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Problems set # 6 **Physics 307** October 26, 2016

Ordinary Differential Equations II

1. Determine the fundamental matrix of the system $\frac{dN_1}{dt} = -k_1 N_1$ $\frac{dN_2}{dt} = k_1 N_1 - k_2 N_2$ and find an expression for the behavior of $N_1(t)$ and $N_2(t)$.

2. Show that:
(i) $\frac{d}{dx}|x| = \text{sgn } x = \Theta(x) - \Theta(-x)$, where $|x| = \begin{cases} x & \text{if } x > 0 \\ -x & \text{if } x < 0 \end{cases}$;
(ii) $\frac{d^2}{dx^2}|x| = \frac{d}{dx}\text{sgn } x = 2\delta(x)$.

3. Evaluate:
(i) $\int_{-\infty}^{\infty} [f(x)\delta(x-1) + f(x)\delta(x+2)]\, dx$;
(ii) $\int_{-\infty}^{\infty} f(x)\, \delta'(x) dx$ (to do this integral use integration by parts);
(iii) $\int_{-\infty}^{\infty} [f(x)\delta(x-a) - f(x)\delta''(x)] dx$;
(iv) $\int_{-\infty}^{\infty} \Theta(x)\, \Theta(1-x)\, f(x)\, dx$;
(v) $\int_{-\infty}^{\infty} \Theta(x)\, \Theta(b-x)\, x\, f(x)\, dx$;
(vi) $\int_{-\infty}^{\infty} [f(x)\, \delta(x-\pi) - f(x)\delta'(x-2\pi) + f(x)\delta''(x-b)] dx$.

4. Use the properties of distributions to show:
(i) $x\delta'''(x) = -3\delta''(x)$;
(ii) $\sin(x)\ \delta(x - \frac{\pi}{2}) = \delta(x - \frac{\pi}{2})$.

5. Consider the class of principal value distributions $T_n(x)$ given by: $\langle T_n, f \rangle = \int_{-1}^{1} \frac{f(x)}{x^n}\, dx \equiv \lim_{\epsilon \to 0^+} (\int_{-1}^{-\epsilon} \frac{f(x)}{x^n}\, dx + \int_{\epsilon}^{1} \frac{f(x)}{x^n}\, dx)$, where $f(x)$ is any test function which vanishes outside some $(a, b) \subset [-1, 1]$. These definitions are used to try to make sense of what are normally regarded as divergent integrals. *(i)* Show that T_2 is not defined for all test functions. To show this, let $f_\star(x)$ be a test function that equals one on $[-\frac{1}{2}, \frac{1}{2}]$ and then show $< T_2, f_\star >$ is undefined. As a side note, T_2 is defined for all test functions having a Taylor series expanion $f(x) = \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots$. *(ii)* One can prove that T_1 is defined for all test functions f. Its distributional derivative T_1' is defined by $\langle T_1', f \rangle = -\langle T_1, f' \rangle$ for the test functions f for which the expression on the right is defined. Show that if T_1' exists, then $T_1' = -T_2$. To show this use $f'(x)$ in the definition of a principal value distribution, with $n = 1$, and integrate by parts while assuming $f(0) = 0$. You need a careful argument as to why the boundary terms in the integration by parts process vanish.