Physics 307

Mathematical Physics

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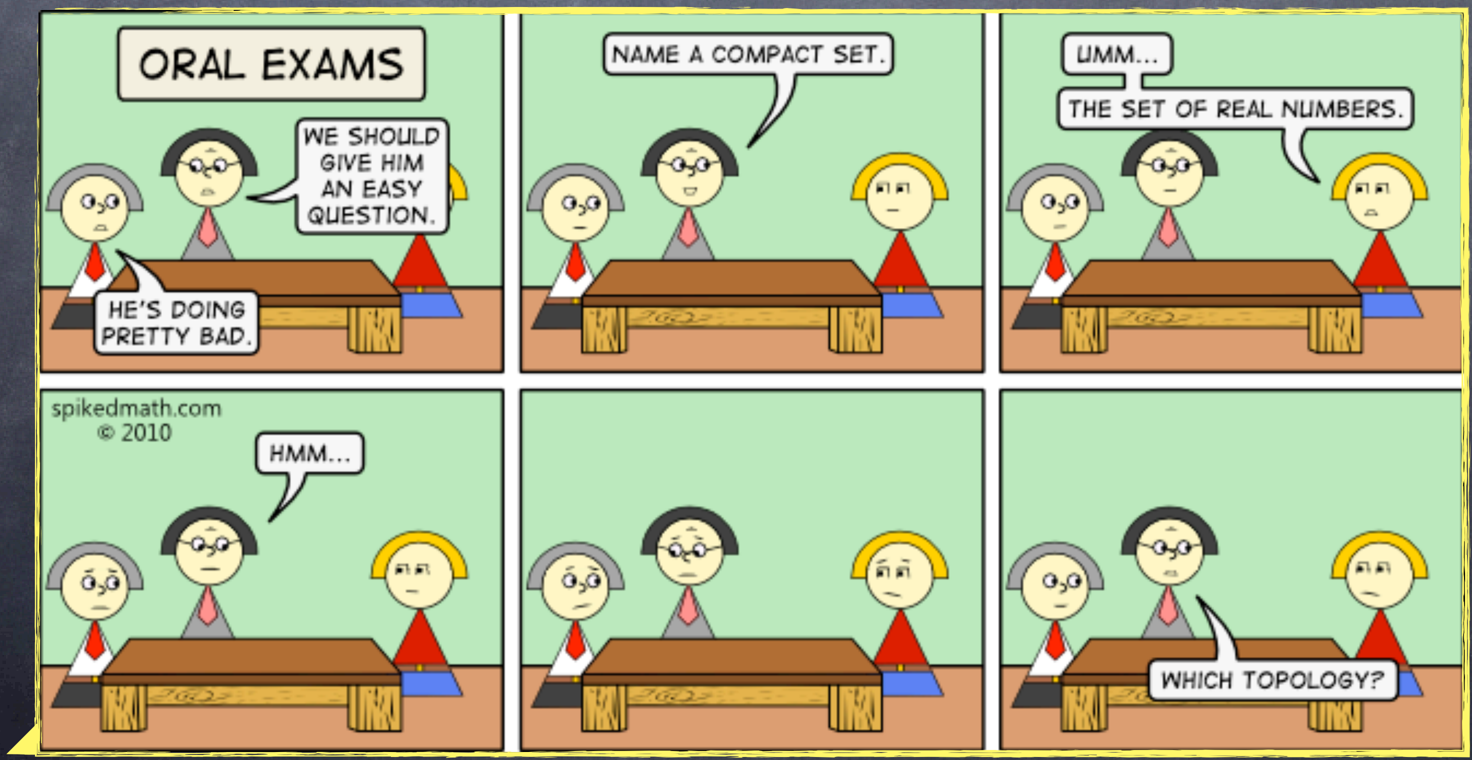
Complex Analysis II

1.1 Complex Algebra ✓

1.2 Functions of a Complex Variable ✓

1.3 Cauchy's Theorem and its Applications

1.4 Isolated Singularities and Residues



Cauchy's Theorem and its Applications

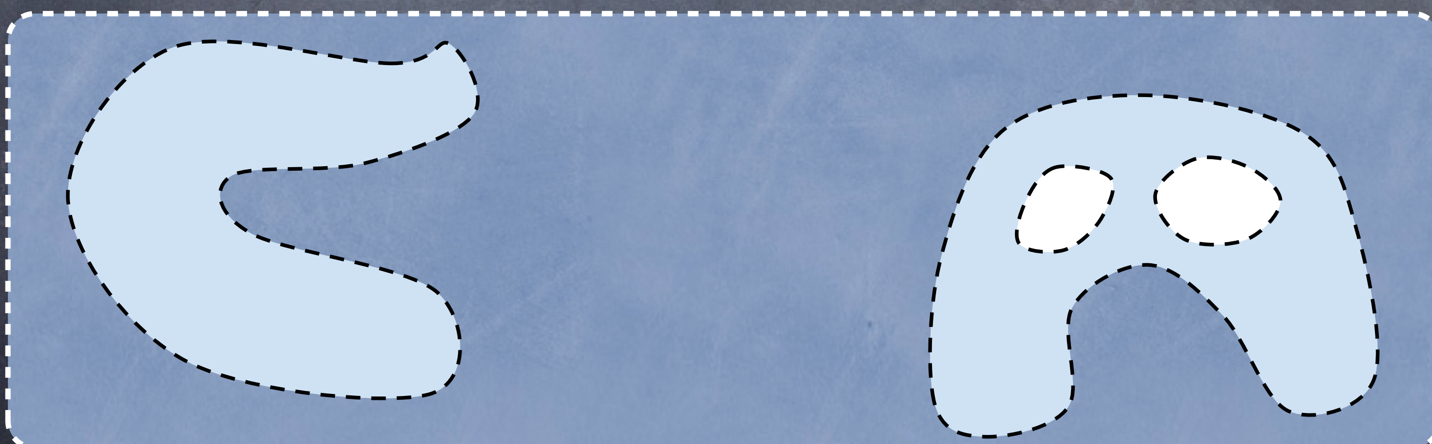
Definition 1.3.3.

A domain D is called simply connected

if every simple closed contour encloses points of D only

A domain D is called multiply connected

if it is not simply connected



Simply connected domain multiply connected domain

Theorem 1.3.2. [Cauchy's Theorem]

If a function $f(z) = u(x, y) + iv(x, y)$

is analytic on a simply connected domain D
and C is a simple closed contour lying in D

then ➡ $\oint_C f(z)dz = 0$

Proof.

with an extra hypothesis

☛ partial derivatives of u and v are continuous

this was originally imposed by Cauchy

but later shown unnecessary by Goursat

RECALL THAT...

STOKES THEOREM RELATES THE SURFACE INTEGRAL OF THE CURL OF A VECTOR FIELD F OVER A SURFACE Σ IN EUCLIDEAN THREE-SPACE

TO THE LINE INTEGRAL OF THE VECTOR FIELD OVER ITS BOUNDARY ∂Σ

$$\iint_{\Sigma} \vec{\nabla} \times \vec{F} \cdot d\vec{\Sigma} = \oint_{\partial\Sigma} \vec{F} \cdot d\vec{r}$$

We prove theorem by direct application of Stokes' theorem

Writing $dz = dx + i\,dy$

$$\oint_C f(z)dz = \oint_C (u+iv)(dx+idy)$$
$$= \oint_C (u\,dx - v\;dy) + i\oint_C (v\;dx + u\;dy) \qquad (1.3.41.)$$

two line integrals are converted to surface integrals

procedure that is justified because we have assumed partial derivatives to be continuous within area enclosed by C

In applying Stokes' theorem

note that final two integrals of (1.3.41.) are real

$$\oint_C (u\,dx - vdy) = -\int_A (v_x + u_y)dxdy \qquad (1.3.42.)$$

and

$$\oint_C (v\,dx + udy) = \int_A (u_x - v_y)dxdy \qquad (1.3.43.)$$

we can rewrite integration around loop (1.3.41.)
as value of surface integral over enclosed area A

$$\oint_C f(z)\, dz = -\int_A (v_x + u_y)\, dx\, dy + i \int_A (u_x - v_y)\, dx\, dy \quad (1.3.44.)$$

Recalling that $f(z)$ has been assumed analytic
we find that both surface integrals in (1.3.44.) are zero
because application of Cauchy-Riemann conditions
makes their integrands vanish

Example 1.3.1.

$\oint_C z^n dz$ where C is a circle of radius $r > 0$ around origin $z = 0$
in positive mathematical sense (counterclockwise)

In polar coordinates $z = re^{i\theta}$ and $dz = ire^{i\theta} d\theta$

For $n \neq -1$
(n an integer)

$$\oint_C z^n dz = ir^{n+1} \int_0^{2\pi} \exp[(i(n+1)\theta] d\theta$$

(1.3.45.)

Because 2π is a period of $e^{i(n+1)\theta}$

$$= ir^{n+1} \left. \frac{e^{i(n+1)\theta}}{i(n+1)} \right]_0^{2\pi} = 0$$

For $n=-1$

$$\oint \frac{dz}{z} = i\int_0^{2\pi} d\theta = i2\pi \qquad (1.3.46.)$$

independent of r but nonzero

Remark

The fact that (1.3.45.) is satisfied for all integers $n \geq 0$
is required by Cauchy's theorem
because for these n values z^n is analytic
for all points within a circle of radius r

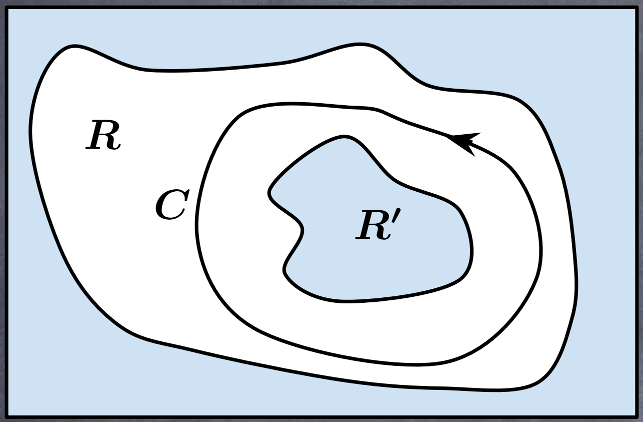
Cauchy's theorem does not apply for any negative integer n
because for these n ☛ z^n is singular at $z=0$

Theorem does not prescribe any particular values
for integrals of negative n

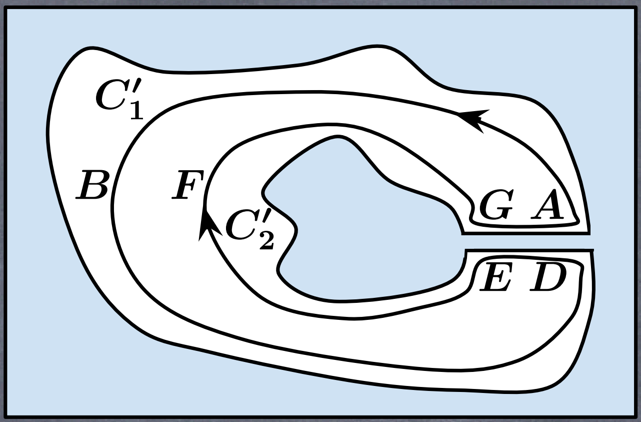
Proposition 1.3.1.

Cauchy's integral theorem demands
a simply connected region of analyticity

This restriction may be relaxed by creation of a barrier:
a narrow region we exclude from the region identified as analytic



Closed contour C in multiply connected region



Conversion of multiply connected region into simply connected region

Cauchy's integral theorem is not valid for contour C
but we can construct a contour C' for which theorem holds

New contour $ABDEFGA$ never crosses barrier

that converts R into a simply connected region

$f(z)$ is in fact continuous across barrier

$$\int_G^A f(z)\, dz = -\int_D^E f(z)\, dz \qquad (1.3.47.)$$

since contour is now within a simply connected region

we use Cauchy's integral theorem and (1.3.47.)

to cancel contribution of segments along barrier

$$\oint_{C'} f(z)dz = \int_{ABD} f(z)dz + \int_{EFG} f(z)dz = 0 \qquad (1.3.48.)$$

Note that A and D are only infinitesimally separated

Renaming $ABDA$ C_1' and $EFGE$ as $-C_2'$ we have

$$\oint_{C_1'} f(z)dz = \oint_{C_2'} f(z)dz \qquad (1.3.49.)$$

C_1' and C_2' are both traversed counterclockwise ☛ positive direction

Corollary 1.3.1

This result calls for some interpretation

We have shown that integral of an analytic function over a closed contour surrounding an island of non-analyticity can be subjected to any continuous deformation within region of analyticity without changing value of integral

The notion of continuous deformation means that the change in contour must be able to be carried out via a series of small steps ☛ which precludes processes whereby we jump over a point or region of non-analyticity

Since we already known that integral of an analytic function over a contour in a simply connected region of analyticity has value zero ☛ we can make more general statement:

The integral of an analytic function over a closed path has a value that remains unchanged over all possible continuous deformations of the contour within the region of analyticity

Corollary 1.3.2.

Using example 1.3.1 + closed contours in a region of analyticity can be deformed continuously without altering value of integral we have valuable and useful result

integral of $(z - z_0)^n$ around any counterclockwise closed path C that encloses z_0 has for any integer n the values

$$\oint_C (z - z_0)^n dz = \begin{cases} 0 & \text{if } n \neq -1 \\ 2\pi i & \text{if } n = -1 \end{cases} \qquad (1.3.50.)$$

Theorem 1.3.3. [Cauchy's Integral Formula]

Let f be analytic in a simply connected domain D and let C be a simple closed contour lying entirely within D

If z_0 is any point within C

$$f(z_0) = \frac{1}{2\pi i} \oint \frac{f(z)}{z - z_0}\, dz \qquad (1.3.51.)$$

Proof.

Let D be a simply connected domain

C a simple closed contour in D

z_0 an interior point of C

C_1 be a circle centered at z_0

with radius small enough that it is interior to C

By principle of deformation of contours ☛ we can write

$$\oint_C \frac{f(z)}{z - z_0} dz = \oint_{C_1} \frac{f(z)}{z - z_0} dz \qquad (1.3.52.)$$

We wish to show that value of integral on right is $2\pi i f(z_0)$

To this end we add and subtract constant $f(z_0)$ in numerator:

$$\oint_{C_1} \frac{f(z)}{z - z_0} dz = \oint_{C_1} \frac{f(z_0) - f(z_0) + f(z)}{z - z_0} dz \qquad (1.3.53.)$$

$$= f(z_0) \oint_{C_1} \frac{1}{z - z_0} dz + \oint_{C_1} \frac{f(z) - f(z_0)}{z - z_0} dz$$

Substituting (1.3.50) into (1.3.53) we obtain

$$\oint_{C_1} \frac{f(z)}{z-z_0} dz = 2\pi i f(z_0) + \oint_{C_1} \frac{f(z)-f(z_0)}{z-z_0} dz \qquad (1.3.54.)$$

Since f is continuous at z_0 ☛ for any arbitrarily small $\epsilon > 0$

there exists a $\delta > 0$ such that $|f(z) - f(z_0)| < \epsilon$ whenever $|z - z_0| < \delta$

In particular ☛ if we choose circle C_1 to be $|z - z_0| = \delta/2 < \delta$

then by ML -inequality

the absolute value of integral on right side of (2.3.54.) satisfies

$$\left| \oint_{C_1} \frac{f(z)-f(z_0)}{z-z_0} dz \right| \leq \frac{\epsilon}{\delta/2} \, 2\pi \, \frac{\delta}{2} = 2\pi\epsilon \qquad (1.3.55.)$$

This can be made arbitrarily small by taking

radius of circle C_1 to be sufficiently small

this can happen only if integral is zero

Cauchy integral formula (1.3.51.)

follows from (1.3.54.) by dividing both sides by $2\pi i$

Corollary 1.3.3.

Cauchy's integral formula may be used to obtain

an expression for derivative of $f(z)$

Differentiating (1.3.51.) with respect to z_0

and interchanging differentiation and integration

$$f'(z_0) = \frac{1}{2\pi i} \oint \frac{f(z)}{(z-z_0)^2} dz \qquad (1.3.56.)$$

$$f''(z_0) = \frac{2}{2\pi i} \oint \frac{f(z)}{(z-z_0)^3} dz \qquad (1.3.57.)$$

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint \frac{f(z)}{(z-z_0)^{n+1}} dz \qquad (1.3.58.)$$

Derivatives of $f(z)$ are automatically analytic at all orders

Summary

Let C be a simple closed curve contained in a simply connected domain and f an analytic function defined on C

$$\oint_C \frac{f(z)}{(z-z_0)^{n+1}} dz = \begin{cases} 2\pi i f(z_0), & \text{if } n = 0 \text{ and } z_0 \text{ is enclosed by } C \\ 2\pi i f^{(n)}(z_0)/n!, & \text{if } n \geq 1 \text{ and } z_0 \text{ is enclosed by } C \\ 0 & \text{if } z_0 \text{ is not enclosed by } C \end{cases}$$

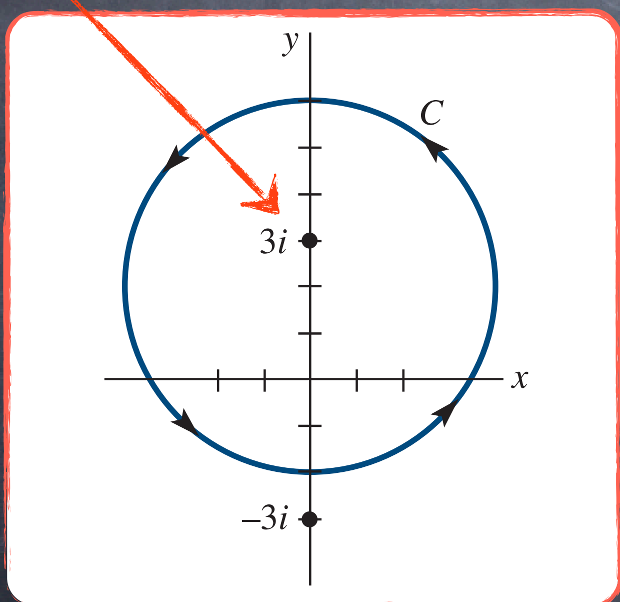
Example 1.3.2. To evaluate

$$\oint_C \frac{z}{z^2+9} dz \text{ with } C \text{ the circle } |z-2i| = 4 \qquad (1.3.59.)$$

we factorize denominator as $z^2 + 9 = (z-3i)(z+3i)$

$3i$ is only point within closed contour

at which the integrand fails to be analytic



by writing $$\frac{z}{z^2+9} = \frac{z/(z+3i)}{z-3i} \qquad (1.3.60.)$$

we can identify $f(z) = z/(z+3i)$

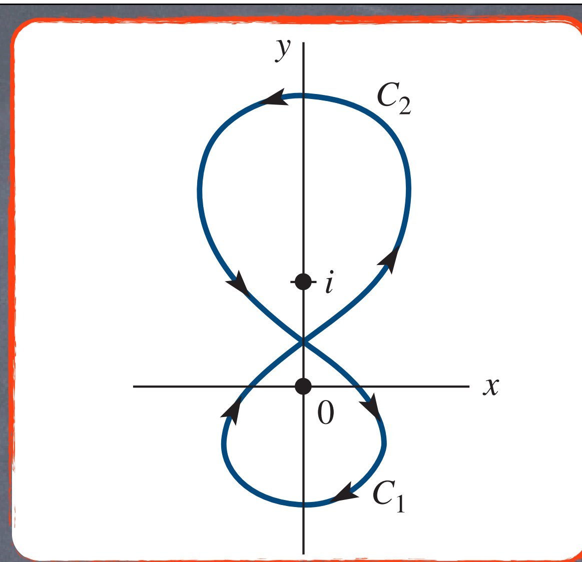
From Cauchy's integral formula we have

$$\oint_C \frac{z}{z^2+9} dz = i\pi \qquad (1.3.61.)$$

Example 1.3.3. To evaluate

$$\oint_C \frac{z^3+3}{z(z-i)^2}\,dz \qquad (1.3.62.)$$

with C



C is not a simple closed contour

but we can think of it as union of 2 simple closed contours $C_1 \cup C_2$

We rewrite (1.3.62.) as

$$\oint_C \frac{z^3+3}{z(z-i)^2}dz = \oint_{C_1} \frac{z^3+3}{z(z-i)^2}dz + \oint_{C_2} \frac{z^3+3}{z(z-i)^2}dz$$

$$= -\oint_{-C_1} \frac{(z^3+3)/(z-i)^2}{z}dz + \oint_{C_2} \frac{(z^3+3)/z}{(z-i)^2}dz$$

where we have considered that C_1 is circulated clockwise

whereas C_2 is circulated counterclockwise

To evaluate first integral

we identify $z_0 = 0, f(z) = (z^3+3)/(z-i)^2$

$$\oint_{-C_1} \frac{(z^3+3)/(z-i)^2}{z} dz = -6\pi i \qquad (1.3.63.)$$

To evaluate second integral

we identify $z_0 = i, n = 1, f(z) = (z^3+3)/z$ and $f'(z) = (2z^3-3)/z^2$

$$\oint_{C_2} \frac{(z^3+3)/z}{(z-i)^2} dz = \frac{2\pi i}{1!}(2i+3) = 2\pi(-2+3i) \qquad (1.3.64.)$$

It follows that

$$\oint_{C} \frac{z^3}{z(z-i)^2} dz = -4\pi + 12\pi i \qquad (1.3.65.)$$

Theorem 1.3.4. [Cauchy's estimate]

Let f be analytic on a simply connected domain D and for some $R > 0$ closed ball $\overline{B_R(z_0)} \subset D$ with $C_R(z_0) = \{z : |z - z_0| = R\}$

If $|f(z)| \leq M_R, \ \forall z \in C_R(z_0)$ ☛ then

$$\left| f^{(n)}(z_0) \right| \leq \frac{n! M_R}{R^n}, \ \forall n \geq 0 \qquad (1.3.66.)$$

Proof.

According to Cauchy's integral formula we have

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint_{C_R(z_0)} \frac{f(z)}{(z - z_0)^{n+1}} dz \qquad (1.3.67.)$$

then

$$|f^{(n)}(z_0)| = \left| \frac{n!}{2\pi i} \oint_{C_R(z_0)} \frac{f(z)}{(z - z_0)^{n+1}} dz \right|$$

$$\leq \left| \frac{n!}{2\pi i} \right| \oint_{C_R(z_0)} \left| \frac{f(z)}{(z - z_0)^{n+1}} \right| |dz| = \frac{n!}{2\pi} \oint_{C_R(z_0)} \frac{|f(z)|}{|z - z_0|^{n+1}} |dz|$$

$$\leq \frac{n!}{2\pi} \oint_{C_R(z_0)} \frac{M_R}{R^{n+1}} |dz| = \frac{n!}{2\pi} \frac{M_R}{R^{n+1}} 2\pi R = \frac{n! M_R}{R^n} \qquad (1.3.68.)$$

Corollary 1.3.4. [Liouville's theorem]

If $f(z)$ is analytic and bounded in entire complex plane then it is a constant

Proof.

To prove this we will prove that f' is the zero function

Suppose $f : \mathbb{C} \to \mathbb{C}$ is everywhere analytic and is bounded by M

i.e. $|f(z)| \leq M$ for every $z \in \mathbb{C}$

Fix an arbitrary point $z_0 \in \mathbb{C}$

Since f is analytic everywhere ☛ it is in particular analytic on a neighborhood of closed ball $\overline{B_R(z_0)}$ for any value of $R > 0$

By Cauchy's estimate there exists $M_R = \max\{|f(z)| \text{ in } |z - z_0| = R\} \leq M$

and thus $$|f'(z_0)| \leq \frac{M_R}{R} \leq \frac{M}{R}, \quad \forall R > 0 \qquad (1.3.69.)$$

Since expression on left is a nonnegative constant

letting $R \to \infty$ on right yields $0 \leq |f'(z_0)| \leq 0$ whence $f'(z_0) = 0$

Theorem 1.3.5. [Fundamental Theorem of Algebra]

Every polynomial $P(z)$ of degree $n \geq 1$ has a root in $\mathbb{C}$

Proof.

Suppose $P(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_0$ is a polynomial with no root in $\mathbb{C}$ then $P^{-1}(z)$ must be analytic on whole $\mathbb{C}$

Since

$$\left|\frac{P(z)}{z^n}\right| = \left|1 + \frac{a_{n-1}}{z} + \cdots + \frac{a_0}{z^n}\right| \to 1, \text{ as } |z| \to \infty \quad (1.3.70.)$$

it follows that $|P(z)| \to \infty$ and hence $|1/P(z)| \to 0$ as $|z| \to \infty$

(prove of a well known fact: polynomials are unbounded functions)

Consequently $P^{-1}(z)$ is a bounded function

Hence ☛ by Liouville's theorem

$P^{-1}(z)$ would have to be constant ➡ which is a contradiction

ML-inequality then gives

$$\rho_n(z) = \left| \frac{(z-z_0)^n}{2\pi i} \oint_C \frac{f(\zeta)}{(\zeta - z)(\zeta - z_0)^n} \, d\zeta \right|$$

$$\leq \frac{d^n}{2\pi} \frac{M}{(R-d)R^n} 2\pi R = \frac{MR}{R-d} \left(\frac{d}{R} \right)^n \qquad (2.3.84.)$$

Because $d < R, (d/R)^n \to 0$ as $n \to \infty$ we conclude that $|\rho_n(z)| \to 0$ as $n \to \infty$

It follows that infinite series

$$f(z_0) + \frac{f'(z_0)}{1!}(z - z_0) + \frac{f''(z_0)}{2!}(z - z_0)^2 + \ldots \qquad (2.3.85.)$$

converges to $f(z)$

In other words result in (2.3.75.)

is valid for any point z interior to C

Theorem 1.3.6. [Taylor's Theorem]

Let f be analytic within a domain D and z_0 be a point in D
Then f has a series representation

$$f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(z_0)}{k!}(z-z_0)^k \quad (1.3.75.)$$

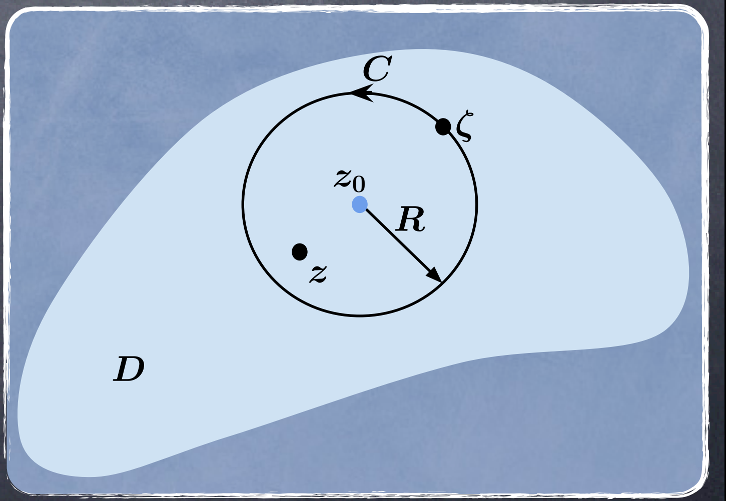
valid for the largest circle C with center at z_0 and radius R that lies entirely within D

Proof.

Let z be a fixed point within circle C
and let ζ denote the integration variable

Circle C is described by

$$|\zeta - z_0| = R$$ ☛



Use Cauchy integral formula to obtain value of f at z

$$f(z) = \frac{1}{2\pi i}\oint \frac{f(\zeta)}{\zeta - z}d\zeta$$

$$= \frac{1}{2\pi i}\oint \frac{f(\zeta)}{(\zeta - z_0) - (z - z_0)}d\zeta \qquad (1.3.76.)$$

$$= \frac{1}{2\pi i}\oint \frac{f(\zeta)}{\zeta - z_0} \frac{1}{1 - (z - z_0)/(\zeta - z_0)}d\zeta$$

We need the following algebraic identity

$$\frac{1}{1-q} = 1 + q + q^2 + \cdots + q^{n-1} + \frac{q^n}{1-q} \qquad (1.3.77.)$$

which follows easily from

$$1 + q + q^2 + \cdots + q^{n-1} = \frac{1-q^n}{1-q} \qquad (1.3.78.)$$

(1.3.78.) follows from

$$\sum_{j=0}^{n} q^j = 1 + q + q^2 + q^3 + \cdots + q^n$$

$$q\sum_{j=0}^{n} q^j = q + q^2 + q^3 + q^4 + \cdots + q^{n+1}$$

$$\sum_{j=0}^{n} q^j - q\sum_{j=0}^{n} q^j = 1 - q^{n+1}$$

$$\sum_{j=0}^{n} q^j = \frac{1 - q^{n+1}}{1 - q}$$

By replacing q by $(z-z_0)/(\zeta-z_0)$ in (1.3.77) we have

$$\left(1-\frac{z-z_0}{\zeta-z_0}\right)^{-1} = 1+\frac{z-z_0}{\zeta-z_0}+\left(\frac{z-z_0}{\zeta-z_0}\right)^2+\cdots+\left(\frac{z-z_0}{\zeta-z_0}\right)^{n-1} + \frac{(z-z_0)^n}{(\zeta-z)(\zeta-z_0)^{n-1}} \quad (1.3.79.)$$

and so (1.3.76.) becomes

$$f(z)=\frac{1}{2\pi i}\oint_C \frac{f(\zeta)}{\zeta-z_0}d\zeta+\frac{z-z_0}{2\pi i}\oint_C\frac{f(\zeta)}{(\zeta-z_0)^2}d\zeta + \frac{(z-z_0)^2}{2\pi i}\oint_C\frac{f(\zeta)}{(\zeta-z_0)^3}d\zeta+\cdots+\frac{(z-z_0)^{n-1}}{2\pi i}\oint_C\frac{f(\zeta)}{(\zeta-z_0)^n}d\zeta + \frac{(z-z_0)^n}{2\pi i}\oint_C\frac{f(\zeta)}{(\zeta-z)(\zeta-z_0)^n}d\zeta \quad (1.3.80.)$$

Utilizing Cauchy's integral formula for derivatives

we can write (1.3.80.) as

$$f(z) = f(z_0) + \frac{f'(z_0)}{1!}(z - z_0) + \frac{f''(z_0)}{2!}(z - z_0)^2 + \dots$$
$$+ \frac{f^{(n-1)}(z_0)}{(n-1)!}(z - z_0)^{n-1} + \rho_n(z) \qquad (1.3.81.)$$

where

$$\rho_n(z) = \frac{(z - z_0)^n}{2\pi i} \oint_C \frac{f(\zeta)}{(\zeta - z)(\zeta - z_0)^n} d\zeta \qquad (1.3.82.)$$

Now ☛ we just need to show that $\lim_{n\to\infty} |\rho_n(z)| = 0$

Since f is analytic in D ☛ $|f(z)|$ has a maximum value M on C

In addition ☛ since z is inside C we have $|z - z_0| < R$

$$|\zeta - z| = |\zeta - z_0 - (z - z_0)| \geq |\zeta - z_0| - |z - z_0| = R - d$$

where $d = |z - z_0|$ ☛ distance from z to z_0 (1.3.83.)

ML-inequality then gives

$$\rho_n(z) = \left| \frac{(z-z_0)^n}{2\pi i} \oint_C \frac{f(\zeta)}{(\zeta - z)(\zeta - z_0)^n} d\zeta \right|$$

$$\leq \frac{d^n}{2\pi} \frac{M}{(R-d)R^n} 2\pi R = \frac{MR}{R-d} \left(\frac{d}{R} \right)^n \qquad (1.3.84.)$$

Because $d < R, (d/R)^n \to 0$ as $n \to \infty$

we conclude that $|\rho_n(z)| \to 0$ as $n \to \infty$

It follows that infinite series

$$f(z_0) + \frac{f'(z_0)}{1!}(z-z_0) + \frac{f''(z_0)}{2!}(z-z_0)^2 + \dots \qquad (1.3.85.)$$

converges to $f(z)$

In other words result in (1.3.75.)

is valid for any point z interior to C

Theorem 1.3.7. [Morera's Theorem]

If f is continuous in a simply connected domain D

and if $\oint f(z)dz = 0$ for every simple closed contour C in D

then f is analytic throughout D

Proof.

To prove theorem we integrate $f(z)$ from z_1 to z_2

Since every closed-path integral of $f(z)$ vanishes
this integral is independent of path
and depends only on its ends points

We may therefore write

$$F(z_2) - F(z_1) = \int_{z_1}^{z_2} f(z)\, dz \quad (1.3.71.)$$

where $F(z)$ can be called the indefinite integral of $f(z)$

We then construct the identity

$$\frac{F(z_2) - F(z_1)}{z_2 - z_1} - f(z_1) = \frac{1}{z_2 - z_1} \int_{z_1}^{z_2} [f(t) - f(z_1)]\, dt \quad (1.3.72.)$$

where we have introduced another complex variable t

Using fact that $f(t)$ is continuous we write

(keeping only terms to first order in $t - z_1$)

$$f(t) - f(z_1) = f'(z_1)(t - z_1) + \ldots \quad (1.3.73.)$$

which implies that

$$\int_{z_1}^{z_2} [f(t) - f(z_1)]\, dt = \int_{z_1}^{z_2} [f'(z_1)(t - z_1) + \ldots]\, dt = \frac{f'(z_1)}{2}(z_2 - z_1)^2 + \ldots$$

Note that right-hand side of (1.3.72.) approaches zero for $z_2 \to z_1$

$$f(z_1) = \lim_{z_2 \to z_1} \frac{F(z_2) - F(z_1)}{z_2 - z_1} = F'(z_1) \quad (1.3.74.)$$

Equation (1.3.74.) shows that $F(z)$ is analytic in D

then so also must be its derivative $f(z)$

thereby proving Morera's theorem


TO BE
CONTINUED....