Physics 307

Mathematical Physics

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Partial Differential Equations I

4.1 Taxonomy

4.2 Wave Equation

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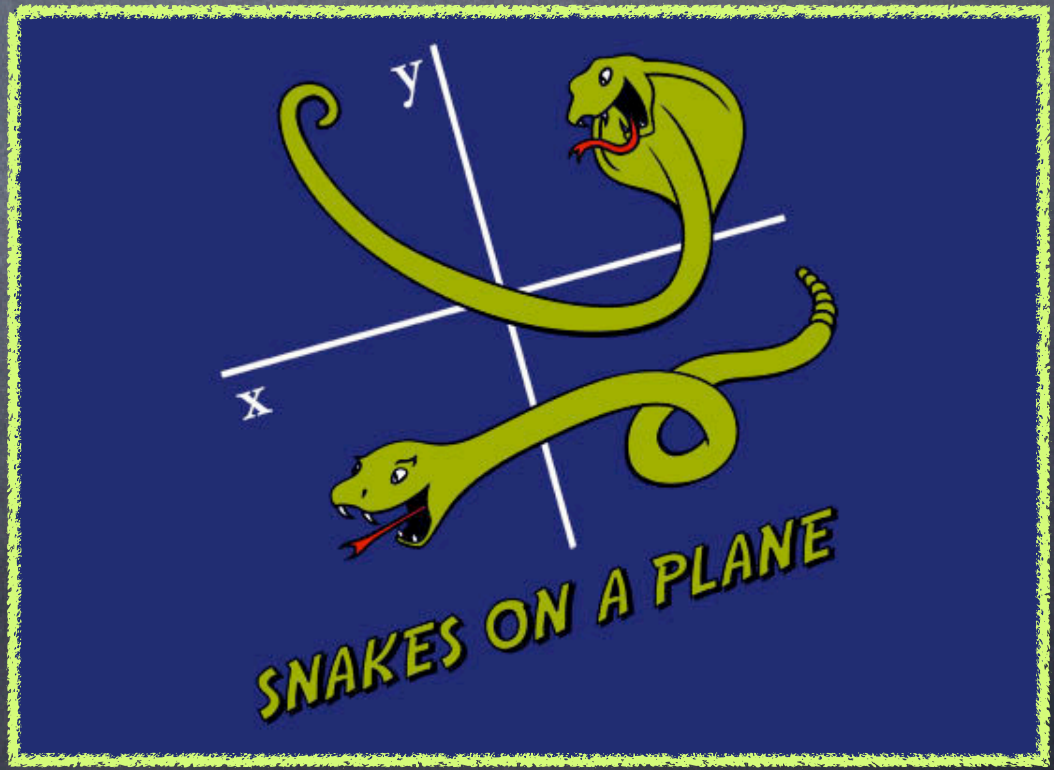
4.4 Laplace Equation



4.1 Taxonomy

A partial differential equation is an equation that involves an unknown function and some of its partial derivatives with respect to two or more independent variables

An n-th order equation has its highest order derivative of order n

A partial differential equation is linear if it is an equation of first degree in the dependent variable and its derivatives

A partial differential equation is homogeneous if every term contains the dependent variable or one of its partial derivatives

Interest here is in linear homogeneous 2nd-order equations the most general of which in two independent variables is given by

$$a\frac{\partial^2 u}{\partial x^2} + 2b\frac{\partial^2 u}{\partial x \partial y} + c\frac{\partial^2 u}{\partial y^2} = \varphi\left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, u\right) \quad (4.1.1)$$

φ is a linear transformation

coefficients a, b, c may be functions of x and y

For simplicity ☛ we assume that these coefficients are constants

Superposition principle holds for linear homogeneous equation that is ☛ its solutions form a linear space

By exhibiting an infinite sequence of independent solutions we will show that dimension of this solution space is infinite

We now introduce new independent variables

$$\xi = \alpha x + \beta y \quad \text{and} \quad \eta = \gamma x + \delta y \tag{4.1.2}$$

$\alpha, \beta, \gamma, \delta$ are constant to be chosen below with ☛ $\alpha\delta - \beta\gamma \neq 0$

It is seen that

$$\begin{aligned} \frac{\partial u}{\partial x} &\equiv u_x = \alpha u_\xi + \gamma u_\eta \\ \frac{\partial u}{\partial y} &\equiv u_y = \beta u_\xi + \delta u_\eta \end{aligned} \tag{4.1.3}$$

and for second derivatives that

$$\begin{aligned} u_{xx} &= \alpha^2 u_{\xi\xi} + 2\gamma\alpha u_{\xi\eta} + \gamma^2 u_{\eta\eta} \\ u_{xy} &= \alpha\beta u_{\xi\xi} + (\alpha\delta + \gamma\beta) u_{\xi\eta} + \gamma\delta u_{\eta\eta} \\ u_{yy} &= \beta^2 u_{\xi\xi} + 2\beta\delta u_{\xi\eta} + \delta^2 u_{\eta\eta} \end{aligned} \tag{4.1.4}$$

Substituting (4.1.3) and (4.1.4) into (4.1.1) we obtain

$$\mathcal{A}\ u_{\xi\xi} + 2\ \mathcal{B}\ u_{\xi\eta} + \mathcal{C}\ u_{\eta\eta} = \tilde{\varphi}(u_\xi, u_\eta, u) \qquad (4.1.5)$$

where

$$\begin{aligned} \mathcal{A} &= a\alpha^2 + 2b\alpha\beta + c\beta^2 \\ \mathcal{B} &= a\gamma\alpha + b(\alpha\delta + \beta\gamma) + c\beta\delta \\ \mathcal{C} &= a\gamma^2 + 2b\gamma\delta + c\delta^2 \end{aligned} \qquad (4.1.6)$$

By suitable choice of $\alpha, \beta, \gamma, \delta$

we can make two of these three coefficients vanish

For example ☛ let us assume that $c \neq 0$

so that roots λ_1 and λ_2 of quadratic $a + 2b\lambda + c\lambda^2 = 0$ (4.1.7)

are both finite

Let us set $\alpha = \gamma = 1, \beta = \lambda_1, \delta = \lambda_2$

so that $\xi = x + \lambda_1 y \quad \text{and} \quad \eta = x + \lambda_2 y$ (4.1.8)

For this choice ☛ $\mathcal{A} = \mathcal{C} = 0$ and therefore (4.1.5) becomes

$$2\left[a + b\underbrace{(\lambda_1 + \lambda_2)}_{-2b/c} + c\underbrace{\lambda_1\lambda_2}_{a/c}\right] u_{\xi\eta} = \frac{4}{c}(ac - b^2)u_{\xi\eta} = \tilde{\varphi}(u_\xi, u_\eta, u) \quad (4.1.9.)$$

Let us assume that only the second derivative terms are present in (4.1.1.) and therefore also in (4.1.9.)

Then ☛ assuming $ac - b^2 \neq 0$ we obtain ➡ $u_{\xi\eta} = 0$ (4.1.10.)

This has obvious general integral ☛ $u = \phi(\xi) + \psi(\eta)$ (4.1.11.)

By analogy with conic sections there are three main cases to be considered according as discriminant $b^2 - ac$ is positive, negative, or zero

Case I: $b^2 - 4ac > 0$

Roots λ_1, λ_2 are real and distinct

Standard form (4.1.10) has general solution (4.1.11)

or by (4.1.8) ☛ $u = \phi(x + \lambda_1 y) + \psi(x + \lambda_2 y)$ (4.1.12.)

In this case (4.1.1.) is said to be hyperbolic

Just as a rotation $\pi/4$ changes rectangular hyperbola $\xi\eta =$ constant

to form $\xi^2 - \eta^2 =$ constant

so rotation ☛ $s = \frac{1}{2}\xi + \frac{1}{2}\eta, \quad t = \frac{1}{2}\xi - \frac{1}{2}\eta$ (4.1.13.)

brings about alternative standard form

$$u_{tt} - u_{ss} = 0$$ (4.1.14.)

Case II: $b^2 - ac < 0$

(4.1.15.)

Roots are conjugate complex: $\lambda_1 = \rho + i\sigma = \lambda_2^*$

$\xi = x + \lambda_1 y = x + \rho y + i\sigma y$ and $\eta = x + \lambda_2 y = x + \rho y - i\sigma y = \xi^*$

Standard form is $u_{\xi\xi^*} = 0$ with general integral $u = \phi(\xi) + \psi(\xi^*)$

Let us now write $\xi = s + it$ with s and t real

$$s = x + \rho y \quad \text{and} \quad t = \sigma y \qquad (4.1.16.)$$

in addition

$$\begin{aligned} s &= \Re\mathrm{e}\,\xi = \frac{1}{2}\xi + \frac{1}{2}\xi^* = \frac{1}{2}\xi + \frac{1}{2}\eta \\ t &= \Im\mathrm{m}\,\xi = \frac{1}{2i}\xi - \frac{1}{2i}\xi^* = \frac{1}{2i}\xi - \frac{1}{2i}\eta \end{aligned} \qquad (4.1.17.)$$

In these variables standard form is seen to be ☛ $u_{tt} + u_{ss} = 0$

In this case equation is said to be elliptic

General solution becomes ☛ $u = \phi(s + it) + \psi(s - it)$ (4.1.19.)

solution is sum of a formal analytic function of $\xi = s + it$

and a formal antianalytic function of $\xi^* = s - it$

Case III: $b^2 - ac = 0$

Roots are real and equal $\lambda_1 = \lambda_2$

Note that transformation (4.1.8.) degenerates if $\eta = \xi$

Instead ☛ we choose for η any combination of x and y
not proportional to ξ

Because of (4.1.7.) ☛ $\mathcal{A}$ still vanishes & mixed coefficient becomes

$$a\gamma + b(\delta + \gamma\lambda) + c\lambda\delta = (a + b\lambda)\gamma + (b + c\lambda)\delta \qquad (4.1.20.)$$

In this case of equal roots $-\lambda = a/b = b/c$
so this expression also vanishes

Therefore ☛ standard form must be $u_{\eta\eta} = 0$ (4.1.21.)

with general integral $u = \phi(\xi) + \eta\,\psi(\xi)$ (4.1.22.)

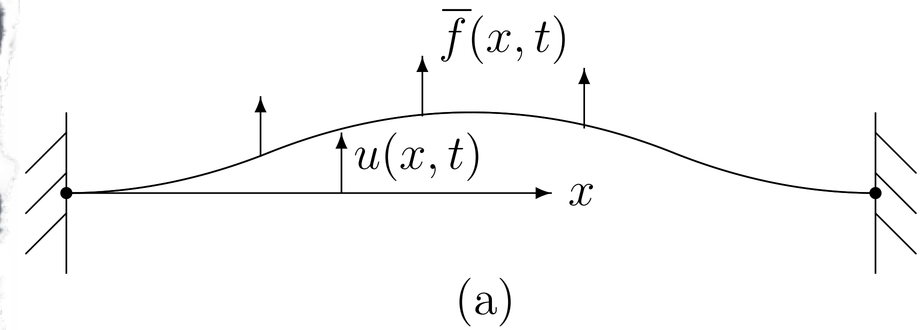
or ☛ $u = \phi(x + \lambda y) + y\,\tilde{\psi}(x + \lambda y)$ (4.1.23.)

this is parabolic case

4.2 WAVE EQUATION

4.2.1. Vibrating string

Consider a string in tension between two fixed end-points and acted upon by transverse forces



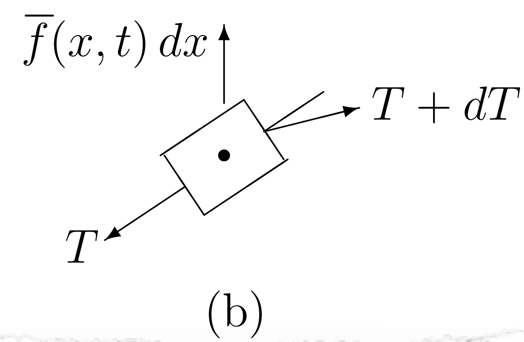
Let $u(x,t)$ denote transverse displacement where x is distance from left end and t is time

Force distribution (of dimension force/length^2) denoted ☛ $\overline{f}(x,t)$

We assume that all motion is vertical and displacement u and slope $\theta = \partial u/\partial x$ are both small

For small θ ☛ $\sin\ \theta \approx \tan\ \theta$

Differential equation of motion is derived by applying Newton's second law to a small differential segment of string



Using 2-term Taylor's series approximation tension per unit length

$$T + dT \approx T + \frac{\partial T}{\partial x}\, dx \qquad (4.2.24.)$$

Likewise for θ

$$\frac{\partial u}{\partial x} + \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) dx = \frac{\partial u}{\partial x} + \frac{\partial^2 u}{\partial x^2} dx \qquad (4.2.25.)$$

we now equate net applied force in vertical direction to ma

$$\left(T + \frac{\partial T}{\partial x} dx\right)\left(\frac{\partial u}{\partial x} + \frac{\partial^2 u}{\partial x^2} dx\right) - T\frac{\partial u}{\partial x} + \overline{f}(x,t)dx = (\rho A dx)\frac{\partial^2 u}{\partial t^2} \qquad (4.2.26.)$$

where ρ is density of string material and A is cross-sectional area

After expanding this equation and eliminating small terms we get linearized partial differential equation for vibrating string

$$u_{xx} + \overline{f}_T(x,t) = \frac{1}{c^2}\, u_{tt} \qquad (4.2.28.)$$

with ☛

$$c = \sqrt{\frac{T}{\rho A}} \quad \text{and} \quad \overline{f}_T(x,t) = \frac{\overline{f}(x,t)}{T}$$

Note that c has units of velocity

For zero force $\overline{f} = 0$ (4.2.28) reduces to $c^2 u_{xx} - u_{tt} = 0$

which is one-dimensional wave equation

Transverse displacement for unforced infinitesimal vibrations

of a stretched string satisfies wave equation

In any number of dimensions ☛ wave equation can be written as

$$\nabla^2 u = \frac{1}{c^2} u_{tt} \quad \text{with} \quad \nabla^2 = \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2} \qquad (4.2.33.)$$

For time-harmonic motion ☛ $u = u_0 \cos(\omega t)$

wave equation simplifies to Helmholtz equation

$$\nabla^2 u_0 + k^2 \, u_0 = 0 \qquad (4.2.34.)$$

u_0 is amplitude of sine wave

$k = \omega / c$ is called wave number

Helmholtz equation is usually referred to as reduced wave equation

4.2.2. d'Alembert solution

The wave equation ☛ $c^2 u_{xx} - u_{tt} = 0$

can be rewritten in canonical form ☛ $u_{\xi\eta} = 0$

by passing to characteristic variables

$$\xi = x + ct \quad \text{and} \quad \eta = x - ct \tag{4.2.35.}$$

Integrating first wrt the variable η and then wrt ξ

we obtain (4.1.11.) which in terms of $x - t$ variables reads

$$u(x,t) = \phi(x - ct) + \psi(x + ct) \tag{4.2.36.}$$

For initial conditions

$$\begin{cases} u(x,0) &= f(x) = \phi(x) + \psi(x) \\ u_t(x,0) &= g(x) = c\,[\psi'(x) - \phi'(x)] \end{cases} \tag{4.2.37.}$$

we can determine functions ϕ and ψ

Integration

$$\int_{-\infty}^{x} \psi'(\alpha)\, d\alpha - \int_{-\infty}^{x} \phi'(\alpha)\, d\alpha = \frac{1}{c}\int_{-\infty}^{x} g(\alpha)\, d\alpha \tag{4.2.39.}$$

leads to ☛

$$\psi(x) - \phi(x) = \frac{1}{c}\int_{0}^{x} g(\alpha)\, d\alpha + \frac{1}{c}\, g(0) \tag{4.2.40.}$$

Combining (4.2.37.) and (4.2.40.) we obtain

$$\psi(x) = \frac{1}{2}f(x) + \frac{1}{2c}\int_0^x g(\alpha)\ d\alpha + \frac{1}{2c}\ g(0)$$

$$\phi(x) = \frac{1}{2}f(x) - \frac{1}{2c}\int_0^x g(\alpha)\ d\alpha - \frac{1}{2c}\ g(0)$$

Substituting these two expressions into (4.2.36.) we get

$$\begin{aligned} u(x,t) &= \frac{1}{2}[f(x-ct) + f(x+ct)] + \frac{1}{2c}\left(\int_0^{x+ct} g(\alpha)\ d\alpha - \int_0^{x-ct} g(\alpha)\ d\alpha\right) \\ &= \frac{1}{2}[f(x-ct) + f(x+ct)] + \frac{1}{2c}\left(\int_0^{x+ct} g(\alpha)\ d\alpha + \int_{x-ct}^{0} g(\alpha)\ d\alpha\right) \\ &= \frac{1}{2}[f(x-ct) + f(x+ct) + \frac{1}{2c}\int_{x-ct}^{x+ct} g(\alpha)\ d\alpha \qquad (4.2.42.) \end{aligned}$$

or equivalent $u(x,t) = \frac{1}{2}[f(x-ct) + f(x+ct) + G(x+ct) - G(x-ct)]$

with $G(x+ct) - G(x-ct) = \frac{1}{c}\int_{x-ct}^{x+ct} g(\alpha)\ d\alpha$

Initial form splits into 2 pulses that travel in opposite directions

Initial velocity originates 2 pulses of different signs $\pm\frac{1}{2}G(x \pm ct)$

Example 4.2.1

For special case $g(x) = 0$ d'Alembert solution simplifies to

$$u(x,t) = \frac{1}{2}\left[f(x-ct) + f(x+ct)\right] \qquad (4.2.45.)$$

e.g. ☛ consider a triangular pulse of width $2b$ and height b

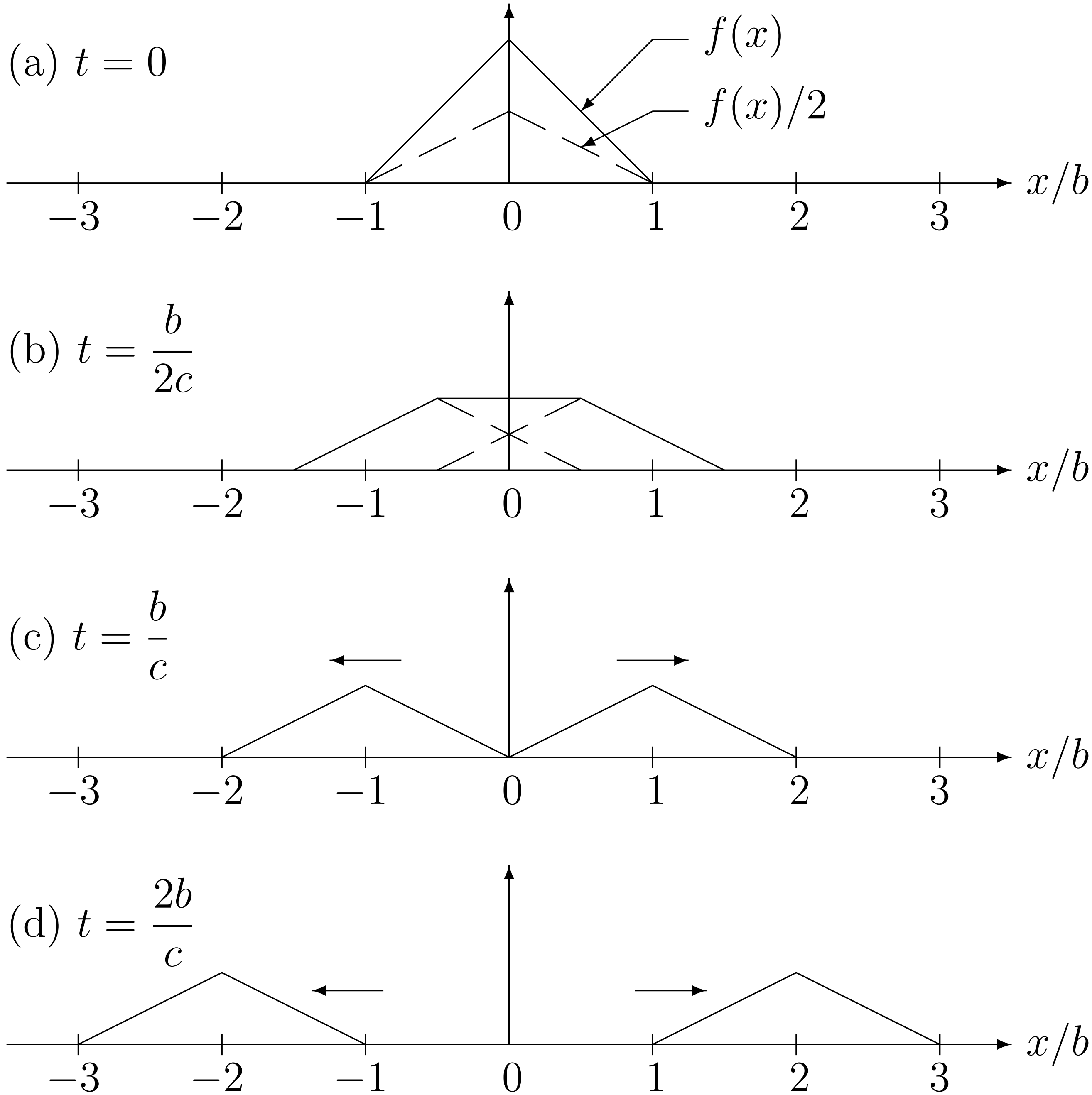
$$f(x) = \begin{cases} -|x| + b & |x| \le b \\ 0 & |x| \ge b \end{cases}$$

Solution is piecewise defined in 4 different regions of x-t half-plane

In order to determine these regions notice that

$$f(x+ct) = \begin{cases} b - |x+ct| & |x+ct| \le b \\ 0 & |x+ct| > b \end{cases}$$

$$f(x-ct) = \begin{cases} b - |x-ct| & |x-ct| \le b \\ 0 & |x-ct| > b \end{cases}$$


(a) $t = 0$
$f(x)$
$f(x)/2$
(b) $t = \frac{b}{2c}$
(c) $t = \frac{b}{c}$
(d) $t = \frac{2b}{c}$
x/b
−3 −2 −1 0 1 2 3

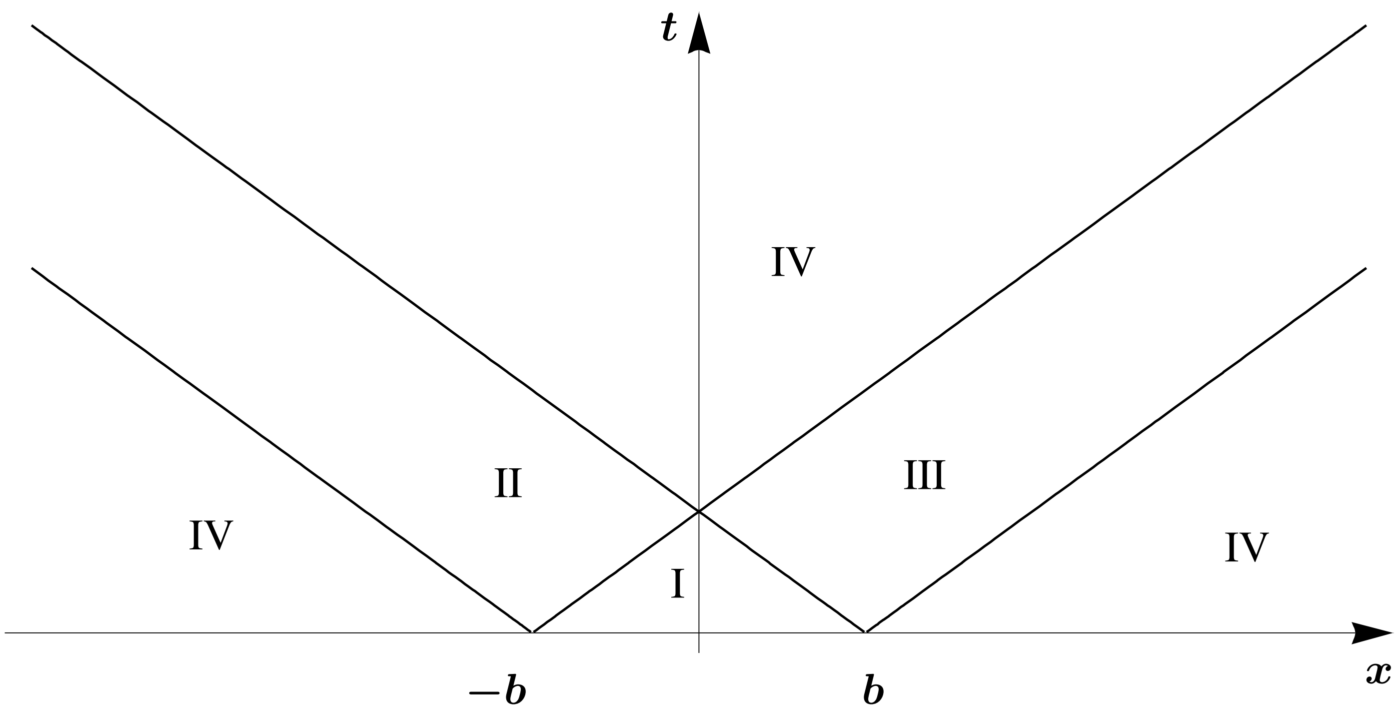
4 regions are given by

$$\text{I}: \quad \{|x+ct| \le b,\ |x-ct| \le b\} \quad \to u(x,t) = b - |x \pm ct|$$

$$\text{II}: \quad \{|x+ct| \le b,\ |x-ct| > b\} \quad \to u(x,t) = \frac{1}{2}\left(b - |x+ct|\right)$$

$$\text{III}: \quad \{|x+ct| > b,\ |x-ct| \le b\} \quad \to u(x,t) = \frac{1}{2}\left(b - |x-ct|\right)$$

$$\text{IV}: \quad \{|x+ct| > b,\ |x-ct| > b\} \quad \to u(x,t) = 0$$



Value of d'Alembert solution at a point (x_0, t_0)

$$u(x_0, t_0) = \frac{1}{2}[f(x_0 - ct_0) + f(x_0 + ct_0)] + \frac{1}{2}\int_{x_0 - ct_0}^{x_0 + ct_0} g(\alpha) d\alpha \quad (4.2.53.)$$

depends on values of f at only two points on x axis

$x_0 + ct_0$ and $x_0 - ct_0$

and values of g only on interval $[x_0 - ct_0, x_0 + ct_0]$

For this reason the interval $[x_0 - ct_0, x_0 + ct_0]$

is called interval of dependence for point (x_0, t_0)

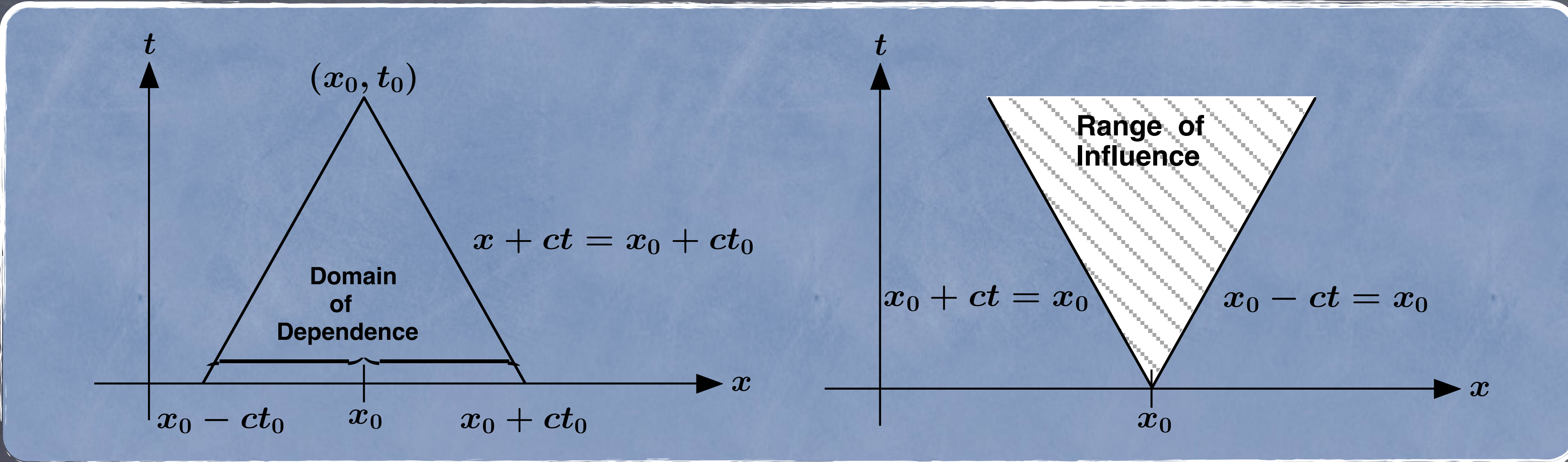
Sometimes entire triangular region

with vertices at $x_0 - ct_0$ and $x_0 + ct_0$ on x axis and vertex (x_0, t_0)

is called domain of dependence of point (x_0, t_0)

Sides of this triangle are segments of characteristic lines

passing through point (x_0, t_0)



An inverse notion to domain of dependence

is notion of range of influence of point x_0 on x axis

This is region in $x - t$ plane consisting of all points

whose domain of dependence contains point x_0

Region has an upside-down triangular shape

with sides being characteristic lines emanating from point x_0

This also means that value of initial data at point x_0

impacts values of solution u at all points in domain of influence

Notice that at a fixed time t_0

only points satisfying $x_0 - ct_0 \leq x \leq x_0 + ct_0$

are influenced by point x_0 on x axis

4.2.3. Linear algebra of space-time

When c is speed of light

range of influence and domain of dependence

can be thought of as future and past light cones

of (1+1)-dimensional Minkowski spacetime $\mathcal{M}\,(\mathbb{R}^2_1, <,>)$

with ☛ $\mathbb{R}^2_1 = \{(t,x) : t, x \in \mathbb{R}\}$

and double-signed inner product $\langle (t_1, x_1), (t_2, x_2) \rangle = c^2 t_1 t_2 - x_1 x_2$

Inner product specifies structure of Minkowski spacetime:

a two-vector $v = (t, x) \in \mathcal{M}$ is said to be:

timelike if $\langle v, v \rangle > 0$ null if $\langle v, v \rangle = 0$ spacelike if $\langle v, v \rangle < 0$

Orthogonal vectors $v, w \in \mathcal{M}$ are defined by $v \perp w \Leftrightarrow \langle v, w \rangle = 0$

Null curves are orthogonal to itself

Minkowski norm ☛ $\|(t,x)\| = \sqrt{c^2 t^2 - x^2}$ (4.2.57.)

ranges over all non-negative real and positive imaginary values

Curves of constant Minkowski norm s satisfy ☛ $c^2 t^2 - x^2 = s^2$

Parameter s determines three families of such curves:

(i) if $s = 0$ ☛ (4.2.57.) defines light cone $x = \pm ct$

(ii) if $s \in \mathbb{R}^+$

☛ (4.2.57.) defines a hyperbola $t^2 - x^2 = s^2$ inside light cone

(iii) if $s = i\zeta \in i\mathbb{R}^+$

☛ (4.2.57.) defines a hyperbola $x^2 - t^2 = \zeta^2$ outside light cone

We can generalize structure of Minkowski spacetime to any arbitrary number of dimensions

Stricktly speaking ☛ light cone is a 3-dimensional surface in (3+1)-dimensional Minkowski spacetime

and events in spacetime may be characterized according to whether they are inside of, outside of, or on light cone

Light cone classification clarifies distinction between

double-signed inner product spacetime

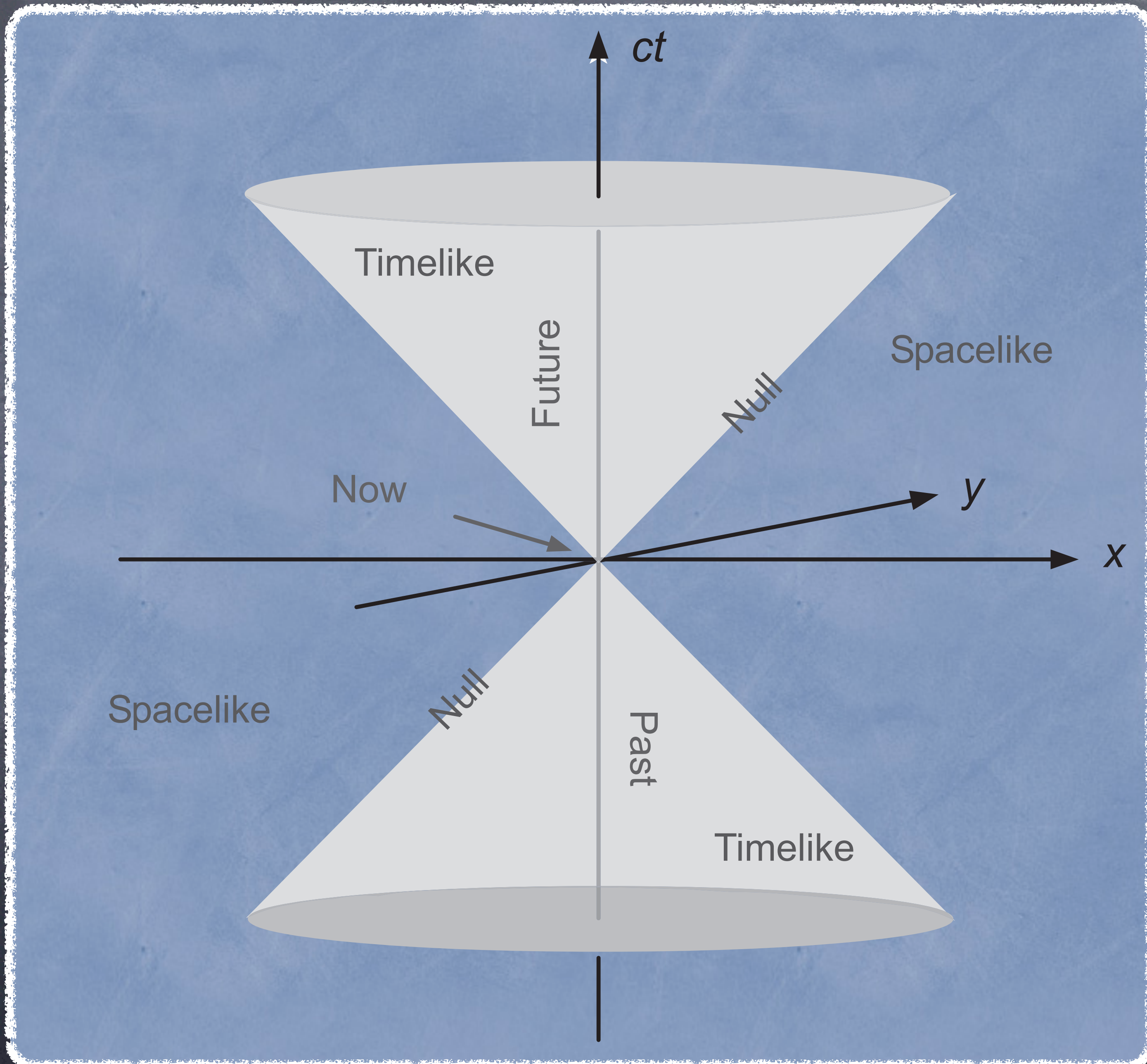
and a genuine inner product space

in that two points in Minkowski spacetime may be separated by a distance whose square could be positive, negative, or zero, which embodies impossibilities for any Euclidean space

Lightlike particles have worldlines confined to light cone

and square of separation of any 2 points on lightlike worldline is zero

Light cone diagram for two space and one time dimension



4.2.4. Causal Green function

Let us begin with problem of initial velocities:

$$u_{tt} - c^2 u_{xx} = 0, \quad u(x,0) = 0, \quad u_t(x,0) = g(x) \quad (4.2.65.)$$

By d'Alembert formula ☛ solution is definite integral

$$u(x,t) = \frac{1}{2c}\int_{x-ct}^{x+ct} g(\alpha)\, d\alpha \quad (4.2.66.)$$

We can express this as a distribution K

$$u(x,t) = K[g(t)] = \int_{-\infty}^{\infty} K(x-x',t)g(x')dx' \quad (4.2.67.)$$

provided we choose

$$K(x-x',t) = \begin{cases} \frac{1}{2c} & |x-x'| < ct \\ 0 & |x-x'| > ct \end{cases} \quad (4.2.68.)$$

Thus ☛ $K = 0$ unless ct exceeds both $x - x'$ and $x' - x$

This step function can be expressed by Heaviside function

$$K(x-x',t) = \frac{1}{2c}\left[\Theta(x-x'+ct) - \Theta(x-x'-ct)\right] \quad (4.2.69.)$$

This is Green function for one-dimensional wave equation

When we regard K as distribution ☛ time t enters as a parameter

Thus we can differentiate $K(x-x')$ with respect to t

$$K_t(x-x',t) = \tfrac{1}{2}\left[\delta(x-x'+ct)\Theta(ct-x+x') + \delta(ct-x+x')\Theta(ct+x-x')\right]$$

For $t>0$ ☛ we can omit Heaviside factors in each term

Note that when $t=0^+$ we have for K initial values

$$\begin{aligned} K(x-x',0) &= 0 \\ K_t(x-x',0) &= \tfrac{1}{2}\left[\delta(x-x')+\delta(x'-x)\right] = \delta(x-x') \end{aligned} \quad (4.2.70.)$$

Now let us examine solution of initial problem

$$u_{tt} - c^2 u_{xx} = 0, \quad u(x,0) = f(x), \quad u_t(x,0) = 0 \quad (4.2.71.)$$

This solution is

$$\begin{aligned} u(x,t) &= \tfrac{1}{2}\left[f(x+ct)+f(x-ct)\right] \\ &= \tfrac{1}{2}\int_{-\infty}^{\infty}\left[\delta(x+ct-x')+\delta(ct-x+x')\right]f(x')\,dx' \\ &= \int_{-\infty}^{\infty} K_t(x-x',t)f(x')\,dx' \end{aligned} \quad (4.2.72.)$$

All in all ☛ we can rewrite d'Alembert solution as

$$u(x,t) = \int_{-\infty}^{\infty} [K_t(x-x',t)f(x') + K(x-x',t)g(x')] \, dx' \quad (4.2.73.)$$

For $t > 0$ initial data that fall outside past cone of point (x,t) do not therefore affect value of $u(x,t)$

Solution of non-homogeneous wave equation with zero initial data is convolution (in space and time) of Green distribution with forcing force

To verify that causal Green function

$$G(x,t) = K(x,t)\Theta(t) \quad (4.2.74.)$$

is a solution of point source wave equation

$$G_{tt} - c^2 G_{xx} = \delta(x) \, \delta(t) \quad (4.2.75.)$$

we should change over to characteristic coordinates

$$\begin{aligned} G_{tt} &= c^2(G_{\xi\xi} + G_{\eta\eta}) - 2c^2 G_{\xi\eta} \\ G_{xx} &= G_{\xi\xi} + G_{\eta\eta} + 2G_{\xi\eta} \end{aligned} \quad (4.2.76.)$$

such that

$$G_{tt} - c^2 G_{xx} = -4c^2 G_{\xi\eta} \quad (4.2.77.)$$

Reason we have chosen $G(x,t)$ to be zero for t negative is that

cause must precede effect

Substituting (4.2.69.) into (4.2.74.) we obtain

$$G(\xi,\eta) = \frac{1}{2c}\Theta(\xi)\Theta(-\eta) \quad (4.2.78.)$$

Now ☛ we take derivatives with respect to ξ and η

$$G_{\xi\eta} = -\frac{1}{2c}\delta(\xi)\delta(\eta) \quad (4.2.79.)$$

and use relation

$$2c\delta(ct+x)\delta(ct-x) = \delta(x)\delta(t) \quad (4.2.80.)$$

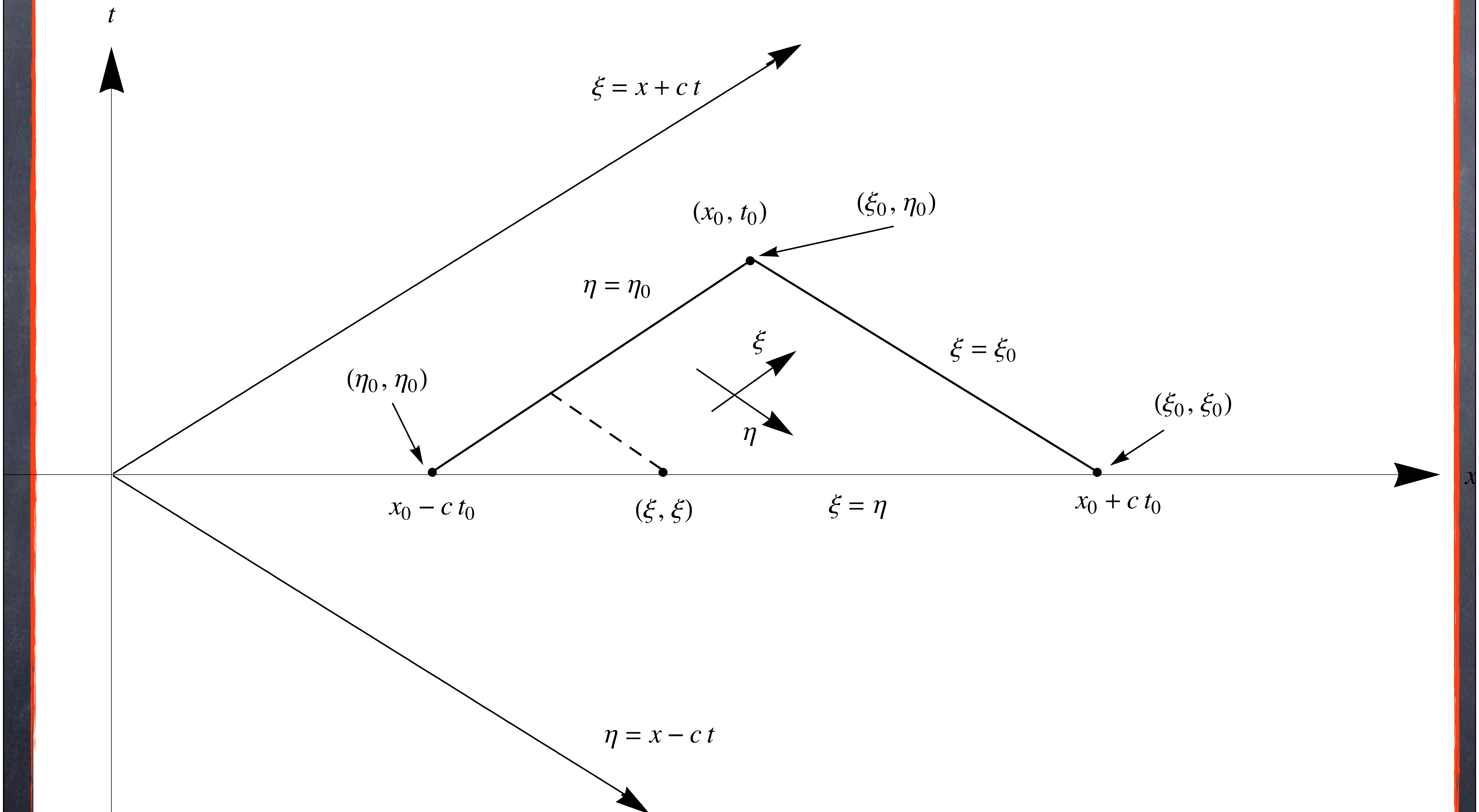
to obtain desired result

For a particular point (x_0,t_0) ☛ solution reads

$$u(\xi_0,\eta_0) = \frac{1}{4c^2}\int_{\eta_0}^{\xi_0}\int_{\eta_0}^{\xi} \overline{f}_T(\xi,\eta)\ d\xi\ d\eta = \frac{1}{4c^2}\iint_{\triangle} \overline{f}_T(\xi,\eta)\ d\xi\ d\eta \quad (4.2.81.)$$

where double integral is taken over triangle of dependence

Triangle of dependence of point (x_0, t_0)



Using change of variable (4.2.35.) and computing Jacobian

$$J = \frac{\partial(\xi,\eta)}{\partial(x,t)} = \begin{vmatrix} 1 & c \\ 1 & -c \end{vmatrix} = -2c \qquad (4.2.82.)$$

we can transform double integral (4.2.81.)

to a double integral in terms of (x,t) variables to get

$$u(x_0,t_0) = \frac{1}{4c^2}\iint_{\triangle} \overline{f}_T(x,t)\ |J|\ dx\,dt = \frac{1}{2c^2}\iint_{\triangle} \overline{f}_T(x,t)\ dx\,dt \qquad (4.2.83.)$$

Finally ☛ rewriting last double integral as an iterated integral

$$u(x,t) = \frac{1}{2c}\int_0^t \int_{x-c(t-t')}^{x+c(t-t')} \overline{f}_T(x',t')dx'dt' \qquad (4.2.84.)$$

General solution of (4.2.28.) with $\overline{f}_T(x,t) = 0$ for $t < 0$
and initial conditions $u(x,0) = f(x), u_t(x,0) = g(x)$
can be written as

$$u(x,t) = \int_{-\infty}^{\infty} [G_t(x-x',t)\; f(x') + G(x-x',t)\; g(x')]\; dx'$$
$$+ \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} G(x-x',t-t')\; \overline{f}_T(x',t')\; dx'\; dt' \quad (4.2.85.)$$

Example 4.2.4.

For a constant field $\overline{f}_T(x,t) = a\;\Theta(t)$
solution of initial value problem (with zero initial data)
is found to be

$$u(x,t) = \frac{a}{2c}\int_0^t \int_{x-c(t-t')}^{x+c(t-t')} dx'dt'$$
$$= \frac{a}{2c}\int_0^t 2c(t-t')dt' \quad (4.2.86.)$$
$$= \frac{1}{2}at^2 \quad (4.2.87.)$$

which corresponds to a constant acceleration


TO BE
CONTINUED....