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Problems set # 11 **Physics 307** November 30, 2016

Partial Differential Equations II

1. Consider the following equation: $u_t = u_x + u^2$.
(i) Find a nonzero steady-state (independent of t) solution u_0 in the half-plane $x > 0$;
(ii) show that $2u_0$ is not a solution;
(iii) use u_0 to show that the equation is not linear.

2. Using separation of variables, find a nonzero solution of the equation $u_t = \alpha u_{xx} - u$, with $\alpha > 0$.

3. Solve the initial-boundary value problem for the heat equation $u_t = \alpha u_{xx}$, with $0 < x < L$, $t > 0$, and the following intial and boundary conditions:
(i) $u(x,0) = 6\sin(9\pi x/L)$, $u(0,t) = u(L,t) = 0$;
(ii) $u(x,0) = 3\sin(\pi x/L) - \sin(3\pi x/L)$, $u(0,t) = u(L,t) = 0$;
(iii) $u(x,0) = 6 + 4\cos(3\pi x/L)$, $u_t(0,t) = u_t(L,t) = 0$;
(iv) $u(x,0) = -3\cos(8\pi x/L)$, $u_t(0,t) = u_t(L,t) = 0$.
(v) Determine the behavior of the solution for $t \to \infty$.

4. *(i)* Find the temperature $u(x,t)$ of a semi-infinite homogeneous bar with the conditions $u_x(t,0) = 0$, $u(x,0) = f(x)$.
(ii) Idem *(i)* for $u(t,0) = 0$, $u(x,0) = f(x)$.

5. Show that if the temperature on the Earth's surface is of the form $u(t) = T_0 + A\cos(\omega t)$ (representing periodic temperature oscillations), then the temperature at depth x below the Earth's surface is

$$u(x,t) = T_0 + Ae^{-\lambda x}\ \cos(\omega t - \lambda x),$$

where T_0, A are constants and $\lambda = \sqrt{\omega/2}$ (assume conditions of semi-infinite bar with $\alpha = 1$). Interpret the result and relate it to the preservation of wine.

[*Hint:* Temperature, the unseen and often neglected factor, can bring the best and worst out of a bottle of wine in 2 areas: storing and serving wine. Sadly, many great wine collections have been damaged because of instability in temperature and moisture. High temperature ($> 20°$C) causes the wine to age prematurely, thus losing its flavor and balance. When chilled too cold, the wine also loses its flavor and aromas. Keep the wines away from sunlight and heat exposure; store them in cellars, wine fridge, and or temperature controlled rooms. Wide fluctuation in temperature will damage the wine and the cork. Ideal range of temperature for storing red wine is $10 - 16°$C. White wines can be stored at lower ranges at $7 - 10°$C. Humidity is another important fine balance. If the humidity is too high, mold can grow and cellar woods and wine labels can be damaged. If the environment is too dry, the cork will crack and air will leak into the bottle, again ruining the

wine. Ideal humidity for wine storage ranges from $60 - 75\%$RH.[1] Wines should be served at a temperature that best reveals its characteristics and aromas. The optimal serving temperature differs for various wines, depending on their grape variety and their region. For example, a rich, intense Bordeaux could be served a few degrees below room temperature ($\sim 18°$C); but a light, fruity red such as Beaujolais is best serve at least eight degrees ($\sim 12°$ C) below room temperature.]

6. Consider the heat equation, $u_t = \alpha(u_{xx} + u_{yy})$, in a two-dimensional rectangular region, $0 < x < L$, $0 < y < H$, subject to the initial condition $u(x, y, 0) = f(x, y)$. Solve the boundary value problem and analyze the temperature as $t \to \infty$ if the boundary conditions are $u_x(0, y, t) = 0$, $u_x(L, y, t) = 0$, $u_y(x, 0, t) = 0$, $u_y(x, H, t) = 0$.

[1]Relative humidity (RH) may be defined as the ratio of the water vapor density (mass per unit volume) to the saturation water vapor density, usually expressed in percent.