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Problems set # 3 **Physics 307** September 21, 2016

Complex Analysis III

1. Let $f(z) = \frac{\pi z(1-z^2)}{\sin(\pi z)}$.
(i) Find all zeros of f and their orders;
(ii) find all singularities of f and classify them;
(iii) discuss the Laurent expansion of f around zero, and calculate the first few non-zero coefficients.

2. *(i)* Give two Laurent series expansions in powers of z for the function

$$f(z) = \frac{1}{z^2(1-z)}$$

and specify the regions in which those expressions are valid.
(ii) Write two Laurent series in powers of z that represent the function

$$f(z) = \frac{1}{z(4+z^2)}$$

in certain domains and specify these domains.

3. Find all the singularities of the following functions, clasify them, and compute the residues at those points:
(i) $f(z) = \frac{z^2}{\sin^2 z}$;
(ii) $f(z) = \frac{z^2-1}{(z^2+1)^2}$;
(iii) $\frac{1}{z^2 \sinh z}$.

4. Evaluate:
(i)

$$\oint_C \frac{2z+6}{z^2+4}\, dz,$$

where the contour C is the circle $|z-i| = 2$;
(ii)

$$\oint_C \frac{e^z}{z^4+5z^3}\, dz,$$

where the contour C is the circle $|z| = 2$;
(iii)

$$\oint_{|z|=8} \tan z \; dz,$$

(iv)

$$\int_0^{2\pi} \frac{1}{(2+\cos\theta)^2} d\theta$$

(v)

$$\int_{-\infty}^{\infty} \frac{1}{(x^2+1)(x^2+9)}\, dx;$$

(vi)

$$\int_{-\infty}^{\infty} \frac{1}{x^4+1}\, dx\,;$$

(vii)

$$\int_0^{\infty} \frac{x\,\sin x}{x^2+9}\, dx\,.$$

5. One of the most important applications of *Bosse-Eisntein* statistics is to investigate the equilibrium properties of the black-body radiation. Consider a gas of photons in a radiation cavity of volume V and temperature T. The energy of a given photon is $\hbar\omega_s$, where ω_s is the angular frequency of the radiation mode. The number of normal modes of vibration per unit volume of the enclosure in the frequency range $(\omega, \omega + d\omega)$ is given by the Rayleigh expression: $\omega^2 d\omega/(\pi^2 c^3)$. The energy density associated to the frequency range $(\omega, \omega + d\omega)$ is given by,

$$u(\omega) d\omega = \frac{\hbar}{\pi^2 c^3} \frac{\omega^3\ d\omega}{e^{\hbar\omega/kT} - 1}\,, \tag{1}$$

which may as well be rewritten in the dimensionless form

$$u'(x) dx = \frac{x^3\ dx}{e^x - 1} \tag{2}$$

where

$$u'(x) = \frac{\pi^2 \hbar^3 c^3}{(kT)^4}\ u(x) \quad \text{and} \quad x = \frac{\hbar\omega}{kT}\,.$$

The total energy density in the radiation cavity

$$\frac{U}{V} = \int_0^{\infty} u(x)\, dx = \frac{(kT)^4}{\pi^2 \hbar^3 c^3} \int_0^{\infty} \frac{x^3\ dx}{e^x - 1} \tag{3}$$

can be evaluated by expanding the integrand in a series. Since $e^{-x} \leq 1$ throughout the range of integration, one can write

$$\begin{aligned} \frac{x^3}{e^x - 1} &= \frac{e^{-x} x^3}{1 - e^{-x}} \\ &= e^{-x} x^3 \left(1 + e^{-x} + e^{-2x} + \ldots\right) \\ &= \sum_{n=1}^{\infty} e^{-nx} x^3\,. \end{aligned} \tag{4}$$

Therefore the integral in (3) becomes

$$\begin{aligned} \int_0^{\infty} \frac{x^3\, dx}{e^x - 1} &= \sum_{n=1}^{\infty} \int_0^{\infty} e^{-nx} x^3 dx \\ &= \sum_{n=1}^{\infty} \frac{1}{n^4} \int_0^{\infty} e^{-y}\ y^3\ dy\,. \end{aligned} \tag{5}$$

(i) Using contour integration in the complex plane show that

$$\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}.$$

(ii) Using the result of *(i)*, show that

$$\frac{U}{V} = \frac{\pi^2 k^4}{15\hbar^3 c^3} T^4 \,. \tag{6}$$

Hint: Consider the integral $\int_0^\infty e^{-x}\, x^n\, dx$. For $n = 0$, the evaluation is trivial

$$\int_0^\infty e^{-x} dx = -\, e^{-x}\big]_0^\infty = 1 \,. \tag{7}$$

More generally, the integral can be simplified using integration by parts. For $n > 0$,

$$\begin{aligned} \int_0^\infty e^{-x}\, x^n dx &= -\int_o^\infty x^n\, d(e^{-x}) \\ &= -\, x^n\, e^{-x}\big]_0^\infty + n \int_0^\infty x^{n-1}\, e^{-x}\, dx \,. \end{aligned}$$

Since the first term on the right vanishes at both limits, one obtains the recurrence relation

$$\int_0^\infty e^{-x}\, x^n\, dx = n \int_0^\infty e^{-x}\, x^{n-1}\, dx \,. \tag{8}$$

If n is a positive integer, one can apply (8) repeatedly to obtain

$$\begin{aligned} \int_o^\infty e^{-x}\, x^n\, dx &= n\,(n-1)\,(n-2)\ldots \\ &= n! \,. \end{aligned} \tag{9}$$