Physics 307

Mathematical Physics

Luis Anchordoqui

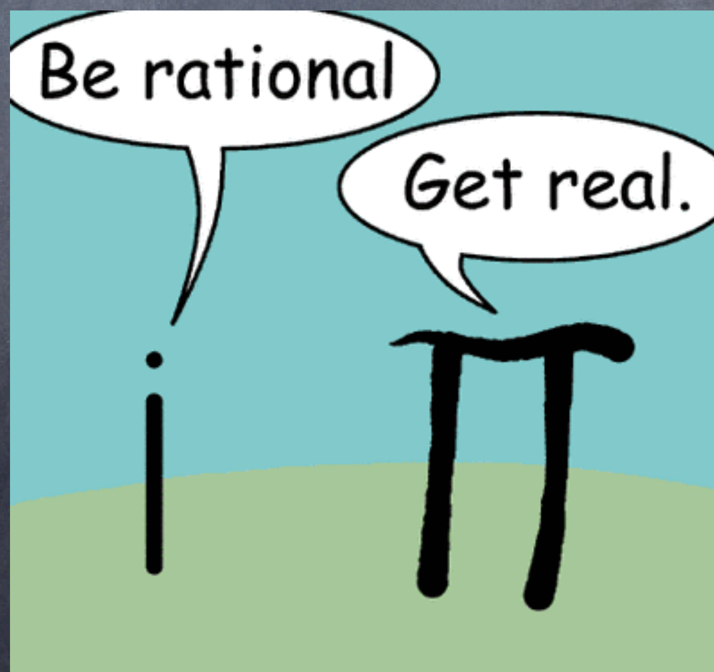
Complex Analysis III

1.1 Complex Algebra ✓

1.2 Functions of a Complex Variable ✓

1.3 Cauchy's Theorem and its Applications ✓

1.4 Isolated Singularities and Residues



Isolated Singularities and Residues

Definition 1.4.1.

Given a function f ☛ a zero of f is a point z_0 such that $f(z_0) = 0$

The zero set is then $\mathcal{Z}_f\{z \in \mathbb{C} : f(z) = 0\}$

there is some similarity between $\mathcal{Z}$ of a polynomial

and that of an analytic function

⇓

Any polynomial of degree n has at most n-zeros

we can construct a ball around a zero where no other zero exists

that is zeros are isolated

An analytic function can have infinitely many zeros

e.g. $f(z) = \sin \lambda z$ but they are still isolated

Theorem 1.4.1

Suppose that f is analytic on $D=\{z\in\mathbb{C}:|z-z_0|<r\}$ & $f(z_0)=0$

Then either (i) $f(z)=0\forall z\in D$

(ii) $\exists\epsilon>0$ such that $\forall z\in B_\epsilon(z_0)$ but $\{z_0\}$, $f(z)\neq 0$

Consequently ☛ if $\exists\ \{z_n\}\subset D$ such that:

(i) $z_n\neq z_0$ for infinitely many n (ii) $f(z_n)=0\,\forall n$ (iii) $z_n\to z_0$

then ☛ $f(z)=0\forall z\in D$

Proof.

Let $f(z)=\displaystyle\sum_{n=1}^{\infty}a_n(z-z_0)^n,\ z\in D$

If $a_n=0\ \forall n$ then ☛ (i) holds

If $\exists n$ such that $a_n\neq 0$ ☛ get smallest n (say n_0) such that $a_n\neq 0$

Then $f(z)=\displaystyle\sum_{n=n_0}^{\infty}a_n(z-z_0)^n$

$$f(z)=(z-z_0)^{n_0}(a_{n_0}+a_{n_0+1}(z-z_0)+a_{n_0+2}(z-z_0)^2+\cdots)=(z-z_0)^{n_0}g(z)$$

where g is analytic on D and $g(z_0)=a_{n_0}\neq 0$

As g is continuos $\exists\epsilon>0$ such that

$g(z)\neq 0\forall z\in B_\epsilon(z_0)\Rightarrow f(z)\neq 0\forall z\in B_\epsilon(z_0)$, but $\{z_0\}$

Theorem 1.4.2. [Laurent's theorem]

Let $f : D \to \mathbb{C}$ be analytic on annulus

$$A_r^R = \{z \in \mathbb{C} : r < |z - z_0| < R\} \subset D \, (0 < r < R < \infty)$$

Then ➡ on A_r^R ☛ $f(z)$ can be expressed by

$$f(z) = \sum_{j=0}^{\infty} a_j (z - z_0)^j + \sum_{j=1}^{\infty} b_j \frac{1}{(z - z_0)^j} \tag{1.4.92}$$

For any choice of simple closed contour $C \subset A_r^R(z_0)$

the coefficients a_j and b_j are given by

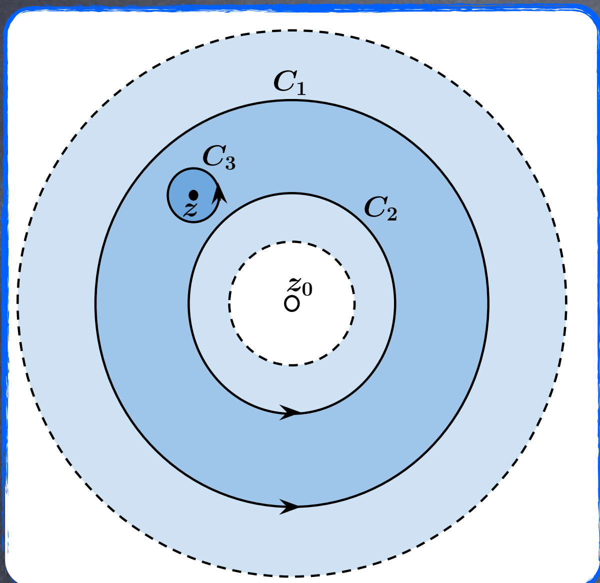
$$a_j = \frac{1}{2\pi i} \oint_C \frac{f(\zeta)}{(\zeta - z_0)^{j+1}} d\zeta, \quad \text{for } j \geq 0 \tag{1.4.93}$$

$$b_j = \frac{1}{2\pi i} \oint_C \frac{f(\zeta)}{(\zeta - z_0)^{-j+1}} d\zeta \quad \text{for } j \geq 1 \tag{1.4.94}$$

Proof.

Without loss of generality we consider $z_0 = 0$ and fix $z \in A_r^R(0)$

Choose circle C_1 and C_2 in $A_r^R(0)$ such that both are centered at 0 with radii R_1 and R_2 satisfying ☛ $r < R_2 < |z| < R_1 < R$



choose a third circle C_3 centered at z and having radius R_3 with R_3 small enough that C_3 is contained in $A_r^R(0)$ and does not intersect C_1 or C_2

then

$$\frac{1}{2\pi i}\oint_{C_1} \frac{f(\zeta)}{\zeta - z} d\zeta = \frac{1}{2\pi i}\oint_{C_3} \frac{f(\zeta)}{\zeta - z} d\zeta + \frac{1}{2\pi i}\oint_{C_2} \frac{f(\zeta)}{\zeta - z} d\zeta$$

$$= f(z) + \frac{1}{2\pi i}\oint_{C_2} \frac{f(\zeta)}{\zeta - z} d\zeta \qquad (1.4.95)$$

where we have used Cauchy's formula to obtain last line

Solving for $f(z)$ gives

$$\begin{aligned} f(z) &= \frac{1}{2\pi i}\oint_{C_1}\frac{f(\zeta)}{\zeta - z}d\zeta - \frac{1}{2\pi i}\int_{C_2}\frac{f(\zeta)}{\zeta - z}d\zeta \\ &= \frac{1}{2\pi i}\oint_{C_1}\frac{f(\zeta)}{\zeta - z}d\zeta + \frac{1}{2\pi i}\int_{C_2}\frac{f(\zeta)}{z-\zeta}d\zeta \qquad (1.4.96) \end{aligned}$$

We will show that first integral on right-hand side of (1.4.96) leads to analytic part of Laurent series expansion for $f(z)$ while second integral on right-hand side leads to singular part

Analysis of first integral proceeds as in proof of Taylor's theorem

$$\frac{1}{2\pi i}\oint_{C_1}\frac{f(\zeta)}{\zeta - z}d\zeta = \sum_{j=0}^{N} z^j \frac{1}{2\pi i}\oint_{C_1}\frac{f(\zeta)}{\zeta^{j+1}}d\zeta + \frac{1}{2\pi i}\oint_{C_1}\left(\frac{z}{\zeta}\right)^{N+1}\frac{f(\zeta)}{\zeta - z}d\zeta$$

In this case however we can no longer expect that

$$\frac{1}{2\pi i}\oint_{c_1}\frac{f(\zeta)}{\zeta^{j+1}}d\zeta = \frac{f^{(j)}(0)}{j!} \qquad (1.4.97)$$

because f is not necessarily differentiable inside C_1

Therefore ☛ we define a_j by

$$a_j \equiv \frac{1}{2\pi i}\oint_{c_1}\frac{f(\zeta)}{\zeta^{j+1}}d\zeta \qquad (1.4.98)$$

for whatever value this integral takes

This yields

$$\frac{1}{2\pi i}\oint_{C_1}\frac{f(\zeta)}{\zeta - z}d\zeta = \sum_{j=0}^{N} a_j\, z^j + \frac{1}{2\pi i}\oint_{C_1}\left(\frac{z}{\zeta}\right)^{N+1}\frac{f(\zeta)}{\zeta - z}d\zeta \qquad (1.4.99)$$

and letting $N \to \infty$ as in proof of Taylor's theorem

gives the analytic part ☛

$$\frac{1}{2\pi i}\oint_{C_1}\frac{f(\zeta)}{\zeta - z}d\zeta = \sum_{j=0}^{N} a_j\, z^j \qquad (1.4.100)$$

To obtain singular part we apply a similar technique

Consider that for $\zeta \in C_2$

$$\begin{aligned}
\frac{f(\zeta)}{z-\zeta} &= \frac{f(\zeta)}{z}\left(\frac{1}{1-\zeta/z}\right) \\
&= f(\zeta)\left[\frac{1}{z}+\frac{\zeta}{z^2}+\cdots+\frac{\zeta^N}{z^{N+1}}+\left(\frac{\zeta}{z}\right)^{N+1}\frac{1}{1-\zeta/z}\right] \\
&= \frac{f(\zeta)}{z}\left[1+\frac{\zeta}{z}+\cdots+\left(\frac{\zeta}{z}\right)^{N}+\left(\frac{\zeta}{z}\right)^{N+1}\frac{1}{z-\zeta}\right] \\
&= \sum_{j=1}^{N+1}\frac{\zeta^{j-1}}{z^j}f(\zeta)+\left(\frac{\zeta}{z}\right)^{N+1}\frac{f(\zeta)}{z-\zeta} \qquad (1.4.101)
\end{aligned}$$

So that

$$\frac{1}{2\pi i}\oint_{C_2}\frac{f(\zeta)}{z-\zeta}d\zeta = \sum_{j=1}^{N+1}\frac{1}{z^j}\left(\frac{1}{2\pi i}\oint_{C_2}\zeta^{j-1}f(\zeta)d\zeta\right)+\frac{1}{2\pi i}\oint_{C_2}\left(\frac{\zeta}{z}\right)^{N+1}\frac{f(\zeta)}{z-\zeta}d\zeta$$

In this case ☛ define

$$b_j \equiv \frac{1}{2\pi i}\int_{C_2} \zeta^{j-1} f(\zeta) d\zeta \qquad (1.4.102)$$

and take limit as $N \to \infty$ to obtain

$$\frac{1}{2\pi i}\oint_{C_2} \frac{f(\zeta)}{z-\zeta} d\zeta = \sum_{j=1}^{\infty} \frac{b_j}{z^j} \qquad (1.4.103)$$

These two series are combined into one series of form

$$f(z) = \sum_{j=-\infty}^{\infty} c_j (z - z_0)^j \qquad (1.4.104)$$

with

$$c_j = \frac{1}{2\pi i}\int_C \frac{f(\zeta)}{(\zeta - z_0)^{j+1}} d\zeta, \text{ for } n \in \mathbb{Z} \qquad (1.4.105)$$

Example 1.4.1.

To find Laurent series that represents the function

$$f(z) = z^2 \sin \frac{1}{z^2} \text{ in the domain } 0 < |z| < \infty$$

(1.4.106.)

Note that

$$\sin w = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} w^{2n+1}$$

(1.4.107.)

for $|w| < \infty$

Substituting z^{-2} for w it follows that

$$z^2 \sin \frac{1}{z^2} = z^2 \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{-4n-2} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \frac{1}{z^{4n}}$$

for $0 < |z| < \infty$ (1.4.108.)

Definition 1.4.2

A point z_0 is called an isolated singular point of a function f if f fails to be analytic at z_0 but is analytic on B_ϵ but $\{z_0\}$ (for some $\epsilon > 0$)

For such a $z_0 \exists$ a unique Laurent expansion of f such that

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z - z_0)^n \quad (1.4.109)$$

Point z_0 is called a pole of order m if $c_{-m} \neq 0$ & $c_n = 0 \forall n < -m$

If $m = 1$ then z_0 is called a simple pole

Theorem 1.4.3.

z_0 is a pole of order m of $f \Leftrightarrow \exists h$ such that

$$f(z) = \frac{h(z)}{(z - z_0)^m} \quad (1.4.110)$$

with h analytic at z_0 and $h(z_0) \neq 0$

Proof.

If z_0 is a pole of order m for f ☛ using Laurent expansion of f we get

$$\begin{aligned} f(z) &= \frac{c_{-m}}{(z-z_0)^m} + \frac{c_{-m+1}}{(z-z_0)^{m-1}} + \cdots + c_0 + c_1(z-z_0) + \ldots \\ &= \frac{1}{(z-z_0)^m}[c_{-m} + c_{-m+1}(z-z_0) + \ldots] \\ &= \frac{h(z)}{(z-z_0)^m} \end{aligned} \qquad (1.4.111)$$

$\therefore h(z_0) = c_m \neq 0$ and h is analytic at z_0

because it has a convergent Taylor series at z_0

Definition 1.4.3.

The complex number c_{-1}

which is coefficient of $(z-z_0)^{-1}$ in Laurent expansion of f

is called residue of f at z_0 and is denoted by $\mathrm{Res} f(z)|_{z=z_0}$

Definition: If $U \subset \mathbb{C}$ is an open subset of the complex plane

$z_0 \in U$ is point of U

and $f : U$ but $\{z_0\} \to \mathbb{C}$ is an analytic function

then z_0 is called a removable singularity for f

if there exists an analytic function $g : U \to \mathbb{C}$

which coincides with f on U but $\{z_0\}$

We say f is analytically extendable over U if such a g exists

Theorem 1.4.4.

If f has a simple pole at $z = z_0$ ☛ then

$$\mathrm{Res} f(z)\big|_{z=z_0} = \lim_{z \to z_0} (z - z_0)\, f(z) \qquad (1.4.112)$$

Proof.

Since $z = z_0$ is a simple pole

Laurent expansion of f about that point has form

$$f(z) = \frac{c_{-1}}{z - z_0} + c_0 + c_1(z - z_0) + c_2(z - z_0)^2 + \dots \qquad (1.4.113)$$

By multiplying both sides by $z - z_0$

and then taking limit as $z \to z_0$ we obtain

$$\lim_{z \to z_0} f(z) = \lim_{z \to z_0} \left[c_{-1} + c_0(z - z_0) + c_1(z - z_0)^2 + \dots \right] = c_{-1} = \mathrm{Res} f(z)\big|_{z=z_0}$$

$(z - z_0)$ ↩

Theorem 1.4.5.

If f has a pole of order n at $z = z_0$ then

$$\mathrm{Res} f(z)|_{z=z_0} = \frac{1}{(n-1)!} \lim_{z \to z_0} \frac{d^{n-1}}{dz^{n-1}} (z - z_0)^n f(z) \qquad (1.4.114)$$

Proof.

Since f is assumed to have a pole of order n

its Laurent expansion for $0 < |z - z_0| < R$ must have form

$$f(z) = \frac{c_{-n}}{(z - z_0)^n} + \dots + \frac{c_{-2}}{(z - z_0)^2} + \frac{c_{-1}}{z - z_0} + c_0 + c_1(z - z_0) + \dots \qquad (1.4.115)$$

We multiply by $(z - z_0)^n$ to obtain

$$\begin{aligned} (z - z_0)^n f(z) &= c_{-n} + \dots + c_{-2}(z - z_0)^{n-2} + c_{-1}(z - z_0)^{n-1} \\ &+ c_0(z - z_0)^n + c_1(z - z_0)^{n+1} + \dots \end{aligned} \qquad (1.4.116)$$

and then differentiate $n - 1$ times

$$\frac{d^{n-1}}{dz^{n-1}} (z - z_0)^n f(z) = (n-1)! c_{-1} + n! c_0 (z - z_0) + \dots \qquad (1.4.117)$$

if $z \to z_0$ ☛ $\lim_{z \to z_0} \frac{d^{n-1}}{dz^{n-1}} (z - z_0)^n f(z) = (n-1)! c_{-1}$ (1.4.118)

Solving for c_{-1} gives (1.4.114)

Example 1.4.2.

The function

$$f(z) = \frac{1}{(z-1)^2(z-3)} \qquad (1.4.119.)$$

has a simple pole at $z = 3$ and a pole of order 2 at $z = 1$

Therefore

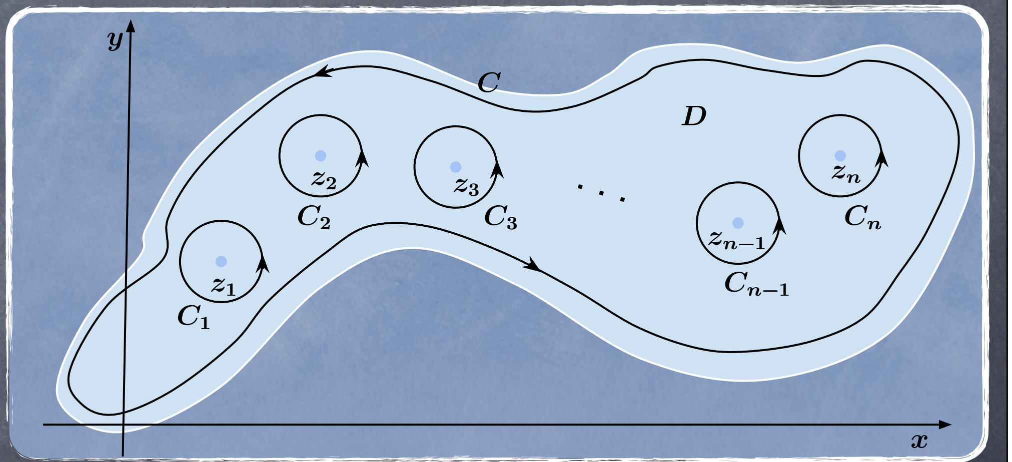
$$\mathrm{Res} f(z)|_{z=3} = \lim_{z\to 3}(z-3)f(z) = \lim_{z\to 3}\frac{1}{(z-1)^2} = \frac{1}{4} \qquad (1.4.120.)$$

at pole of order 2 we have

$$\begin{aligned} \mathrm{Res} f(z)|_{z=1} &= \frac{1}{1!}\lim_{z\to 1}\frac{d}{dz}(z-1)^2 f(z) \\ &= \lim_{z\to 1}\frac{d}{dz}\frac{1}{z-3} \\ &= \lim_{z\to 1} -\frac{1}{(z-3)^2} = -\frac{1}{4} \qquad (1.4.121.) \end{aligned}$$

Theorem 1.4.6. [Cuachy's residue theorem]

Let D be a simply connected domain
and C a simple closed contour lying entirely within D
If a function f is analytic on and within C
except at a finite number of singular points $z_1, z_2, ..., z_n$ within C
then ☛ $\oint_C f(z)dz = 2\pi i \sum_{k=1}^{n} \text{Res} \, f(z)|_{z=z_k}$ (1.4.122)



Proof.

$C_1, C_2, \ldots, C_n$ are circles
centered at $z_1, z_2, \ldots, z_n$
each circle C_k
has a radius r_k small enough
so that $C_1, C_2, \ldots, C_n$ are mutually disjoint
and are interior to simple closed curve C

then ☛ $\oint_C f(z)dz = \sum_{k=1}^{n} \oint_{C_k} f(z)dz = 2\pi i \sum_{k=1}^{n} \text{Res} \, f(z)|_{z=z_k}$

Example 1.4.3.

Evaluate

$$\oint_C \frac{1}{(z-1)^2(z-3)}dz \qquad (1.4.124.)$$

where (i) contour C is rectangle defined by $x=0, x=4, y=-1, y=1$

and (ii) contour C is circle $|z|=2$

Since in $C_{(i)}$ both poles $z=1$ and $z=3$ lie within square, we have

$$\begin{aligned}\oint_{C_{(i)}} \frac{1}{(z-1)^2(z-3)}dz &= 2\pi i[\text{Res} f(z)|_{z=1} + \text{Res} f(z)|_{z=3}] \\ &= 2\pi i\left[-\frac{1}{4}+\frac{1}{4}\right] = 0 \qquad (1.4.125.)\end{aligned}$$

For $C_{(ii)}$ only pole $z=1$ lies within circle $|z|=2$

$$\oint_{C_{(ii)}} \frac{1}{(z-1)^2(z-3)}dz = -\frac{\pi}{2}i \qquad (1.4.126.)$$

Corollary 1.4.1.

Residue theory can be used to evaluate real integrals of forms

$$\int_0^{2\pi} f(\cos\theta, \sin\theta)d\theta \qquad (1.4.133)$$

Proof.

Basic idea here is to convert an integral of form (1.4.133.) into complex integral where contour C is unit circle centered at origin

This contour can be parameterized by $z = \cos\theta + i\sin\theta = e^{i\theta},\ 0 \le \theta \le 2\pi$

Using

$$dz = ie^{i\theta}d\theta, \quad \cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}, \quad \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i} \qquad (1.4.134)$$

we replace in turn $d\theta, \cos\theta$ and $\sin\theta$ by

$$d\theta = \frac{dz}{iz}, \quad \cos\theta = \frac{1}{2}(z + z^{-1}), \quad \sin\theta = \frac{1}{2i}(z - z^{-1}) \qquad (1.4.135)$$

integral (2.4.133) then becomes

$$\int_C f\left(\frac{1}{2}(z + z^{-1}), \frac{1}{2i}(z - z^{-1})\right)\frac{dz}{iz} \qquad (1.4.136)$$

where C is $|z| = 1$

Lemma 1.4.1. [Jordan's lemma]

Consider definite integrals of form

$$I = \int_{-\infty}^{\infty} f(x)\, e^{iax}\, dx \qquad (1.4.144)$$

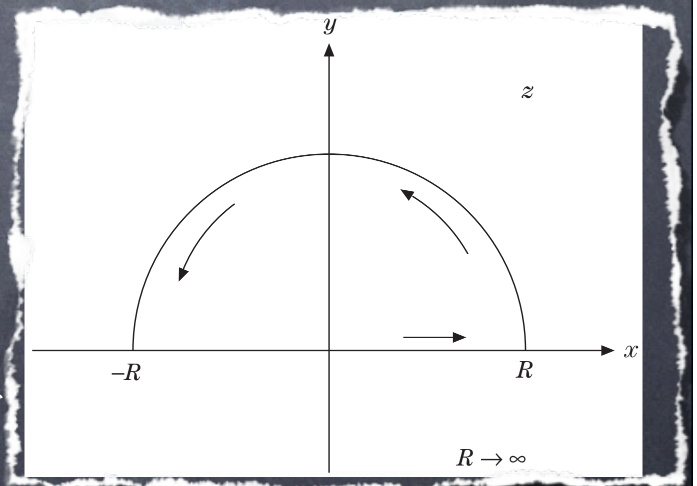
with a real and positive

We assume two following conditions are satisfied:

(i) $f(z)$ is analytic in upper half-plane except for finite # of poles

(ii) for $0 \leq \arg z \leq \pi$ ☛ $\lim_{|z|\to\infty} f(z) = 0$ (1.4.145)

Then $I = \oint f(z)\, dz$ on the contour ☛



because the integral I

is given by integration over real axis

Note that $R \to \infty$

integration over arc gives no contribution

$$I_R = \int_0^{\pi} f(Re^{i\theta}) e^{iaR\cos\theta - aR\sin\theta} iRe^{i\theta}\, d\theta \qquad (1.4.146)$$

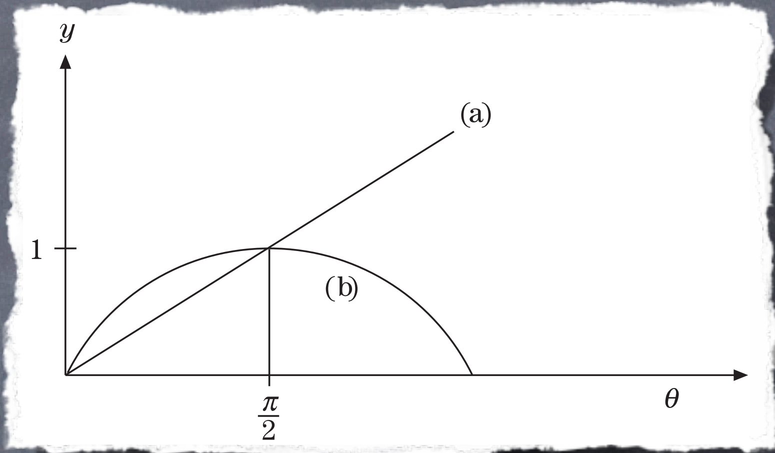
exponential factor goes rapidly to zero in upper half-plane

Let R be so large that $|f(z)| = |f(Re^{i\theta})| < \epsilon$

Then

$$|I_R| \le \epsilon R \int_0^{\pi} e^{-aR\sin\theta} d\theta = 2\epsilon R \int_0^{\pi/2} e^{-aR\sin\theta} d\theta$$

In range $[0, \pi/2], 2\theta/\pi \le \sin\theta$ ☛



(a)$y = 2\theta/\pi$ & (b)$y = \sin\theta$

therefore

$$|I_R| \le 2\epsilon R \int_0^{\pi/2} e^{-aR2\theta/\pi} d\theta$$

integration leads to

$$|I_R| \le 2\epsilon R \frac{1 - e^{-aR}}{2aR/\pi}$$

Finally ☛ $\lim_{R\to\infty} |I_R| \le \frac{\pi}{a}\epsilon$

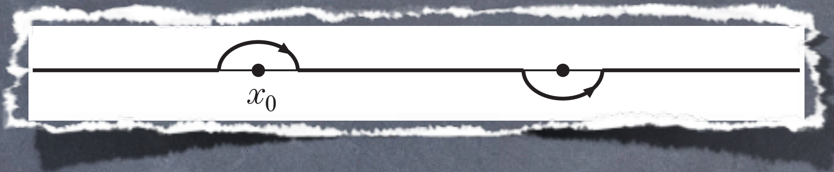
From condition (1.4.145) $\epsilon \to 0$ as $R \to \infty$ and $\lim_{R\to\infty} |I_R| = 0$

therefore

$$\int_{-\infty}^{\infty} f(x)e^{iax}dx + \lim_{R\to\infty} I_R = 2\pi i \sum \text{residues (upper half-plane)}$$

Corollary 1.4.2. [Cauchy principal value]

If first-order pole is directly on contour of integration we may deform contour to include or exclude residue as desired by including a semicircular detour of infinitesimal radius



taking ☛ $z - x_0 = \delta e^{i\phi}, dz = i\delta e^{i\phi} d\phi$

integration over semicircle then gives

$$\int \frac{dz}{z - x_0} = i \int_{\pi}^{2\pi} d\phi = i\pi, \quad \text{if conunterclockwise}$$

$$\int \frac{dz}{z - x_0} = i \int_{\pi}^{2\pi} d\phi = -i\pi, \qquad \text{if clockwise}$$

This contribution (+ or −) should be added to LHS of (1.4.122)

If detour is clockwise ☛ residue would not be enclosed
and there would be no corresponding term on RHS of (1.4.122)

If detour is counterclockwise ☛ residue would be enclosed by C
and a term $2\pi i \,\mathrm{Res}\, f(z)|_{z=x_0}$ would appear on RHS of (1.4.122)

Net result for either a clockwise or counterclockwise detour
is that a simple pole on the contour
is counted as one-half what it would be if it were within contour

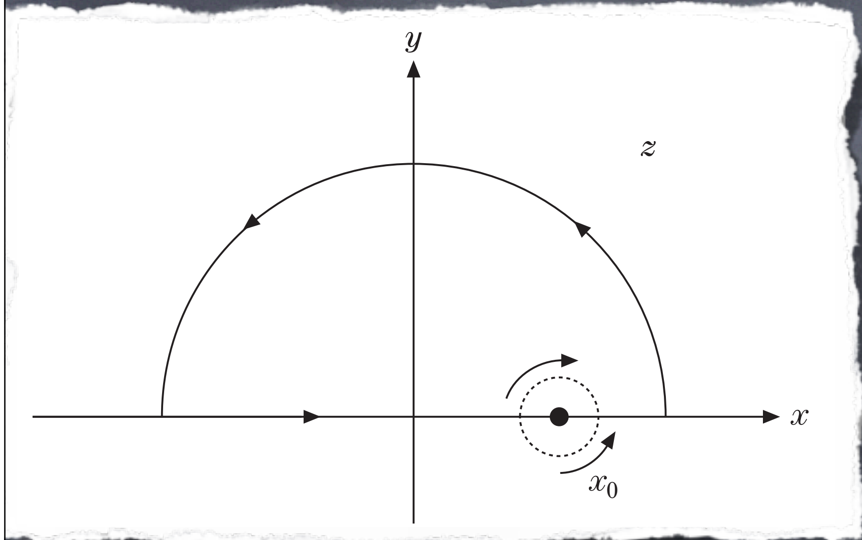
For example ☛ let us suppose that $f(z)$ with a simple pole at $z = x_0$
is integrated over entire real axis
assuming $|f(z)| \to 0$ for $|z| \to \infty$ that relevant integrals are finite
(faster than $1/|z|$)

Contour is closed with infinite semicircle in upper half-plane ☛

$$\oint f(z)dz = \int_{-\infty}^{x_0-\delta} f(x)dx + \int_{C_{x_0}} f(z)dz$$

$$+ \int_{x_0+\delta}^{\infty} f(x)dx + \int_{C \text{ infinite semicircle}}$$

$$= 2\pi i \sum \text{enclosed residues} \quad (1.4.153)$$



Integrals along x-axis may be combined and semicircle radius permitted to approach zero

We therefore define Cauchy principal value ☛ P.V. $\int$

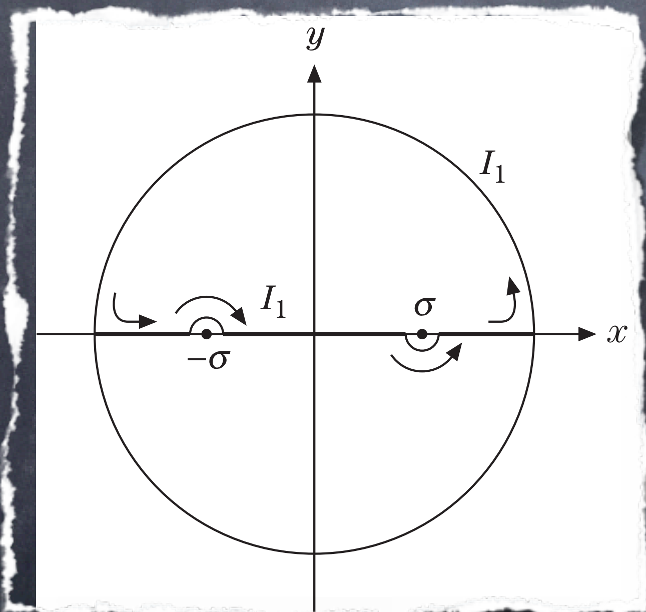
$$\lim_{\delta \to 0} \left\{ \int_{-\infty}^{x_0-\delta} f(x)dx + \int_{x_0+\delta}^{\infty} f(x)dx \right\} = \text{P.V.} \int_{-\infty}^{\infty} f(x)dx$$

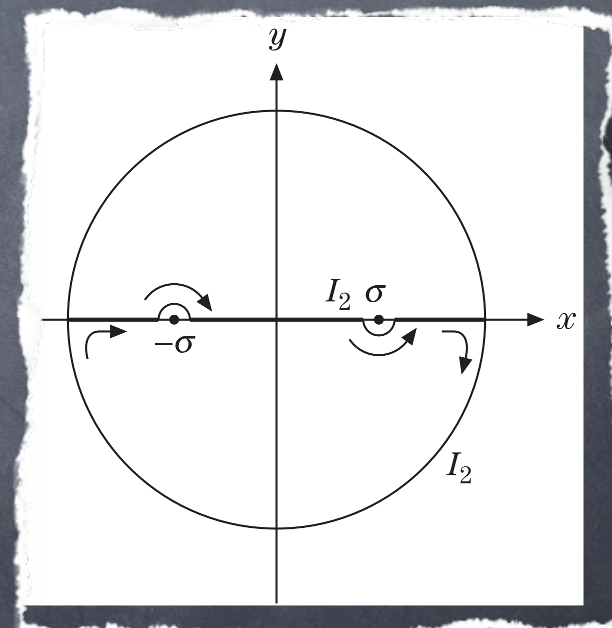
Example 1.4.7.

Evaluate $\displaystyle\int_{-\infty}^{\infty} \frac{x \sin x}{x^2 - \sigma^2}\, dx$ ☛ σ is real and positive (1.4.155.)

Using $\sin z = (e^{iz} - e^{-iz})/2i$ we rewrite (1.4.155.) in complex plane

$$\int_{-\infty}^{\infty} \frac{x \sin x}{x^2 - \sigma^2}\, dx = \frac{1}{2i}\int_{-\infty}^{\infty} \frac{z e^{iz}}{z^2 - \sigma^2}\, dz - \frac{1}{2i}\int_{-\infty}^{\infty} \frac{z e^{-iz}}{z^2 - \sigma^2}\, dz \quad (1.4.156.)$$





To compute first term in (1.4.156.) complete contour with infinite semicircle in upper half-plane

To compute second term in (1.4.156) we complete contour with infinite semicircle in lower half-plane

By Jordan's lemma

neither semicircle contributes anything to integral

We find poles at $z=\sigma$ and $z=-\sigma$

for $z=\sigma$ residue $e^{i\sigma}/2$

for $z=-\sigma$ residue $e^{-i\sigma}/2$

detouring around poles we find that residue theorem yields

$$\frac{1}{2i}\,\mathrm{P.V.}\int_{-\infty}^{\infty}\frac{ze^{iz}}{z^2-\sigma^2}\,dz-\pi i\;\frac{1}{2i}\frac{e^{-i\sigma}}{2}+\pi i\;\frac{1}{2i}\;\frac{e^{i\sigma}}{2}=2\pi i\;\frac{1}{2i}\;\frac{e^{i\sigma}}{2}$$

recalling contour for second integral is clockwise

$$-\frac{1}{2i}\,\mathrm{P.V.}\int_{-\infty}^{\infty}\frac{ze^{-iz}}{z^2-\sigma^2}\,dz+\pi i\;\frac{1}{2i}\frac{e^{i\sigma}}{2}-\pi i\;\frac{1}{2i}\;\frac{e^{-i\sigma}}{2}=2\pi i\;\frac{1}{2i}\;\frac{e^{i\sigma}}{2}$$

adding ➡ $$\mathrm{P.V.}\int_{-\infty}^{\infty}\frac{x\sin x}{x^2-\sigma^2}=\frac{\pi}{2}\left(e^{i\sigma}+e^{-i\sigma}\right)=\pi\cos\sigma$$


TO BE
CONTINUED....

Theorem 2.3.7. [Taylor's Theorem]

Let f be analytic within a domain D and z_0 be a point in D
Then f has a series representation

$$f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(z_0)}{k!}(z - z_0)^k \quad (2.3.75.)$$

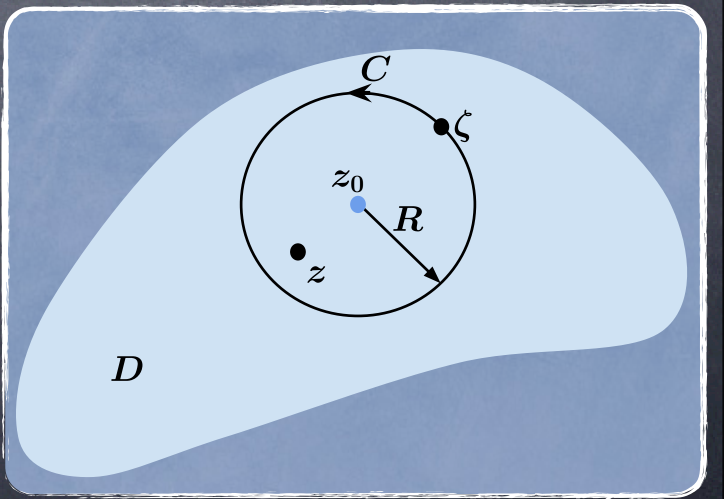
valid for the largest circle C with center at z_0 and radius R that lies entirely within D

Proof.

Let z be a fixed point within circle C
and let ζ denote the integration variable

Circle C is described by

$$|\zeta - z_0| = R$$ ☛



Use Cauchy integral formula to obtain value of f at z

$$f(z) = \frac{1}{2\pi i}\oint \frac{f(\zeta)}{\zeta - z}d\zeta$$

$$= \frac{1}{2\pi i}\oint \frac{f(\zeta)}{(\zeta - z_0) - (z - z_0)}d\zeta \qquad (2.3.76.)$$

$$= \frac{1}{2\pi i}\oint \frac{f(\zeta)}{\zeta - z_0}\ \frac{1}{1 - (z - z_0)/(\zeta - z_0)}d\zeta$$

We need the following algebraic identity

$$\frac{1}{1-q} = 1 + q + q^2 + \cdots + q^{n-1} + \frac{q^n}{1-q} \qquad (2.3.77.)$$

which follows easily from

$$1 + q + q^2 + \cdots + q^{n-1} = \frac{1-q^n}{1-q} \qquad (2.3.78.)$$

By replacing q by $(z - z_0)/(\zeta - z_0)$ in (2.3.77) we have

$$\left(1 - \frac{z - z_0}{\zeta - z_0}\right)^{-1} = 1 + \frac{z - z_0}{\zeta - z_0} + \left(\frac{z - z_0}{\zeta - z_0}\right)^2 + \cdots + \left(\frac{z - z_0}{\zeta - z_0}\right)^{n-1} + \frac{(z - z_0)^n}{(\zeta - z)(\zeta - z_0)^{n-1}} \quad (2.3.79.)$$

and so (2.3.76.) becomes

$$\begin{aligned} f(z) = {} & \frac{1}{2\pi i}\oint_C \frac{f(\zeta)}{\zeta - z_0} d\zeta + \frac{z - z_0}{2\pi i}\oint_C \frac{f(\zeta)}{(\zeta - z_0)^2} d\zeta \\ & + \frac{(z - z_0)^2}{2\pi i}\oint_C \frac{f(\zeta)}{(\zeta - z_0)^3} d\zeta + \cdots + \frac{(z - z_0)^{n-1}}{2\pi i}\oint_C \frac{f(\zeta)}{(\zeta - z_0)^n} d\zeta \\ & + \frac{(z - z_0)^n}{2\pi i}\oint_C \frac{f(\zeta)}{(\zeta - z)(\zeta - z_0)^n} d\zeta \end{aligned} \quad (2.3.80.)$$

Utilizing Cauchy's integral formula for derivatives

we can write (2.3.80.) as

$$f(z) = f(z_0) + \frac{f'(z_0)}{1!}(z - z_0) + \frac{f''(z_0)}{2!}(z - z_0)^2 + \dots$$

$$+ \frac{f^{(n-1)}(z_0)}{(n-1)!}(z - z_0)^{n-1} + \rho_n(z) \qquad (2.3.81.)$$

where

$$\rho_n(z) = \frac{(z - z_0)^n}{2\pi i} \oint_C \frac{f(\zeta)}{(\zeta - z)(\zeta - z_0)^n} d\zeta \qquad (2.3.82.)$$

Now ☛ we just need to show that $\lim_{n\to\infty} |\rho_n(z)| = 0$

Since f is analytic in D ☛ $|f(z)|$ has a maximum value M on C

In addition ☛ since z is inside C we have $|z - z_0| < R$

$$|\zeta - z| = |\zeta - z_0 - (z - z_0)| \geq |\zeta - z_0| - |z - z_0| = R - d$$

where $d = |z - z_0|$ ☛ distance from z to z_0 (2.3.83.)

ML-inequality then gives

$$\rho_n(z) = \left| \frac{(z-z_0)^n}{2\pi i} \oint_C \frac{f(\zeta)}{(\zeta - z)(\zeta - z_0)^n} \, d\zeta \right|$$
$$\leq \frac{d^n}{2\pi} \frac{M}{(R-d)R^n} 2\pi R = \frac{MR}{R-d} \left(\frac{d}{R} \right)^n \qquad (2.3.84.)$$

Because $d < R, (d/R)^n \to 0$ as $n \to \infty$
we conclude that $|\rho_n(z)| \to 0$ as $n \to \infty$

It follows that infinite series

$$f(z_0) + \frac{f'(z_0)}{1!}(z-z_0) + \frac{f''(z_0)}{2!}(z-z_0)^2 + \dots \qquad (2.3.85.)$$

converges to $f(z)$

In other words result in (2.3.75.)

is valid for any point z interior to C