

Ordinary Differential Equations IV

1. Use a power series to solve the following differential equations:

(i)

$$u'' - k^2 u = 0;$$

(ii)

$$u'' + xu' + u = 0.$$

2. Show that:

(i) Rodrigues formula is a solution of Legendre equation;

(ii)

$$P_l^m(x) = (1-x^2)^{m/2} \frac{d^m}{dx^m} P_l(x);$$

(iii)

$$\int_0^a J_n^2(k_m^n r/a) r dr = \frac{1}{2} a^2 J_{n+1}(k_m^n).$$

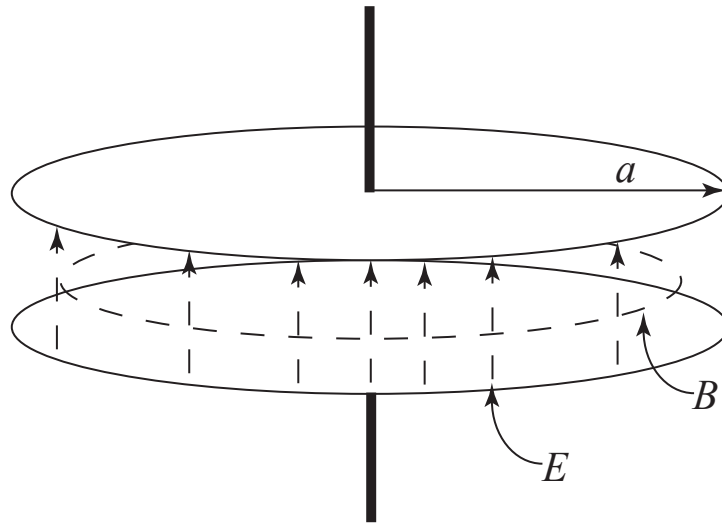
3. Suppose that we have a cylindrical capacitor, as seen in the figure. Suppose further that we put an AC current across the plates, starting at a low frequency, ω . As the voltage alternates, the positive charge on the top plate is taken off and negative charge is put on. While that is happening, the electric field disappears and then builds up in the opposite direction. As the charge sloshes back and forth slowly, the electric field follows. At each instant the electric field is uniform, as shown in the figure, except for some edge effects which we are going to disregard. We can write the electric field as $E = E_0 \cos(\omega t)$, where $E_0 = Q_0/\epsilon_0 A$ is constant, and $A = \pi a^2$ is the area of the plate. Now will this continue to be right as the frequency goes up? No, because as the electric field is going up and down, there is a flux of electric field through any circular loop, say Γ , of radius r inside the capacitor. And, as you know, a changing electric field acts to produce a magnetic field. From Maxwell's equations, the magnetic field is given by a

$$c^2 \oint \vec{B} \cdot d\vec{\ell} = \frac{d}{dt} \int \vec{E} \cdot d\vec{A} \Rightarrow c^2 B 2\pi r = \frac{d}{dt} (E \pi r^2),$$

or

$$B = -\frac{\omega r}{2c^2} E_0 \sin(\omega t).$$

So, the changing electric field has produced a magnetic field circulating around inside the capacitor, and oscillating at the same frequency as the electric field. Now, are we done? No! This magnetic field also oscillates, which produces a new electric field! The uniform field, $E_1 = E_0 \cos(\omega t)$, is only the first term! The changing magnetic field produces a new electric field, E_2 , such that the total field is $E = E_1 + E_2$. Now, in general, E_2 is also oscillating! This means that there will be a new magnetic field from E_2 , which will be oscillating, which will create a new electric field, E_3 , which will create a new magnetic field.... Your task is to calculate the first four terms of the series,



enough to get the pattern, and write the total electric field, taking the field at the center of the capacitor to be exactly $E_0 \cos(\omega t)$, (i.e., there is no correction at the center). Then, compare your result with Bessel functions, and see if you can write the full electric field you find in terms of one of the Bessel functions, in closed form. Can you find an exact expression in terms of one of the Bessel functions for the magnetic field?