Modern Physics

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Lesson IV
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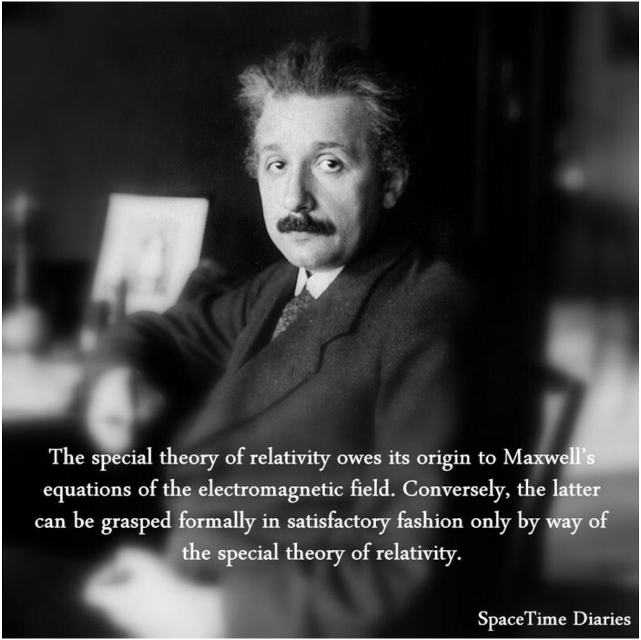

The special theory of relativity owes its origin to Maxwell's equations of the electromagnetic field. Conversely, the latter can be grasped formally in satisfactory fashion only by way of the special theory of relativity.
SpaceTime Diaries

Maxwell's Equations

1. All known laws of electricity and magnetism are summarize in

$$\vec{\nabla} \cdot \vec{E}(\vec{r},t) = \frac{1}{\epsilon_0}\rho(\vec{r},t) \tag{1}$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}(\vec{r},t)}{\partial t} \tag{2}$$

$$\vec{\nabla} \cdot \vec{B}(\vec{r},t) = 0 \tag{3}$$

$$\vec{\nabla} \times \vec{B}(\vec{r},t) = \mu_0 \vec{j}(\vec{r},t) + \mu_0\epsilon_0 \frac{\partial \vec{E}(\vec{r},t)}{\partial t} \tag{4}$$

and associated force law

$$\vec{F} = q\vec{E} + q\vec{v} \times \vec{B} \tag{5}$$

2. Careful dimensional analysis ☞ $c = (\mu_0\epsilon_0)^{-1/2}$

Giving a quick rundown of the $\vec{E} \leftrightharpoons \vec{B}$ dilemma

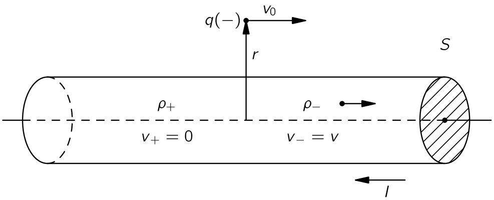
- When we said that magnetic force on charge was proportional to its velocity ☞ you may have wondered:
  1. What velocity?
  2. With respect to which reference frame?
- From definition of $\vec{B}$ ☞ what this vector is depends on our choice of reference frame for specification of velocity of charges
- But we have said nothing about which is the proper frame for specifying the magnetic field
- It turns out that any inertial frame will do
- Although static Maxwell's equations separate into $\vec{E}$ and $\vec{B}$ with no apparent connection between the two fields ☞ in nature there is intimate relation between them that arises from relativity principle
- Let's see what our knowledge of relativity would tell us about magnetic forces if we assume that relativity principle is applicable – as it is – to electromagnetism

Feynman's example

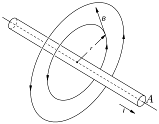
- Think about what happens when negative charge moves with velocity v_0 parallel to current-carrying wire
- Try to understand what goes on in two reference frames: one fixed wrt wire (S) and one fixed wrt particle (S')
- In S-frame ☞ there is magnetic force on particle
- Force is directed toward wire ☞ if charge were moving freely we would see it curve in toward wire
- But in S'-frame there can be no magnetic force on particle ☞ because its velocity is zero
- Does it then stay where it is?
- Would we see different things happening in the two systems?
- Principle of relativity would say that in S' we should also see particle move closer to wire
- We must try to understand why that would happen

Atomic description of current-carrying wire in S-frame

- In conductor electric currents come from motion of negative conduction electrons while positive nuclear charges and remainder of electrons stay fixed in body of material
- ρ_- ☞ charge density of conduction electrons of velocity v
- ρ_+ ☞ density of charges at rest $= \rho_-$ ☞ wire is uncharged
- There is no $\vec{E}$ field outside wire
- Force on moving particle ☞ $\vec{F} = q\vec{v}_0 \times \vec{B}$



- Recall Ampère's law ☞ $\oint \vec{B} \cdot d\vec{\ell} = \mu_0 I_{\rm encl}$
- $\vec{B}$ field at distance r from axis of wire: $B = \frac{1}{4\pi\epsilon_0 c^2}\frac{2I}{r}$
- $c = 1/\sqrt{\mu_0\epsilon_0}$

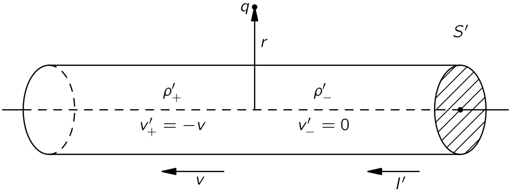


Force in S-frame

- Conclude that:
  1. Force on particle is directed toward wire
  2. Force has magnitude ☞ $F = \frac{1}{4\pi\epsilon_0 c^2}\frac{2Iqv_0}{r}$
- Since $I = \rho_- vA$ ☞ $F = \frac{1}{4\pi\epsilon_0 c^2}\frac{2q\rho_- Avv_0}{r}$
- We could continue to treat general case of arbitrary velocities but it will be just as good to look at special case $v_0 = v$
- Taking $v_0 = v$ ☞ $F = \frac{q}{2\pi\epsilon_0}\frac{\rho_- A}{r}\frac{v^2}{c^2}$

What happens in S'?

- Particle is at rest and wire is running past with speed v
- Positive charges moving with wire will make some B' at particle
- But particle is now at rest ☞ there is no magnetic force on it!
- If there is any force on particle it must come from $\vec{E}$
- It must be that moving wire has produced an $\vec{E}$
- But it can do that only if it appears charged ☞ it must be that neutral wire with current appears to be charged when set in motion



Charge density in S and S'

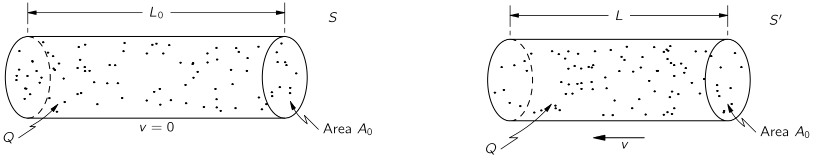
- Compute ρ of wire in S' from what's known about it in S
- Aren't ρ and ρ' the same?
- Charge q on particle is invariant scalar quantity
  ☞ independent of reference frame
- In any frame ☞ charge density of distribution of electrons is proportional to number of electrons per unit volume
- BUT we know that lengths are changed between S and S'
  ☞ so volumes will change also
- Since charge densities depend on volume occupied by charges
  ☞ densities will change too
- Must calculate:
  volume changes because of relativistic contraction of distances

Length contraction of current-carrying wire

- Take length L_0 of wire with charge density ρ_0 of stationary charges
- Total charge $Q = \rho_0 L_0 A_0$
- If same charges are observed in different frame moving v ☞ they will all be found in piece of material with shorter length

$$L = L_0 \sqrt{1 - v^2/c^2}$$

but same area



Current and Charge Distribution within Wire

- ρ ☞ density of charges in S
- Charge conservation implies:
  $Q = \rho L A_0 = \rho_0 L_0 A_0$ ☞ $\rho L = \rho_0 L_0 \Rightarrow \rho = \rho_0 / \sqrt{1 - v^2/c^2}$
- ρ_+ charges are at rest in S ☞ BUT move with spped v in S'
  $$\rho'_+ = \rho_+ / \sqrt{1 - v^2/c^2} \equiv \gamma \rho_+$$
- Negative charges are at rest in S' ☞ rest density $\equiv \rho_0 = \rho'_-$
  because they have density ρ_- when wire is at rest in S
  where speed of negative charges is v
- For conductor electrons ☞ $\rho_- = \gamma \rho'_- \Rightarrow \rho'_- = \rho_- \sqrt{1 - v^2/c^2}$
- In S' we have a net charge ☞ $\rho' = \rho'_+ + \rho'_- \neq 0$
  $$\rho' = \rho_+ \frac{1}{\sqrt{1 - v^2/c^2}} + \rho_- \sqrt{1 - v^2/c^2}$$
- Since stationary wire is neutral ☞ $\rho_- = -\rho_+ \Rightarrow \rho' = \rho_+ \frac{v^2/c^2}{\sqrt{1 - v^2/c^2}}$

- Recall Gauss' law ☞ $\oiint \vec{E} \cdot d\vec{A} = Q/\epsilon_0$
- Take $Q = \rho AL$ and $A = 2\pi rL$
- $\vec{E}$ field at distance r from axix of wire ☞ $E' = \frac{\rho' A}{2\pi\epsilon_0 r} = \frac{\rho_+ A v^2/c^2}{2\pi\epsilon_0 r\sqrt{1-v^2/c^2}}$

Force in S'-frame

- Force on negatively charged particle in S' is also towards wire
- Magnitude of force in S' ☞ $F' = \frac{q}{2\epsilon_0}\frac{\rho_+ A}{r}\frac{v^2/c^2}{\sqrt{1-v^2/c^2}}$
- Comparing F with F' ☞ $F' = \frac{F}{\sqrt{1-v^2/c^2}}$
- For small velocities ☞ $F = F'$!
- Conclude that ☞ for low velocities electricity and magnetism are just "two ways of looking at the same stuff"
- But wait ☞ things are even better than that!!!

No Contraction in Orthogonal Directions

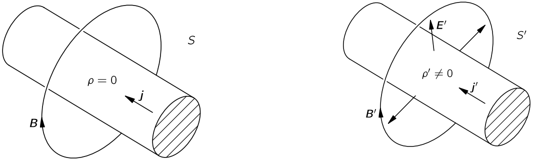
- What transverse momentum will particle have after force has acted for little while?
- Transverse momentum of particle should be the same in both S- and S'-frames
- Calling transverse coordinate y ☞ $\Delta p_y = F \Delta t$ and $\Delta p'_y = F' \Delta t'$
- We must compare Δp_y and $\Delta p'_y$ for time intervals Δt and $\Delta t'$
- Since particle is initially at rest in S' ☞ for small time interval

$$\Delta t = \frac{\Delta t'}{\sqrt{1 - v^2/c^2}}$$

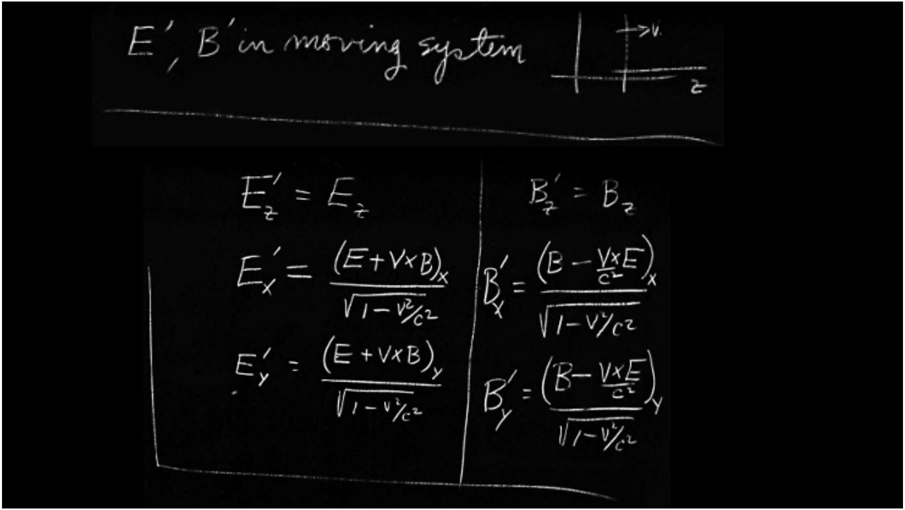
- We conclude that

$$\frac{\Delta p'_y}{\Delta p_y} = \frac{F' \ \Delta t'}{F \ \Delta t} = 1!!!$$

Relativity of Electric and Magnetic Fields



- In S frame
  1. Charge density is zero and current density is J
  2. There is only $\vec{B}$ field
- In S' frame
  1. There is charge density $\rho' \neq 0$ and different current density J'
  2. $\vec{B}'$ field is different and there is $\vec{E}'$ field
- We must not attach too much reality to $\vec{E}$ and $\vec{B}$ "lines" ☞ they may disappear if we observe them from different coordinate system
- Conclude that ☞ **electricity and magnetism are just "two ways of looking at the same stuff"**


E′, B′ in moving system
v
z
$E'_z = E_z$
$B'_z = B_z$
$E'_x = \frac{(E + V \times B)_x}{\sqrt{1 - v^2/c^2}}$
$B'_x = \frac{\left(B - \frac{V \times E}{c^2}\right)_x}{\sqrt{1 - V^2/c^2}}$
$E'_y = \frac{(E + V \times B)_y}{\sqrt{1 - v^2/c^2}}$
$B'_y = \frac{\left(B - \frac{V \times E}{c^2}\right)_y}{\sqrt{1 - V^2/c^2}}$

Lorentz boost in the x-direction

$$E'_x = E_x \qquad B'_x = B_x$$

$$E'_y = \gamma(E_y - vB_z) \qquad B'_y = \gamma(B_y + vE_z)$$

$$E'_z = \gamma(E_z + vB_y) \qquad B'_z = \gamma(B_z - vE_y)$$

$I_a = \vec{E} \cdot \vec{B}$ is Lorentz invariant

$$\begin{aligned}
\vec{E}' \cdot \vec{B}' &= E_x B_x + \gamma^2 (E_y - v B_z)(B_y + v E_z/c^2) + \gamma^2 (E_z + v B_y) \\
&\times (B_z - v E_y/c^2) \\
&= E_x B_x + \gamma^2 (E_y B_y - v^2/c^2 E_z B_z - v B_z B_y + v E_y E_z/c^2) \\
&+ \gamma^2 (E_z B_z - v^2/c^2 E_y B_y + v B_y B_z - v E_z E_y/c^2) \\
&= \vec{E} \cdot \vec{B}
\end{aligned}$$

$I_b = E^2 - c^2 B^2$ is Lorentz invariant

$$
\begin{aligned}
E'^2 - c^2 B'^2 &= E_x^2 + \gamma^2 (E_y - vB_z)^2 + \gamma^2 (E_z + vB_y)^2 \\
&- c^2 B_x^2 - c^2 \gamma^2 (B_y + vE_z/c^2)^2 - c^2 \gamma^2 (B_z - vE_y/c^2)^2 \\
&= E_x^2 + E_y^2 + E_z^2 - c^2 B_x^2 - c^2 B_y^2 - c^2 B_z^2 \\
&+ 2\gamma^2 v(-E_y B_z + E_z B_y) - 2\gamma^2 v(B_y E_z - B_z E_y) \\
&= E^2 - c^2 B^2
\end{aligned}
$$

Could an electromagnetic field appear as pure electric field in one frame and purely magnetic field in another one?

- $\vec{B} = 0$ at P in inertial system S
- This implies $I_a = 0$ and $I_b > 0$ at P
- Can $\vec{E} = 0$ at P in some inertial system?
- Answer: No!
- If $\vec{E} = 0$ at P, then $I_b < 0$ at P
  contradicting the fact that I_b is Lorentz invariant!

A magnetic field is NOT an electric field in another frame!

E **and** B **are components of electromagnetic tensor** $F^{\mu\nu}$


TO BE
CONTINUED...