Modern Physics

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Lesson V
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Table of Contents

1. Minkowski Spacetime
    - Causal structure
    - Lorentz invariance

$(1,3)$ spacetime

- Primary unit in special relativity ☞ event
- *Arena* $\forall$ events in universe ☞ Minkowski spacetime
- Events take place in a four dimensional structure that contains 3-dimensional Euclidean space + 1 time dimension
- Interval

$$\Delta s^2 = c\Delta t^2 - \Delta x^2 - \Delta y^2 - \Delta z^2 \tag{1}$$

invarinat measure for Lorentz-Poincaré transformations

Lorentz boosts + rotations

- Galilean transformations preserve the usual Pythagorian distance

$$\Delta\vec{r}^{\,2} = \Delta x^2 + \Delta y^2 + \Delta z^2$$

which is invariant under rotations and spatial translations

- Lorentz transformations (boosts + rotations) preserve interval

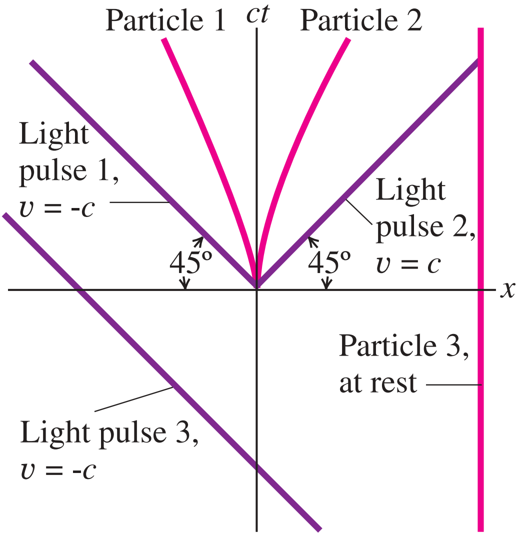
$$\Delta s^2 = c^2 \Delta t^2 - \Delta x^2 - \Delta y^2 - \Delta z^2$$

- Since interval is defined by differences in coordinates
  it is also invariant under translations in spacetime
  (Poincare invariance)

- $(1,3)$ spacetime $\equiv$ $(1,1)$ spacetime $\oplus$ rotations

- Trajectory ☞ connected set of events
  representing places + times through which particle moves
- Worldlines ☞ trajectories of massive particles and observers
- At any event on trajectory ☞ slope is inverse of velocity
  relative to inertial observer that has $x = 0$ straight up
  and with perpendicular set of simultaneity lines of constant ct
- We use ct rather than t so that both scales can have same unit
- Particle path forms worldline as particle moves in 1-dimension
- At any point ☞ slope of worldline is $d(ct)/dx = (cdt)/(vdt) = c/v$
- Light pulse with $\pm c$ speed has slope of ± 1
  giving angles of 45° with the $\pm x$-axes
- Since massive particles have speeds less than c
  all worldlines are steeper than those 45° angle
- Nothing known has worldline with slope between -1 and 1
- Worldline of particle at rest is vertical and so has infinite slope

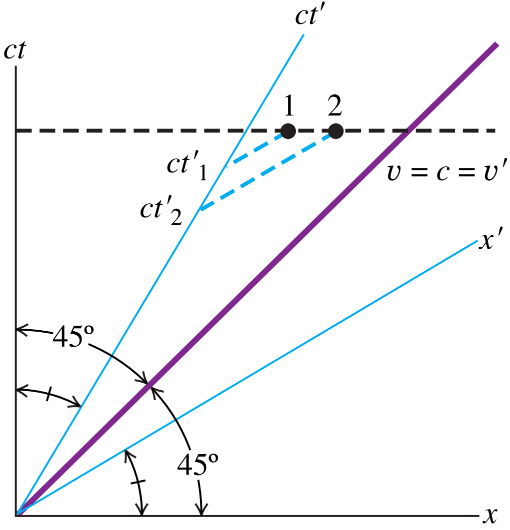
3 worldlines and 3 lightlines



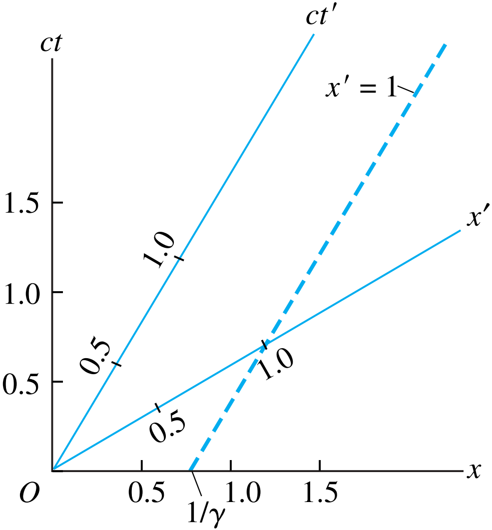
How does the S' reference frame appear on a ct-x spacetime diagram?

- Recall we always set $x' = 0$ at $x = 0$ when $t' = 0 = t$
  and let S' move at speed v in $+x$-direction
- $x' = 0$ all along ct'-axis ☞ ct'-axis have worldline of slope c/v
- E.g. $v = 0.600c$ ☞ ct'-axis is at $\tan^{-1}(1/0.600) = 59.0^\circ$ from x-axis
- x'-axis is not drawn perpendicular to the ct'-axis
- Since $ct' = 0$ all along x'-axis ☞ use Lorentz transformation for t'
  $(t - vx/c^2) = 0$ or $ct = (v/c)x$ for x'-axis
- x'-axis is drawn with slope of v/c on ct-x spacetime diagram
- E.g. $v = 0.600c$ ☞ x'-axis is at $\tan^{-1}(0.600) = 31.0^\circ$ from x-axis
- x'-axis makes same angle with x-axis
  as ct'-axis makes with ct-axis

Simultaneity on a spacetime diagram

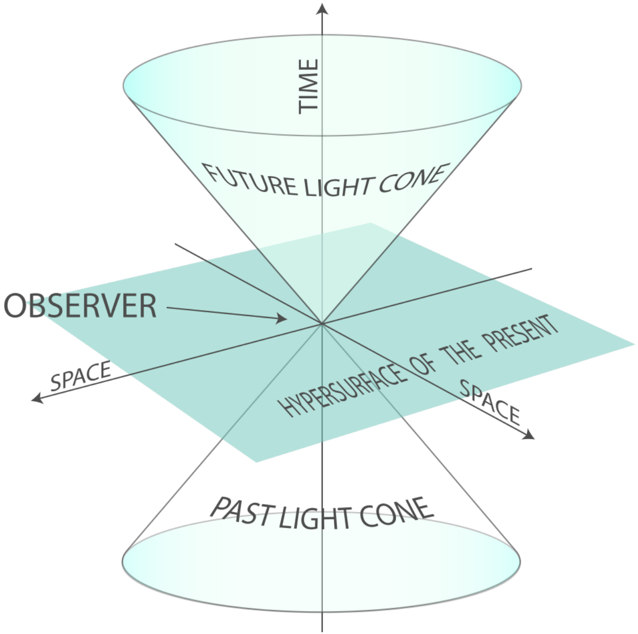


Length contraction on a spacetime diagram



Future, past, and elsewhere

- Spacetime around any one event is divided into regions separated by trajectories of light rays emanating from that event
- Separation of events is same for all Lorentz observers since light rays are unchanged by Lorentz transformations
- All events in upper light cone are future of event in question
- From origin event and any event in future ☞ there exists inertial observer for whom interval between events is a pure time $(\tau, \vec{0})$ and time of other event is after the now of our original event $\tau > 0$.
- Events in backward light cone from original event are in the past
- There exists inertial observer for whom second event is pure time $(\tau, \vec{0})$ ☞ but in this case $\tau < 0$
- All future and past events relative to original event with intervals in any inertial coordinate system have $\Delta s^2 > 0$.
- For any elsewhere event $\Delta s^2 < 0$ ☞ there exists Lorentz observer for whom events are separated by spatial interval $(0, \vec{r})$


TIME
FUTURE LIGHT CONE
OBSERVER
SPACE
HYPERSURFACE OF THE PRESENT
SPACE
PAST LIGHT CONE

4-vectors

- Event ☞ a place and a time
- Set of 4 numbers $(t, \vec{x})$ specifies event in coordinate system
- Designate coordinates with index x^μ

$$x^0 = ct,\ x^1 = x,\ x^2 = y,\ x^3 = z$$

- In this notation ☞ a Lorentz transformations is expressed by

$$x'^\mu = \sum_{\nu=0}^{3} \Lambda^\mu{}_\nu \, x^\nu \tag{2}$$

$$\Lambda^0{}_0 = \gamma \quad \Lambda^i{}_0 = \gamma v^i/c \quad \Lambda^i{}_j = \delta^i{}_j + (\gamma - 1)\frac{v^i\, v_j}{v^2} \quad \Lambda^0{}_j = \gamma v_j/c$$

- Einstein summation convention:
  eliminates summation symbol if same index appears up and down

$$x'^\mu = \Lambda^\mu{}_\nu \, x^\nu \tag{3}$$

Spacetime interval

- Given two events ☞ there is 4 vector

$$\Delta x^\mu = (c(t_2 - t_1), (x_2 - x_1), (y_2 - y_1), (z_2 - z_1)) \tag{4}$$

- Invariant interval is now expressed by

$$\begin{aligned} \Delta s^2 &= g_{\mu\nu}\, \Delta x^\mu\, \Delta x^\nu \\ &= c^2 \Delta t^2 - \Delta x^2 - \Delta y^2 - \Delta z^2 \end{aligned} \tag{5}$$

- Metric of Minkowski spacetime:

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{6}$$

Metric tensor

- Metric arises directly from physics of spacetime
- It can be used to lower indices ☞ $x_\mu = g_{\mu\nu}\, x^\nu$
- Inverse metric $g^{\mu\nu}$ is defined so that $g^{\mu\nu} g_{\nu\zeta} = \delta^\mu{}_\zeta$
- Inverse metric raises indices ☞

$$g^{\mu\nu}\, x_\nu = g^{\mu\nu}(g_{\nu\zeta}\, x^\zeta) = (g^{\mu\nu} g_{\nu\zeta}) x^\zeta = \delta^\mu{}_\zeta\, x^\zeta = x^\mu$$

- Invariance of interval $\Delta s^2 = \Delta s'^2$ places constraint on form of $\Lambda^\mu{}_\nu$

$$g_{\mu\nu} x'^\mu x'^\nu = g_{\mu\nu}\, \Lambda^\mu{}_\alpha\, \Lambda^\nu{}_\beta\, x^\alpha x^\beta = g_{\alpha\beta} x^\alpha x^\beta \tag{7}$$

which implies

$$g_{\alpha\beta} = g_{\mu\nu} \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta \tag{8}$$

- (8) can be used to define Lorentz transformation
- Since $g_{\mu\nu}$ is symmetric there are only ten independent equations
- Only 6 free parameters: 3 to label velocity and 3 to label rotations

Proper time

- Consider time-like (or light-like) interval ☞ $\Delta s^2 = g_{\mu\nu}\Delta x^\mu \Delta x^\nu$ in any Lorentz frame separating points on particle's trajectory
- Same interval can be expressed in coordinates such that at each moment particle is at rest
- Such a frame is called ☞ instantaneous rest frame
- Since in instantaneous rest frame particle is at rest using interval invariance

$$\Delta s^2 = c^2\, \Delta\tau^2 \tag{9}$$

- Because interval is assumed time-like (or light-like) we may take square root of (9) to define proper time interval

$$\Delta\tau = \frac{1}{c}\,\Delta s \tag{10}$$

Proper time (cont'd)

- If curved trajectory is time-like ☞ each segment must be time-like
- Cumulative time is assigned to time-like trajectory $(t_0, x_0; t_f, x_f)$

$$\begin{aligned}\tau[\text{traj}] &= \sum_{i=0}^{f-1} \left[(t_{i+1} - t_i)^2 - \frac{1}{c^2}(x_{i+1} - x_i)^2 \right. \\ &- \left. \frac{1}{c^2}(y_{i+1} - y_i)^2 - \frac{1}{c^2}(z_{i+1} - z_i)^2 \right]^{1/2} \qquad (11)\end{aligned}$$

- In limit of small segments ☞ proper time over the trajectory

$$\tau[\text{traj}] = \int_{(t_0,x_0)}^{(t_f,x_f)} \frac{1}{c}\, ds \qquad (12)$$

- Wordline can be specified by $x^i(t)$ in particular inertial frame
- Alternatively ☞ worldline specified by $x^\mu = x^\mu(\tau)$

3-velocity vector $\vec{u}$

- Trajectory ☞ connected set of events coordinatized by some inertial observer as $(t, \vec{x}(t))$ can be parametrized by proper time of trajectory ☞ $(t(\tau), \vec{x}(\tau))$

$$\begin{aligned} \tau[\text{traj}] &= \int_{(t_0,x_0)}^{(t_f,x_f)} \sqrt{1 - \frac{1}{c^2}\frac{d\vec{x}}{dt}\cdot\frac{d\vec{x}}{dt}}\, dt \\ &= \int_{(t_0,x_0)}^{(t_f,x_f)} \sqrt{1 - \frac{\vec{u}\cdot\vec{u}}{c^2}}\, dt \end{aligned} \qquad (13)$$

- Elapsed proper time ☞ functional of the trajectory but function of labels of events at end points of the integral
- Since it is a function of time on trajectory we can derive a differential form of (13)

$$\frac{d\tau}{dt} = \sqrt{1 - \frac{\vec{u}\cdot\vec{u}}{c^2}} \qquad (14)$$

4-velocity vector $\boldsymbol{U}$

- For a time-like trajectory ☞ define a four vector velocity

$$\boldsymbol{U} \equiv U^{\mu} = \frac{dx^{\mu}}{d\tau} = \frac{dx^{\mu}/dt}{d\tau/dt} \tag{15}$$

- 4-velocity $\boldsymbol{U}$ is tangent to world line at each point
  because displacement is given by $\Delta x^{\mu} = U^{\mu}\Delta\tau$

- 4-components of 4-velocity vector $\boldsymbol{U}$
  can be expressed in terms of 3-velocity $\vec{u}$

$$\frac{dx^0}{d\tau} = U^0 = c\frac{dt}{d\tau} = \frac{c}{(1-u^2/c^2)^{1/2}} \tag{16}$$

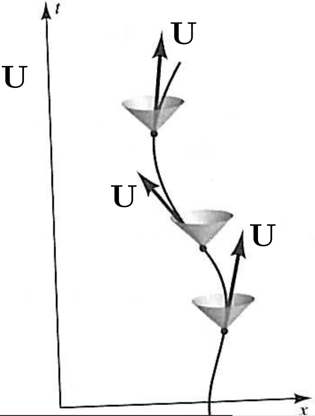
$$\vec{U} = \frac{1}{(1-u^2/c^2)^{1/2}}\frac{d\vec{x}}{dt} = \frac{1}{(1-u^2/c^2)^{1/2}}\,\vec{u} \tag{17}$$

Since ☞

$$\boldsymbol{U} \equiv U^\mu = (\gamma c, \gamma \vec{u}) \tag{18}$$

4-velocity is always time-like and future-pointing 4-vector

$$\boldsymbol{U} \cdot \boldsymbol{U} = g_{\mu\nu} U^\mu U^\nu = g_{\mu\nu} \frac{(dx^\mu/dt)(dx^\nu/dt)}{(d\tau/dt)^2} = c^2 \tag{19}$$



- @ any point along curve $x^\mu(\tau)$
- U^μ time-like tangent 4-vector
- $\boldsymbol{U}$ lies inside lightcone of point

Relation between the 4-acceleration vector

$$\boldsymbol{A} = \frac{d\boldsymbol{U}}{d\tau} \equiv \frac{d^2 x^\mu}{d\tau^2} \tag{20}$$

and 3-aceleration vector

$$\vec{a} = \frac{d^2 x^i}{dt^2} \tag{21}$$

is more complicated

$$\begin{aligned} \boldsymbol{A} &= \frac{d\boldsymbol{U}}{d\tau} = \gamma \frac{d\boldsymbol{U}}{dt} = \gamma \frac{d}{dt}(\gamma c, \gamma \vec{u}) \\ &= \gamma \left(\frac{d\gamma}{dt} c, \frac{d\gamma}{dt} \vec{u} + \gamma \vec{a} \right) \end{aligned} \tag{22}$$

- Note that since

$$\frac{d\gamma}{dt} = \frac{\vec{u} \cdot d\vec{u}/dt}{c^2\ (1 - u^2/c^2)^{3/2}} \tag{23}$$

in instantaneous rest frame of particle $(\vec{u} = 0)$ ☞ $\boldsymbol{A} = (0, \vec{a})$

- $\boldsymbol{A} = 0 \Leftrightarrow |\vec{a}| = \alpha = 0$
- For A^2 being an invariant ☞ evaluate it in rest frame

$$\boldsymbol{A} \cdot \boldsymbol{A} = -\alpha^2 \tag{24}$$

- From (24) we see that $\boldsymbol{A}$ is space-like vector
- By same articfice ☞ from (18) and (22)

$$\boldsymbol{U} \cdot \boldsymbol{A} = 0 \tag{25}$$

4-acceleration vector is always orthogonal to 4-velocity

Thought-provoking observation

Consider Lorentz transformation ☞ new frame (prime coordinates) moves with velocity v along z axis of original frame (unprimed coordinates)

Take ☞ $\cosh(\vartheta) = (1 - v^2/c^2)^{-1/2}$

$$\begin{aligned} ct' &= \cosh(\vartheta)\ ct - \sinh(\vartheta)\ z \\ z' &= -\sinh(\vartheta)\ ct + \cosh(\vartheta)\ z \\ x' &= x \\ y' &= y \end{aligned} \tag{26}$$

Because $\cos(i\vartheta) = \cosh(\vartheta)$ and $\sin(i\vartheta) = \sinh(\vartheta)$
Lorentz boost may be regarded as rotation through imaginary angle $i\vartheta$ in itc-z plane