) leads to

$$\ln|u| = q\ln|t| + c' \qquad (3.2.68.)$$

that is ☛ if $t > 0$ then $u(t) = ct^q$ (3.2.69.)

If $t_0 = 0$ and $u_0 = 0$

(3.2.69.) is solution of (3.2.67.) for any value of c (including $c = 0$)

There is not a unique solution ☛ but a family of solutions

On other hand ☛ if $t_0 = 0$ and $u_0 \neq 0$ there is no solution


TO BE
CONTINUED....