Luis Anchordoqui
Lehman College
City University of New York

Thermodynamics and Statistical Mechanics

Statistical Mechanics VI
December 2014

- **Fermi-Dirac gas**
- **White dwarfs**

FERMI-DIRAC GAS

Properties of Fermi gas are different from those of Bose gas
because exclusion principle prevents multi-occupancy of quantum states

As a result ☛ no condensation at ground state occurs at low temperatures
For macroscopic system chemical potential can be found at all temperatures using

$$N = \int_0^\infty d\varepsilon \rho(\varepsilon) f(\varepsilon) = \int_0^\infty d\varepsilon \frac{\rho(\varepsilon)}{e^{\beta(\varepsilon-\mu)} + 1} \quad (92)$$

This is a nonlinear equation for μ that in general can be solved only numerically

In limit $T \to 0$ fermions fill certain number of low-lying energy levels
to minimize total energy while obeying exclusion principle

Chemical potential of fermions is positive at low temperatures ☛ $\mu > 0$

For $T \to 0$ (i. e. $\beta \to \infty$) it follows that ☛ $\begin{cases} e^{\beta(\varepsilon-\mu)} \to 0 & \text{if} \quad \varepsilon < \mu \\ e^{\beta(\varepsilon-\mu)} \to \infty & \text{if} \quad \varepsilon > \mu \end{cases}$

yielding ☛ $f(\varepsilon) = \begin{cases} 1, & \varepsilon < \mu \\ 0, & \varepsilon > \mu \end{cases}$

FERMI TEMPERATURE

Zero-temperature value μ_0 defined

$$N = \int_0^{\mu_0} d\varepsilon \rho(\varepsilon)$$

Fermions are mainly electrons having spin $\frac{1}{2}$ and correspondingly degeneracy 2 because of two states of spin

In three dimensions ☛ using (39) with additional factor 2 for degeneracy

$$N = \frac{2V}{(2\pi)^2} \left(\frac{2m}{\hbar^2}\right)^{3/2} \int_0^{\mu_0} d\varepsilon \sqrt{\varepsilon} = \frac{2V}{(2\pi)^2} \left(\frac{2m}{\hbar^2}\right)^{3/2} \frac{2}{3} \mu_0^{3/2}$$

It follows that ☛ $\mu_0 = \dfrac{\hbar^2}{2m} (3\pi^2 n)^{2/3} = \varepsilon_F$

Fermi energy

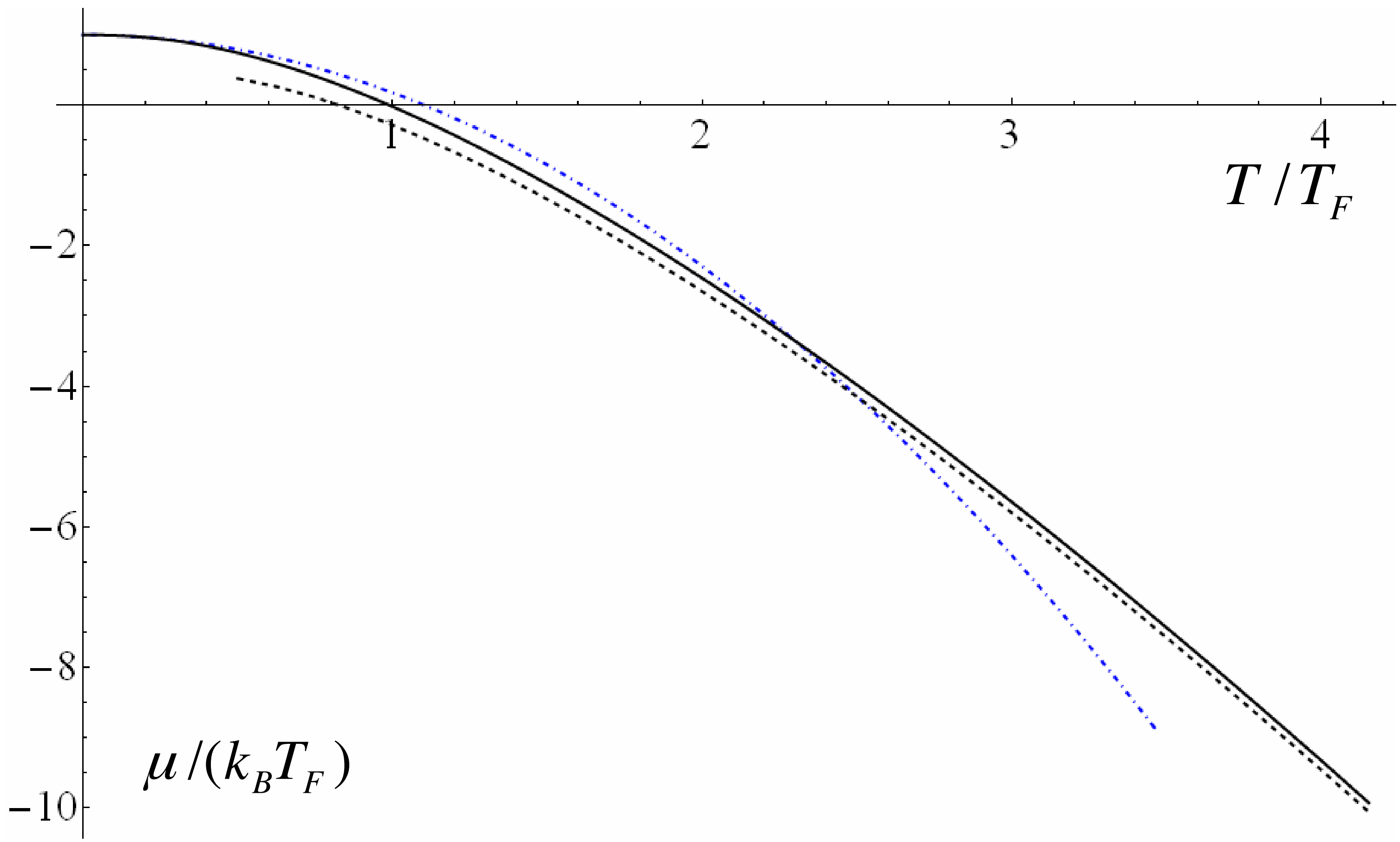
Convenient to introduce Fermi temperature ☛ $k_B T_F = \varepsilon_F$ (93)

Note that T_F has same structure as T_B defined by (91)

In typical metals $T_F \sim 10^5\, K$ so that at room temperatures ☛ $T \ll T_{\rm F}$ and electron gas is degenerate

Luis Anchordoqui

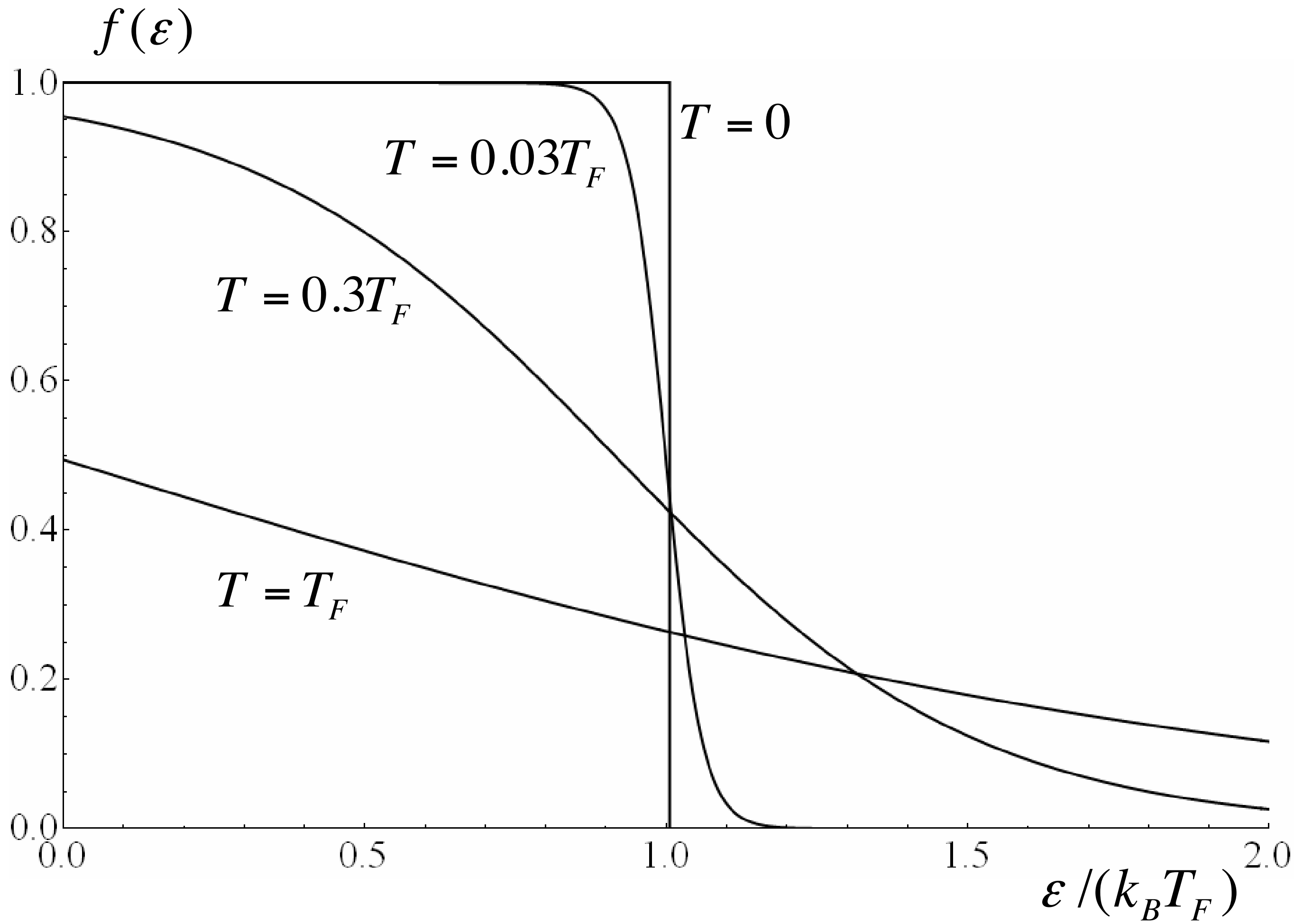
CHEMICAL POTENTIAL OF FERMI-DIRAC GAS



Dashed line: High-temperature asymptote corresponding to Boltzmann statistics

Dashed-dotted line: Low-temperature asymptote

FERMI-DIRAC DISTRIBUTION FUNCTION



INTERNAL ENERGY AND PRESSURE

Convenient to express density of states (39) in terms of ε_F

$$\rho(\varepsilon) = \frac{3}{2} N \frac{\sqrt{\varepsilon}}{\varepsilon_F^{3/2}} \tag{94}$$

Internal energy at $T = 0$

$$U = \int_0^{\mu_0} d\varepsilon \rho(\varepsilon)\varepsilon = \frac{3}{2} \frac{N}{\varepsilon_F^{3/2}} \int_0^{\varepsilon_F} d\varepsilon\, \varepsilon^{3/2} = \frac{3}{2} \frac{N}{\varepsilon_F^{3/2}} \frac{2}{5} \varepsilon_F^{5/2} = \frac{3}{5} N \varepsilon_F \tag{95}$$

We cannot calculate heat capacity C_V from (95)

as it requires taking into account small temperature- dependent corrections in U

We can calculate pressure at low temperatures since S should be small and

$$F = U - TS \cong U$$

$$P = -\left(\frac{\partial F}{\partial V}\right)_{T=0} \simeq -\left(\frac{\partial U}{\partial V}\right)_{T=0} = -\frac{3}{5} N \frac{\partial \varepsilon_F}{\partial V}$$

$$= -\frac{3}{5} N \left(-\frac{2}{3} \frac{\varepsilon_F}{V}\right) = \frac{2}{5} n \varepsilon_F = \frac{\hbar}{2m} \frac{2}{5} (3\pi^2)^{2/3} n^{5/3} \tag{96}$$

TO COMPUTE CORRECTIONS...

we will need integral of a general type

$$M_\eta = \int_0^\infty d\varepsilon \, \varepsilon^\eta \, f(\varepsilon) = \int_0^\infty d\varepsilon \, \frac{\varepsilon^\eta}{e^{(\varepsilon-\mu)/(k_BT)}+1} \qquad (97)$$

From (93) it follows that

$$N = \frac{3}{2} \frac{N}{\varepsilon_F^{3/2}} M_{1/2} \qquad (98)$$

$$U = \frac{3}{2} \frac{N}{\varepsilon_F^{3/2}} M_{3/2} \qquad (99)$$

It is easily seen that for $k_BT \ll \mu$

expansion of M_η up to quadratic terms has form

$$M_\eta = \frac{\mu^{\eta+1}}{\eta+1}\left[1 + \frac{\pi^2\eta(\eta+1)}{6}\left(\frac{k_BT}{\mu}\right)^2\right] \qquad (100)$$

CHEMICAL POTENTIAL

(98) becomes

$$\varepsilon_F^{3/2} = \mu^{3/2}\left[1 + \frac{\pi^2}{8}\left(\frac{k_B T}{\mu}\right)^2\right]$$

that defines $\mu(T)$ up to terms of order T^2

$$\mu = \varepsilon_F\left[1 + \frac{\pi^2}{8}\left(\frac{k_B T}{\mu}\right)^2\right]^{-2/3} \cong \varepsilon_F\left[1 - \frac{\pi^2}{12}\left(\frac{k_B T}{\mu}\right)^2\right] \cong \varepsilon_F\left[1 - \frac{\pi^2}{12}\left(\frac{k_B T}{\varepsilon_F}\right)^2\right]$$

or using (93) ☛

$$\mu = \varepsilon_F\left[1 - \frac{\pi^2}{12}\left(\frac{T}{T_F}\right)^2\right] \qquad (101)$$

It is not surprising that chemical potential decreases with temperature because at high temperatures it takes large negative values

(99) becomes

$$U = \frac{3}{2}\frac{N}{\varepsilon_F^{3/2}}\frac{\mu^{5/2}}{(5/2)}\left[1 + \frac{5\pi^2}{8}\left(\frac{k_B T}{\mu}\right)^2\right] \cong \frac{3}{5}N\frac{\mu^{5/2}}{\varepsilon_F^{3/2}}\left[1 + \frac{5\pi^2}{8}\left(\frac{k_B T}{\varepsilon_F}\right)^2\right]$$

HEAT CAPACITY

Using (101)

$$U = \frac{3}{5} N \varepsilon_F \left[1 - \frac{\pi^2}{12}\left(\frac{T}{T_F}\right)^2\right]^{5/2} \left[1 + \frac{5\pi^2}{8}\left(\frac{T}{T_F}\right)^2\right]$$

$$\cong \frac{3}{5} N \varepsilon_F \left[1 - \frac{5\pi^2}{24}\left(\frac{T}{T_F}\right)^2\right] \left[1 + \frac{5\pi^2}{8}\left(\frac{T}{T_F}\right)^2\right]$$

that yields

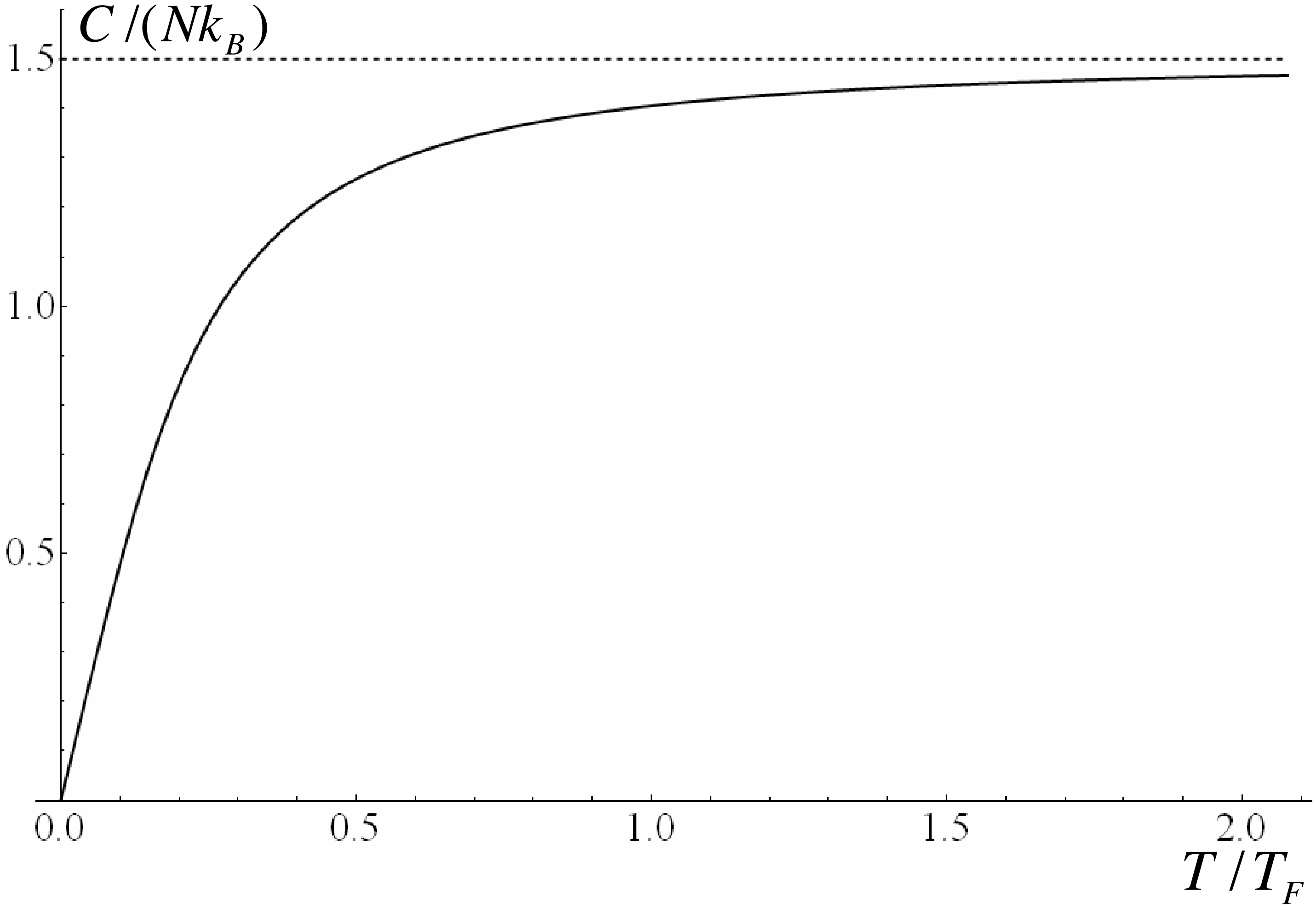
$$U = \frac{3}{5} N \varepsilon_F \left[1 + \frac{5\pi^2}{12}\left(\frac{T}{T_F}\right)^2\right]$$

At $T = 0$ this formula reduces to (94)

Heat capacity ☛ $$C_V = \left(\frac{\partial U}{\partial T}\right)_V = N k_B T \frac{\pi^2}{2} \frac{T}{T_F}$$

is small at $T \ll T_F$

HEAT CAPACITY OF IDEAL FERMI-DIRAC GAS



HOW TO GET ONE HUNDRED

Integrating (97) by parts

$$M_\eta = \frac{\varepsilon^{\eta+1}}{\eta+1} f(\varepsilon)\Big|_0^\infty - \int_0^\infty d\varepsilon \, \frac{\varepsilon^{\eta+1}}{\eta+1} \frac{\partial f(\varepsilon)}{\partial \varepsilon} \qquad (102)$$

First term of this formula is zero

At low temperatures

$f(\varepsilon)$ is close to step function fast changing from 1 to 0 in vicinity of $\varepsilon = \mu$

Thus

$$\frac{\partial f(\varepsilon)}{\partial \varepsilon} = \frac{\partial}{\partial \varepsilon} \frac{1}{e^{\beta(\varepsilon-\mu)}+1} = -\frac{\beta e^{\beta(\varepsilon-\mu)}}{[e^{\beta(\varepsilon-\mu)+1}]^2} = -\frac{\beta}{4\cosh^2[\beta(\varepsilon-\mu)/2]}$$

has a sharp negative peak at $\varepsilon = \mu$

MORE ON HOW TO GET ONE HUNDRED

$\varepsilon^{\eta+1}$ is a slow function of ε that can be expanded in Taylor series near $\varepsilon = \mu$

Up to second order

$$\varepsilon^{\eta+1} = \mu^{\eta+1} + \frac{\partial \varepsilon^{\eta+1}}{\partial \varepsilon}\Big|_{\varepsilon=\mu} (\varepsilon - \mu) + \frac{1}{2}\frac{\partial^2 \varepsilon^{\eta+1}}{\partial \varepsilon^2}\Big|_{\varepsilon=\mu} (\varepsilon - \mu)^2$$

$$= \mu^{\eta+1} + (\eta + 1)\,\mu^{\eta}\,(\varepsilon - \mu) + \frac{1}{2}\eta\,(\eta + 1)\,\mu^{\eta-1}\,(\varepsilon - \mu)^2$$

Introducing $x \equiv \beta(\varepsilon - \mu)/2$

and formally extending integration in (102) from $-\infty$ to ∞

$$M_\eta = \frac{\mu^{\eta+1}}{\eta + 1}\int_0^\infty dx \left[\frac{1}{2} + \frac{\eta + 1}{\beta\mu}x + \frac{\eta(\eta + 1)}{\beta^2\mu^2}x^2\right]\frac{1}{\cosh^2(x)}$$

Contribution of the linear x term vanishes by symmetry

Using integrals

$$\int_{-\infty}^{\infty} dx\,\frac{1}{\cosh^2(x)} = 2 \qquad \int_{-\infty}^{\infty} dx\,\frac{x^2}{\cosh^2(x)} = \frac{\pi^2}{6}$$

you arrive at (100)

WHITE DWARF STARS

Consider mass $M \sim 10^{33}$ g of helium

at nuclear densities of $\rho \sim 10^7 \text{ g/cm}^3$ and temperature $T \sim 10^7$ K

This temperature is much larger than ionization energy of ^4He,

hence we may safely assume that all helium atoms are ionized

If there are N electrons ☛ number of α particles (i.e. ^4He nuclei) must be $\frac{1}{2}N$

Mass of α particle ☛ $m_\alpha \approx 4m_\text{p}$

Total stellar mass M is almost completely due to α particle cores

using ☛ $$M = Nm_e + \frac{1}{2}N4m_p$$

electron density ☛ $$n = \frac{N}{V} = \frac{2\ M/(4m_p)}{V} = \frac{\rho}{2m_p} \approx 10^{30}\ \text{cm}^{-3}$$

RELATIVISTIC ELECTRON GAS

Since electrons are degenerate we estimate p to be order of uncertainty in momentum Δp

Δx is order of average distance between electrons ☛ approximately $n^{-1/3}$

$$\Delta p \, \Delta x \sim \hbar$$

from number density n we find Fermi momentum of electron gas

$$p_F = \hbar(3\pi^2 n)^{1/3} \approx 2.26 \times 10^{-17} \text{ g cm/s}$$

$$mc = (9.1 \times 10^{-28}\text{g})\,(3 \times 10^{10}\,\text{m/s}) = 2.7 \times 10^{-17}\,\text{g cm/s}$$

Since $p_{\rm F} \sim mc$ ☛ electrons are relativistic

Fermi temperature will then be $T_{\rm F} \sim mc^2 \sim 10^6\,\text{eV} \sim 10^{12}\,\text{K}$

$T \ll T_F$ ☛ electron gas is degenerate and considered to be at $T \sim 0$

So we need to understand ground state properties of relativistic electron gas

kinetic energy ☛ $$\varepsilon(\vec{p}) = \sqrt{\vec{p}^{\,2} c^2 + m^2 c^4} - mc^2$$

velocity ☛ $$\vec{v} = \frac{\partial \varepsilon}{\partial \vec{p}} = \frac{\vec{p}c^2}{\sqrt{p^2 c^2 + m^2 c^4}}$$

GROUND STATE PRESSURE

Pressure in ground state is

$$\begin{aligned} P_0 &= \frac{1}{3} n \langle \vec{v} \cdot \vec{p} \rangle \\ &= \frac{1}{3\pi^2 \hbar^3} \int_0^{p_\mathrm{F}} dp\, p^2 \cdot \frac{p^2 c^2}{\sqrt{p^2 c^2 + m^2 c^4}} \\ &= \frac{m^4 c^5}{3\pi^2 \hbar^3} \int_0^{\theta_\mathrm{F}} d\theta \, \sinh^4 \theta \\ &= \frac{m^4 c^5}{96\pi^2 \hbar^3} \left(\sinh(4\,\theta_\mathrm{F}) - 8 \sinh(2\,\theta_\mathrm{F}) + 12\,\theta_\mathrm{F} \right) \end{aligned} \qquad (103)$$

we used substitution

$$p = mc \sinh \theta \qquad v = c \tanh \theta \implies \theta = \frac{1}{2} \ln\left(\frac{c+v}{c-v}\right)$$

$$p_F = \hbar (3\pi^2 n)^{1/3}$$

$$n = \frac{M}{2 m_\mathrm{p} V} \implies 3\pi^2 n = \frac{9\pi}{8} \frac{M}{R^3 m_\mathrm{p}}$$

BALANCE EQUATION

In equilibrium pressure ☛ $dU_0 = -P_0 dV = -P_0(R) \cdot 4\pi R^2 dR$

is balanced by gravitational pressure ☛ $dU_g = \gamma \cdot \dfrac{GM^2}{R^2} dR$

depends on radial mass distribution

Equilibrium then implies ☛ $P_0(R) = \dfrac{\gamma}{4\pi} \dfrac{GM^2}{R^4}$

To find relation $R = R(M)$ we must solve

$$\frac{\gamma}{4\pi} \frac{gM^2}{R^4} = \frac{m^4 c^5}{96\pi^2 \hbar^3} \left(\sinh(4\theta_F) - 8\sinh(2\theta_{\rm F}) + 12\,\theta_{\rm F}\right)$$

Note that

$$\sinh(4\theta_{\rm F}) - 8\sinh(2\theta_{\rm F}) + 12\theta_{\rm F} = \begin{cases} \frac{96}{15}\,\theta_{\rm F}^5 & \theta_{\rm F} \to 0 \\ \frac{1}{2}\,e^{4\theta_{\rm F}} & \theta_{\rm F} \to \infty \end{cases} \qquad (104)$$

CHANDRASEKHAR LIMIT

We may write

$$P_0(R) = \frac{\gamma}{4\pi}\frac{gM^2}{R^4} = \begin{cases} \frac{\hbar^2}{15\pi^2 m}\left(\frac{9\pi}{8}\frac{M}{R^3 m_{\rm p}}\right)^{5/3} & \theta_{\rm F} \to 0 \\ \frac{\hbar c}{12\pi^2}\left(\frac{9\pi}{8}\frac{M}{R^3 m_{\rm p}}\right)^{4/3} & \theta_{\rm F} \to \infty \end{cases}$$

In limit $\theta_{\rm F} \to 0$ ☛ we solve for $R(M)$ and find

$$R = \frac{3}{40\gamma}(9\pi)^{2/3}\frac{\hbar^2}{G\, m_{\rm p}^{5/3}\, mM^{1/3}} \propto M^{-1/3}$$

In limit $\theta_{\rm F} \to \infty$ ☛ $R(M)$ factors divide out and we obtain

$$M = M_0 = \frac{9}{64}\left(\frac{3\pi}{\gamma^3}\right)^{1/2}\left(\frac{\hbar c}{G}\right)^{3/2}\frac{1}{m_{\rm p}^2}$$

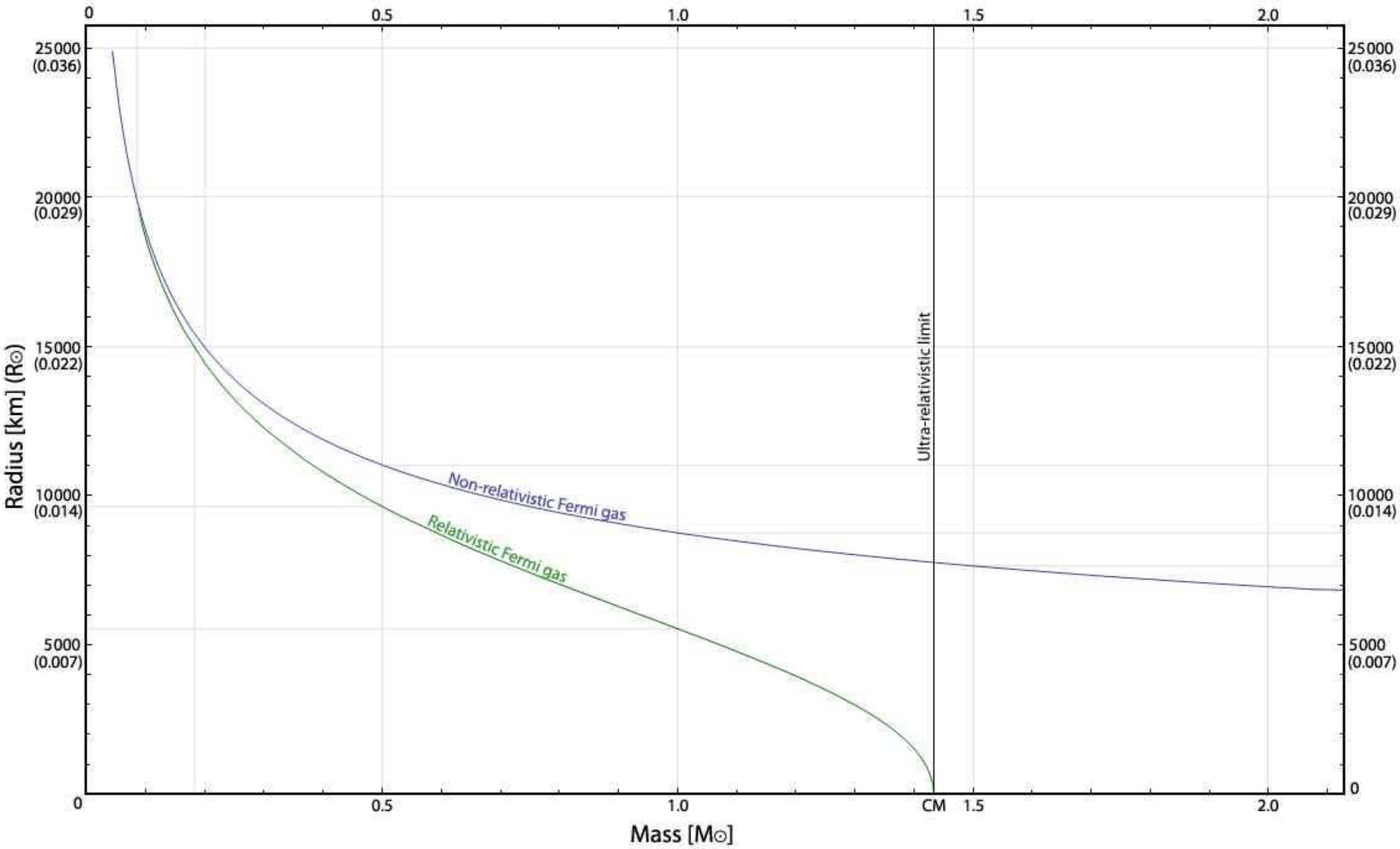
To find $\mathcal{R}$ dependence ☛ we must go beyond lowest order expansion of (104)

we obtain $$R = \left(\frac{9\pi}{8}\right)^{1/3}\left(\frac{\hbar}{mc}\right)\left(\frac{M}{m_{\rm p}}\right)^{1/3}\left[1 - \left(\frac{M}{M_0}\right)^{2/3}\right]^{1/2}$$

Value M_0 is limiting size for a white dwarf

It is called **Chandrasekhar limit**

MASS-RADIUS RELATIONSHIP FOR WHITE DWARF STARS



non-relativistic calculation follows from (96) instead of (103)