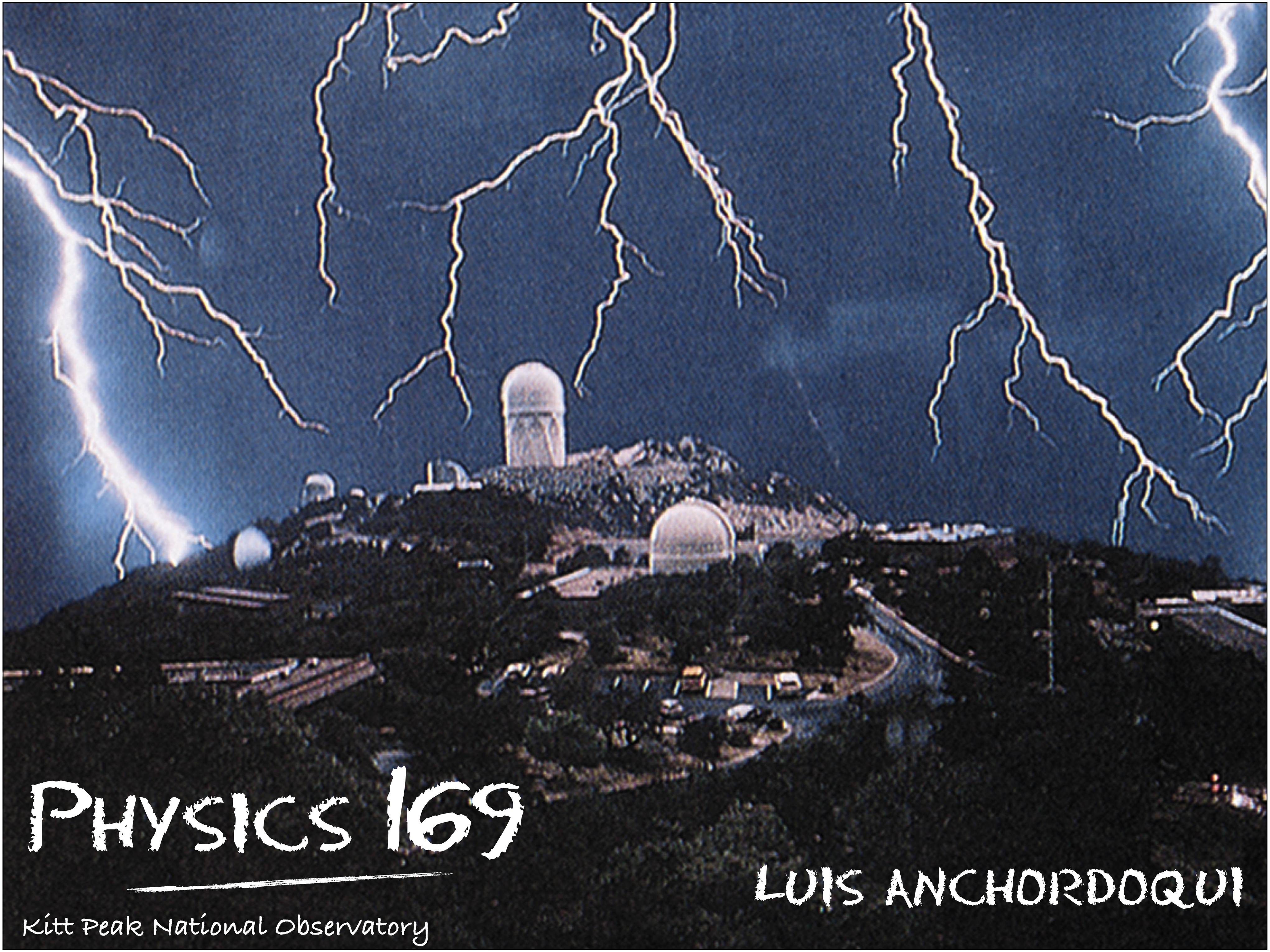


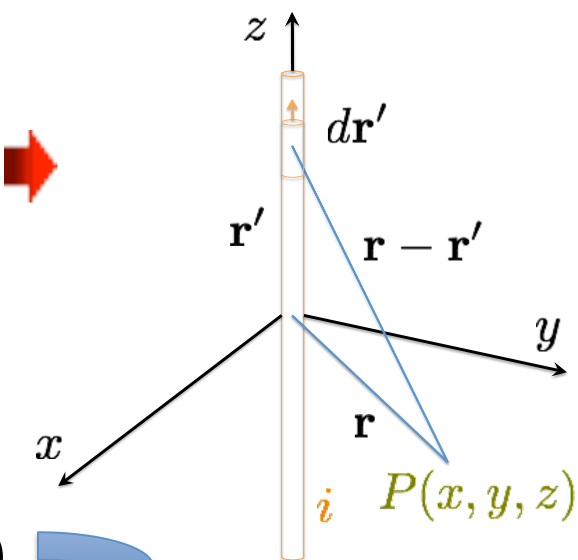
Physics 169
Luis Anchordoqui
Kitt Peak National Observatory

6.1 Magnetic Field

It was discovered experimentally that
moving electrical charges (i.e. current) create a magnetic field

Consider small piece of wire $d\mathbf{r}'$ with current i flowing through it
contribution it makes to magnetic field $d\mathbf{B}$ @ point P a distance $\mathbf{r}$ is
given by Biot-Savart Law:

$$d\mathbf{B} = \frac{\mu_0}{4\pi} \, i \, d\mathbf{r}' \times \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3}$$



Magnetic field $\mathbf{B}$: Unit = Tesla (T)

$$1\ \mathrm{T} = 1\ \mathrm{N\ m^{-1}\ A^{-1}} = 1\ \mathrm{kg\ s^{-2}\ A^{-1}}$$

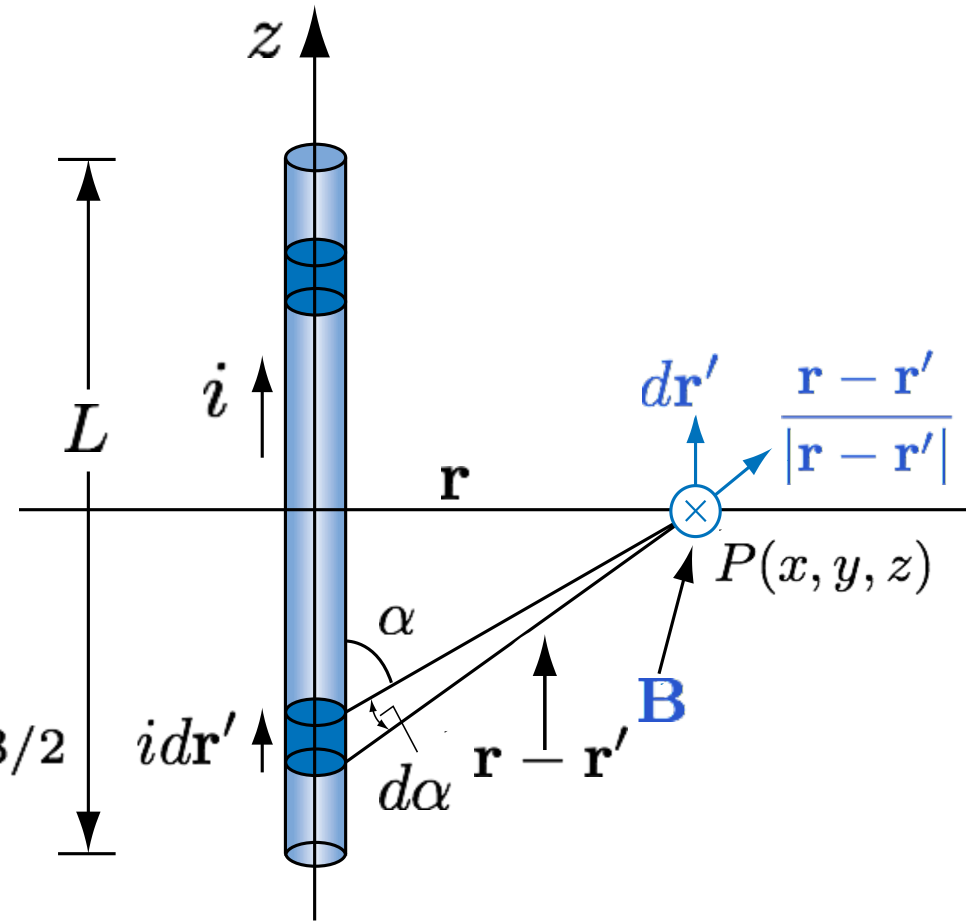
Permeability of free space (magnetic constant)

$$\mu_0 = 4\pi \times 10^{-7}\ \mathrm{T \cdot m/A}\ (\text{or equivalently } \mathrm{N/A^2})$$

$$\mathrm{Gauss\ (G)} = 10^{-4}\ \mathrm{T} \sim$$ magnetic field on Earth's surface

Example 1: Magnetic field @ P due to infinite straight wire

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int_{\text{wire}} i \, d\mathbf{r}' \times \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3}$$



$$i \, d\mathbf{r}' = +i \, dz' \, \hat{\mathbf{z}}$$

$$\mathbf{r} - \mathbf{r}' = x \, \hat{\mathbf{x}} + y \, \hat{\mathbf{y}} + (z - z') \, \hat{\mathbf{z}}$$

$$|\mathbf{r} - \mathbf{r}'|^2 = x^2 + y^2 + (z - z')^2$$

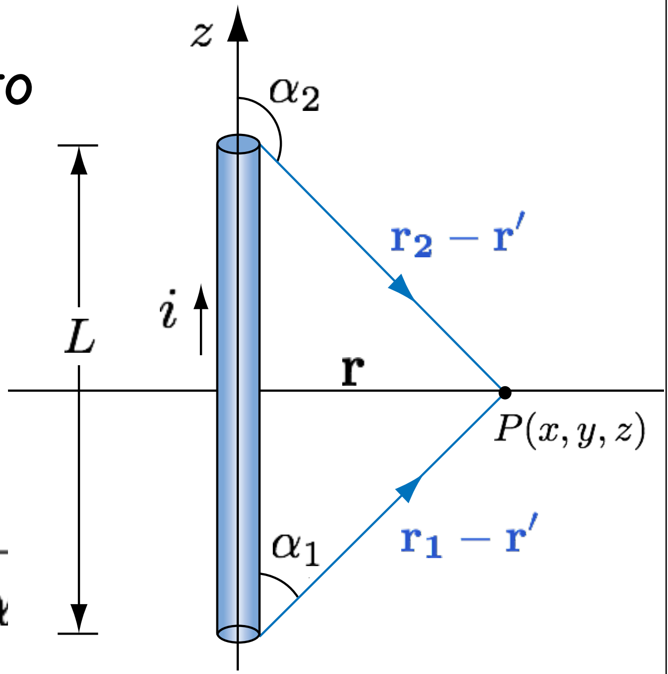
$$|\mathbf{r} - \mathbf{r}'|^3 = \left[x^2 + y^2 + (z - z')^2\right]^{3/2}$$

$$i \, dz' \, \hat{\mathbf{z}} \times (\mathbf{r} - \mathbf{r}') = i \, dz' \, (x \, \hat{\mathbf{y}} - y \, \hat{\mathbf{x}})$$

$$\mathbf{B}(x, y, z) = \frac{\mu_0}{4\pi} i (x \, \hat{\mathbf{y}} - y \, \hat{\mathbf{x}}) \int_{-\infty}^{+\infty} \frac{dz'}{[(x^2 + y^2 + (z - z')^2]^{3/2}}$$

To evaluate integral we change variable from z' to

$$\alpha = \arctan[s/(z-z')] \Rightarrow z' = z - s/\tan\alpha$$



$$dz' = +\frac{s\ d\alpha}{\sin^2\alpha} \qquad s = \sqrt{x^2+y^2}$$

$$x^2 + y^2 + (z-z')^2 = s^2 + \frac{s^2}{\tan^2\alpha} = \frac{s^2}{\sin^2\alpha}$$

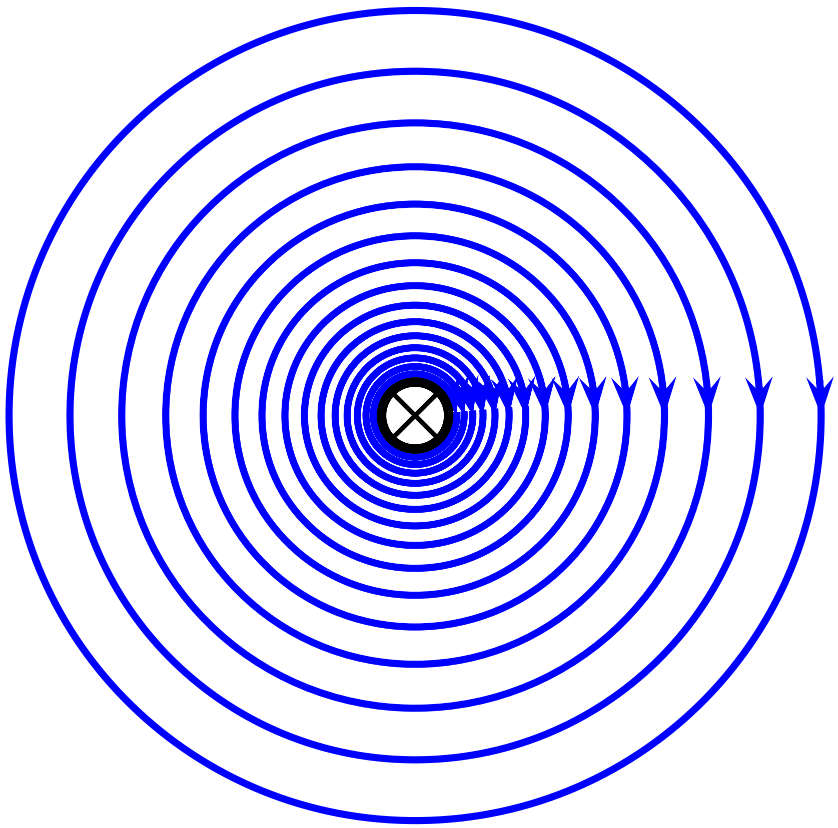
$$\frac{dz'}{[x^2+y^2+(z-z')^2]^{3/2}} = \frac{\sin\alpha\ d\alpha}{s^2} = \frac{d(-\cos\alpha)}{x^2+y^2}$$

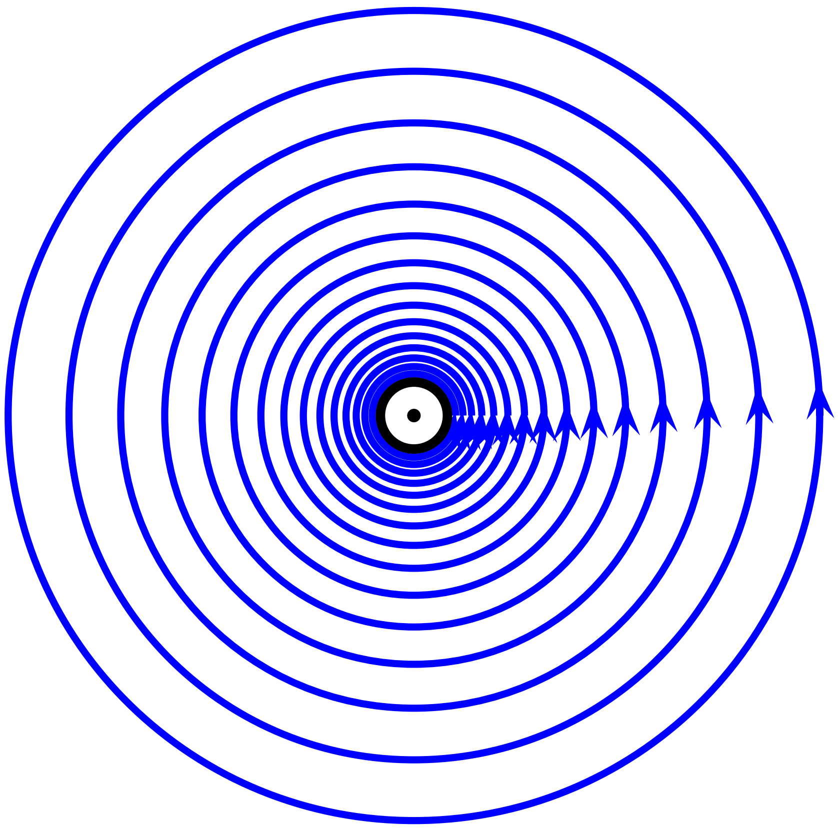
$$\int_{-\infty}^{+\infty} \frac{dz'}{[x^2+y^2+(z-z')^2]^{3/2}} = \int_0^\pi \frac{d(-\cos\alpha)}{x^2+y^2} = \frac{2}{x^2+y^2}$$

$$\mathbf{B}(x,y,z) = \frac{\mu_0 i}{2\pi}\ \frac{x\ \hat{\mathbf{y}} - y\ \hat{\mathbf{x}}}{x^2+y^2} = \frac{\mu_0 i}{2\pi}\ \frac{\hat{\phi}}{s}$$

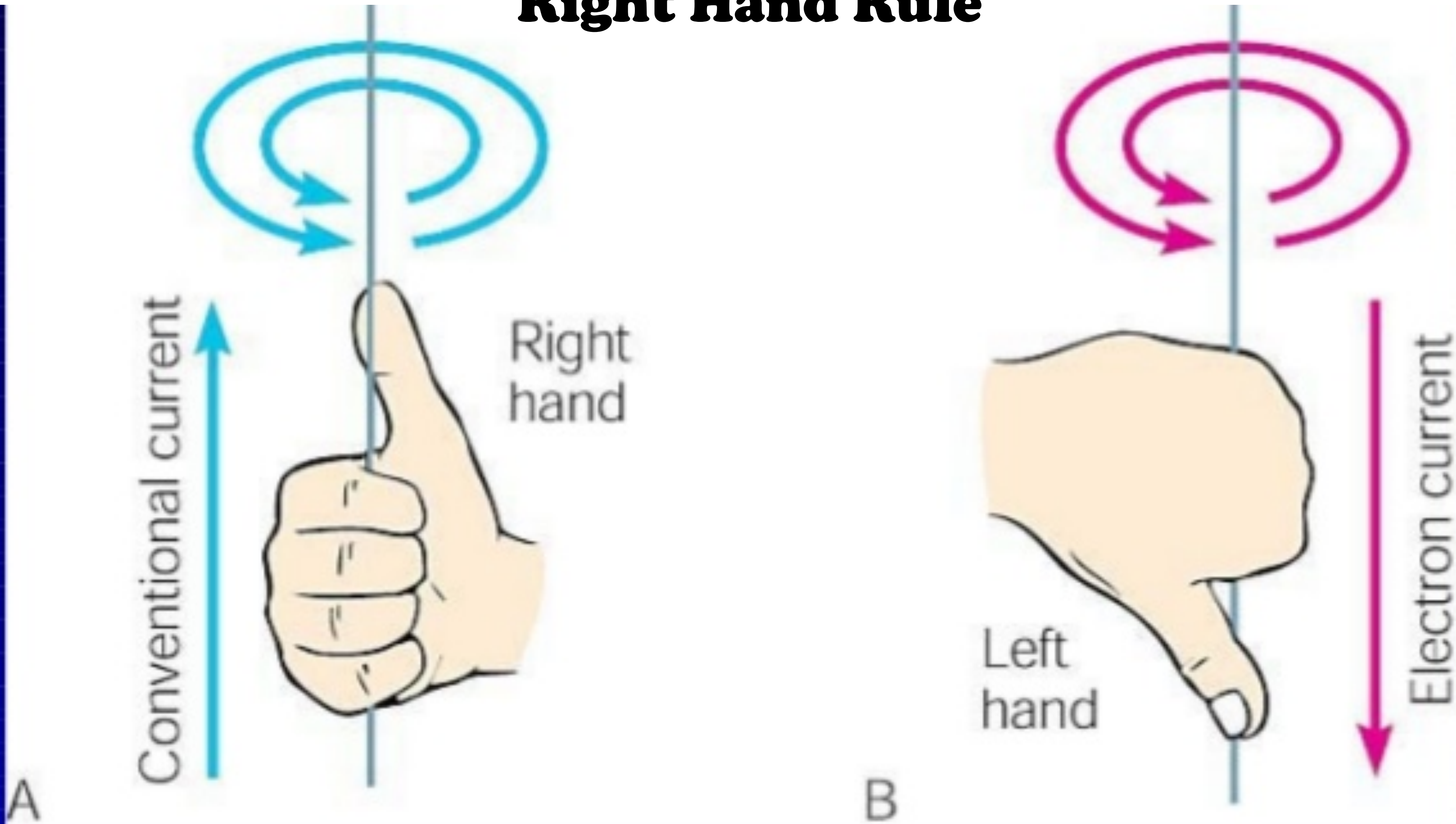
$\hat{\phi}$ ☛ unit vector in circular direction around wire in plane perpendicular to wire

If we are looking down the wire and current flows away from us then circular direction $\hat{\phi}$ of magnetic field is clockwise while if current flows toward us then magnetic field is counterclockwise



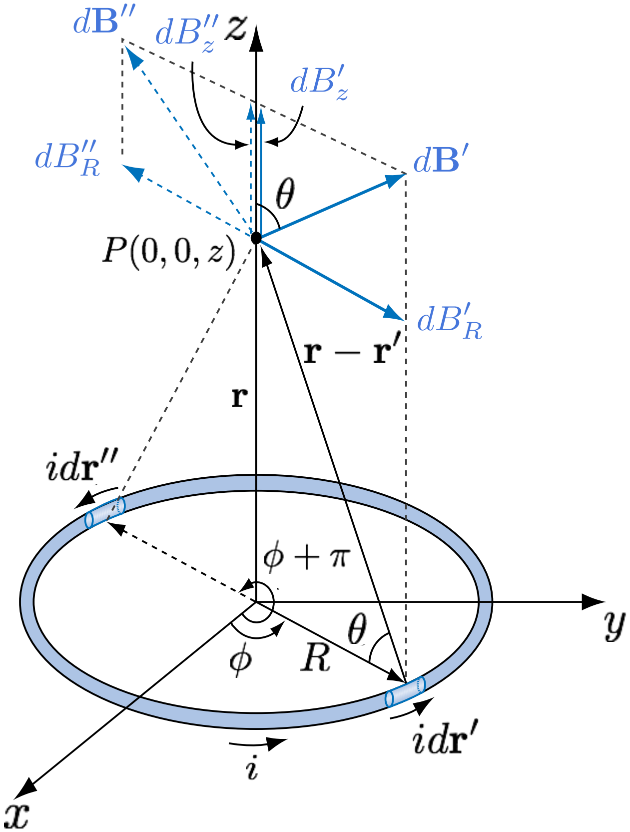


Right Hand Rule



Use (A) a right-hand rule of thumb to determine the direction of a magnetic field around a conventional current and (B) a left-hand rule of thumb to determine the direction of a magnetic field around an electron current

Example 2: Circular current loop



$$\mathbf{r}' = R\cos\phi\ \hat{\mathbf{x}} + R\sin\phi\ \hat{\mathbf{y}}$$

$$d\mathbf{r}' = R(-\sin\phi\ \hat{\mathbf{x}} + \cos\phi\ \hat{\mathbf{y}})d\phi$$

$$\mathbf{r} - \mathbf{r}' = -R\cos\phi\ \hat{\mathbf{x}} - R\sin\phi\ \hat{\mathbf{y}} + z\ \hat{\mathbf{z}}$$

$$|\mathbf{r} - \mathbf{r}'|^2 = R^2 + z^2$$

$$|\mathbf{r} - \mathbf{r}'|^3 = (R^2 + z^2)^{3/2}$$

$$(-\sin\phi\hat{\mathbf{x}} + \cos\phi\hat{\mathbf{y}}) \times (-\cos\phi\hat{\mathbf{x}} - \sin\phi\hat{\mathbf{y}}) = \sin^2\phi(\hat{\mathbf{x}}\times\hat{\mathbf{y}}) - \cos^2\phi(\hat{\mathbf{y}}\times\hat{\mathbf{x}}) = \hat{\mathbf{z}}$$

$$(-\sin\phi\hat{\mathbf{x}} + \cos\phi\hat{\mathbf{y}}) \times \hat{\mathbf{z}} = -\sin\phi(\hat{\mathbf{x}}\times\hat{\mathbf{z}}) + \cos\phi(\hat{\mathbf{y}}\times\hat{\mathbf{z}}) = \cos\phi\hat{\mathbf{x}} + \sin\phi\hat{\mathbf{y}}$$

$$id\mathbf{r}' \times (\mathbf{r} - \mathbf{r}') = iR(z\cos\phi\ \hat{\mathbf{x}} + z\sin\phi\ \hat{\mathbf{y}} + R\ \hat{\mathbf{z}})\ d\phi$$

$$\mathbf{B}(0,0,z) = \frac{\mu_0}{4\pi}\int_{\text{loop}} i\ d\mathbf{r}' \times \frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3}$$

$$= \frac{\mu_0}{4\pi}\ i\ \frac{R}{(R^2+z^2)^{3/2}}\int_0^{2\pi} d\phi(z\cos\phi\hat{\mathbf{x}} + z\sin\phi\hat{\mathbf{y}} + R\hat{\mathbf{z}})$$

Integral evaluates to

$$\int_0^{2\pi} d\phi(z\cos\phi\hat{\mathbf{x}} + z\sin\phi\hat{\mathbf{y}} + R\hat{\mathbf{z}}) = z\hat{\mathbf{x}}\int_0^{2\pi} d\phi\cos\phi + z\hat{\mathbf{y}}\int_0^{2\pi} d\phi\sin\phi + R\hat{\mathbf{z}}\int_0^{2\pi} d\phi$$
$$= 2\pi R\hat{\mathbf{z}}$$

$$\mathbf{B}(0,0,z) = \frac{\mu_0 i}{2}\frac{R^2}{(R^2+z^2)^{3/2}}\hat{\mathbf{z}}$$

Limiting Cases

① B-field at center of loop ☛ $\mathbf{B}(0,0,0) = \dfrac{\mu_0 i}{2R}\hat{\mathbf{z}}$

② For $z \gg R$ ☛ $\mathbf{B}(0,0,z) = \dfrac{\mu_0 i}{2}\dfrac{R^2}{z^3\left(1+\frac{R^2}{z^2}\right)^{3/2}}\hat{\mathbf{z}} \approx \dfrac{\mu_0 i R^2}{2z^3}\hat{\mathbf{z}}$

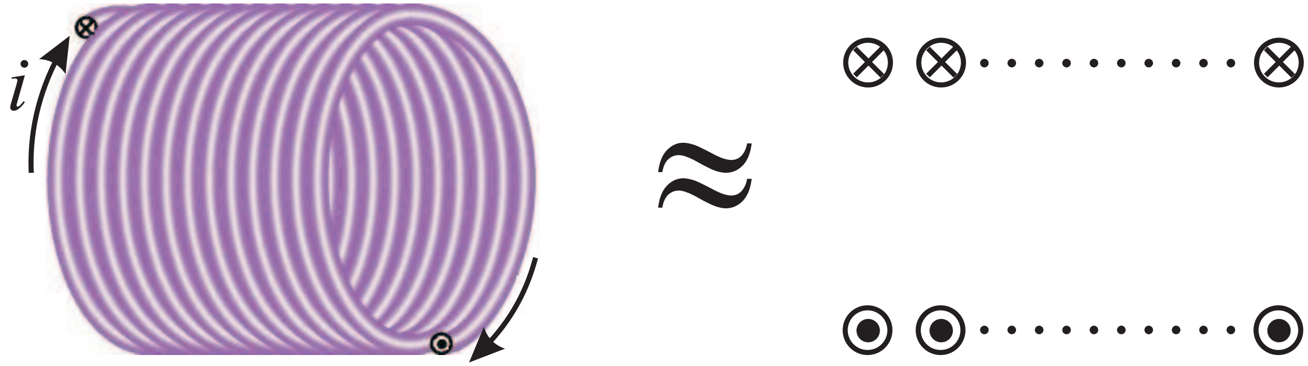
Recall E-field for an electric dipole ☛ $|\mathbf{E}| \propto \dfrac{1}{x^3}$

A circular current loop is also called a **magnetic dipole**

Example 3: Circular solenoid

Consider a solenoid of length L consisting of $N = nL$ turns of wire

Solenoid can produce a **strong and uniform** magnetic field inside its coils



Solenoid

Tightly-packed coils of wire

Start with magnetic field due to circular loop

$$\mathbf{B}(0,0,z) = \frac{\mu_0 i}{2} \frac{R^2}{(R^2 + z^2)^{3/2}} \hat{\mathbf{z}}$$

Construct bundle j of $\Delta N = n\Delta z = n(z_{j+1} - z_j)$ loops

$$\mathbf{B}_j(0,0,z) = \frac{\mu_0 n i R^2}{2} \frac{1}{(R^2 + z_j^2)^{3/2}} \left(z_{j+1} - z_j\right) \hat{\mathbf{z}}$$

Sum over bundles

$$\mathbf{B}(0,0,z) = \sum_{j=1}^{nL} \frac{\mu_0 n i R^2}{2} \frac{1}{(R^2 + z_j^2)^{3/2}} \,(z_{j+1} - z_j)\ \hat{\mathbf{z}}$$

For $nL \to \infty \Rightarrow \Delta z \to dz$

$$\mathbf{B}(0,0,z) = \int_{-z_1}^{z_2} dz\ \frac{\mu_0 n i R^2}{2} \frac{1}{(R^2 + z^2)^{3/2}}\ \hat{\mathbf{z}}$$

Change of variables $z = R\ \tan\alpha$

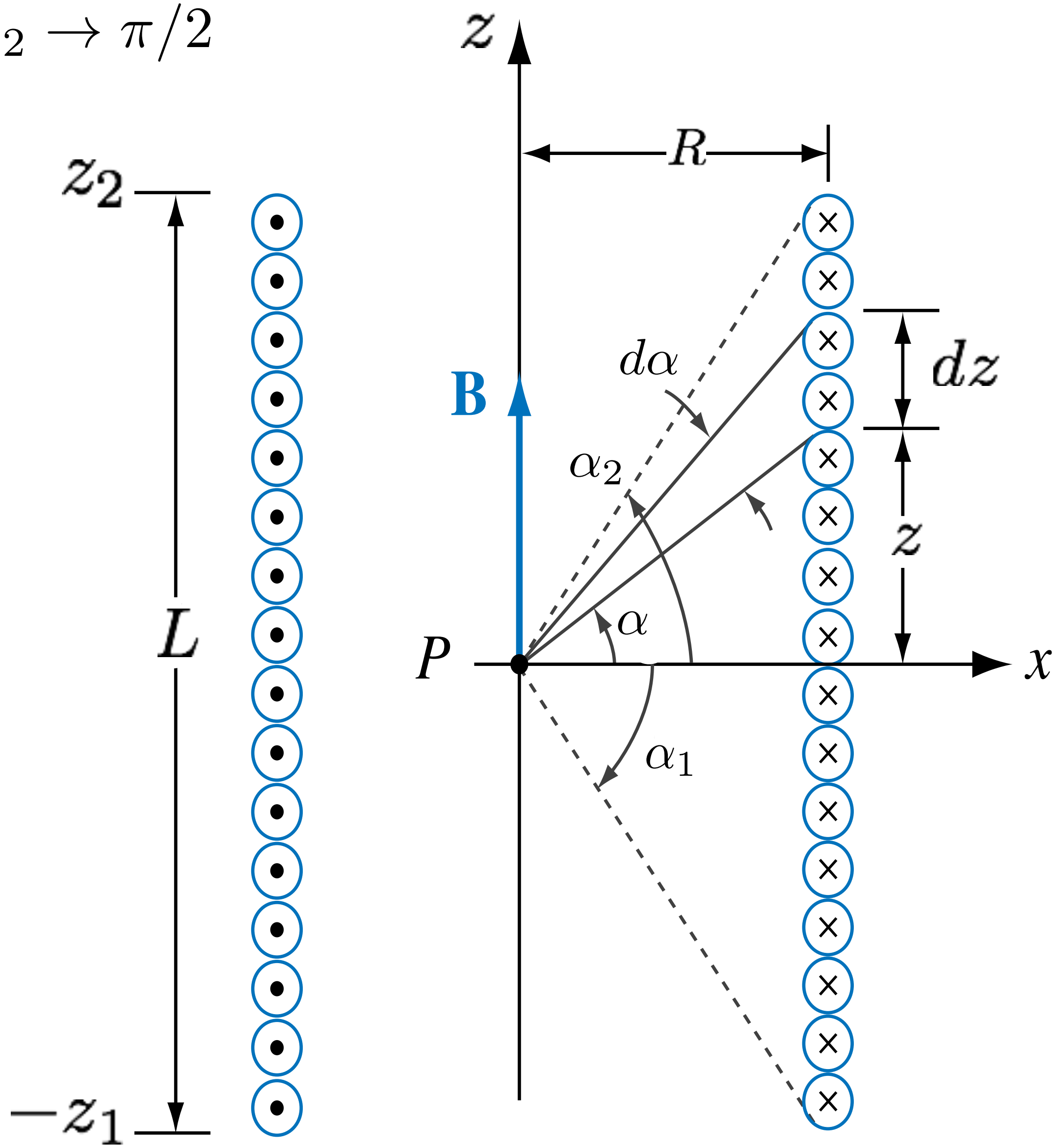
$$\mathbf{B}(0,0,z) = \frac{\mu_0 n i}{2} \int_{\alpha_1}^{\alpha_2} \frac{\sec^2\alpha\ d\alpha}{(1+\tan^2\alpha)^{3/2}}\ \hat{\mathbf{z}} = \frac{\mu_0 n i}{2} \int_{\alpha_1}^{\alpha_2} \cos\alpha\ d\alpha\ \hat{\mathbf{z}}$$

$$= \frac{\mu_0 n i}{2} (\sin\alpha_2 - \sin\alpha_1)\ \hat{\mathbf{z}}$$

Limiting Case

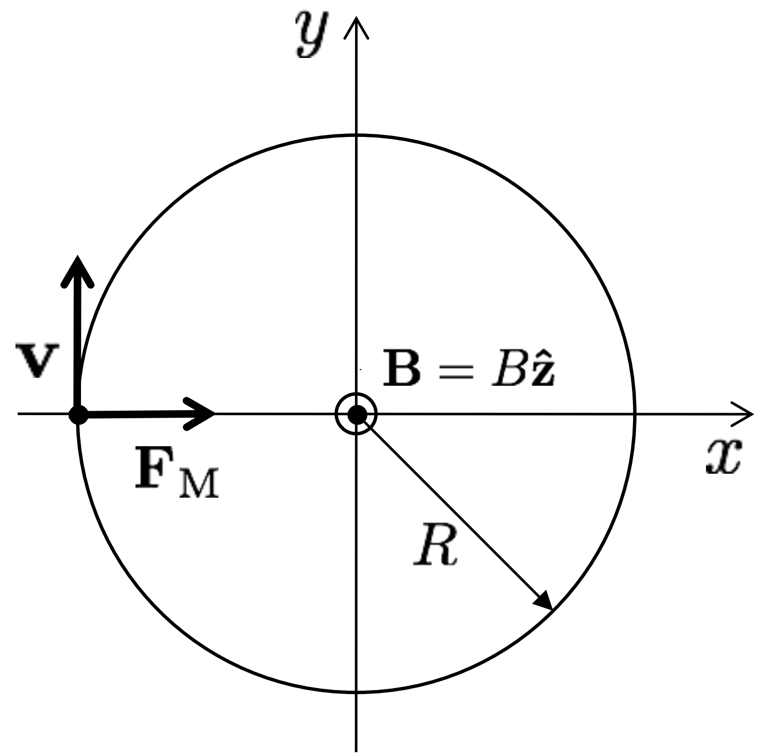
For $L \gg R \Rightarrow \alpha_1 \to -\pi/2$ & $\alpha_2 \to \pi/2$

$$\mathbf{B}(0,0,z) = \mu_0 n i\, \hat{\mathbf{z}}$$



6.2 Motion of a Point Charge in Magnetic Field

$$\mathbf{F}_{\mathrm{M}} = q\,(\mathbf{v} \times \mathbf{B})$$



Cyclotron motion

$$\mathbf{v} = v\,\hat{\mathbf{y}} \;\&\; \mathbf{r} = -R\,\hat{\mathbf{x}} \;@\; t = 0$$

$$m\frac{dv_x}{dt} = qBv_y$$

$$m\frac{dv_y}{dt} = -qBv_x$$

$$\frac{d^2v_x}{dt^2} = \frac{qB}{m}\frac{dv_y}{dt} = -\left(\frac{qB}{m}\right)^2 v_x = -\omega_c^2 v_x$$

$$\omega_c = qB/m$$

$$\frac{d^2v_y}{dt^2} = -\omega_c^2\, v_y$$

General solution

$$v_{x,y} = C_1 \cos(\omega_c t) + C_2 \sin(\omega_c t)$$

Imposing initial condition

$$v_x(t) = v \sin(\omega_c t)$$

$$v_y(t) = v \cos(\omega_c t)$$

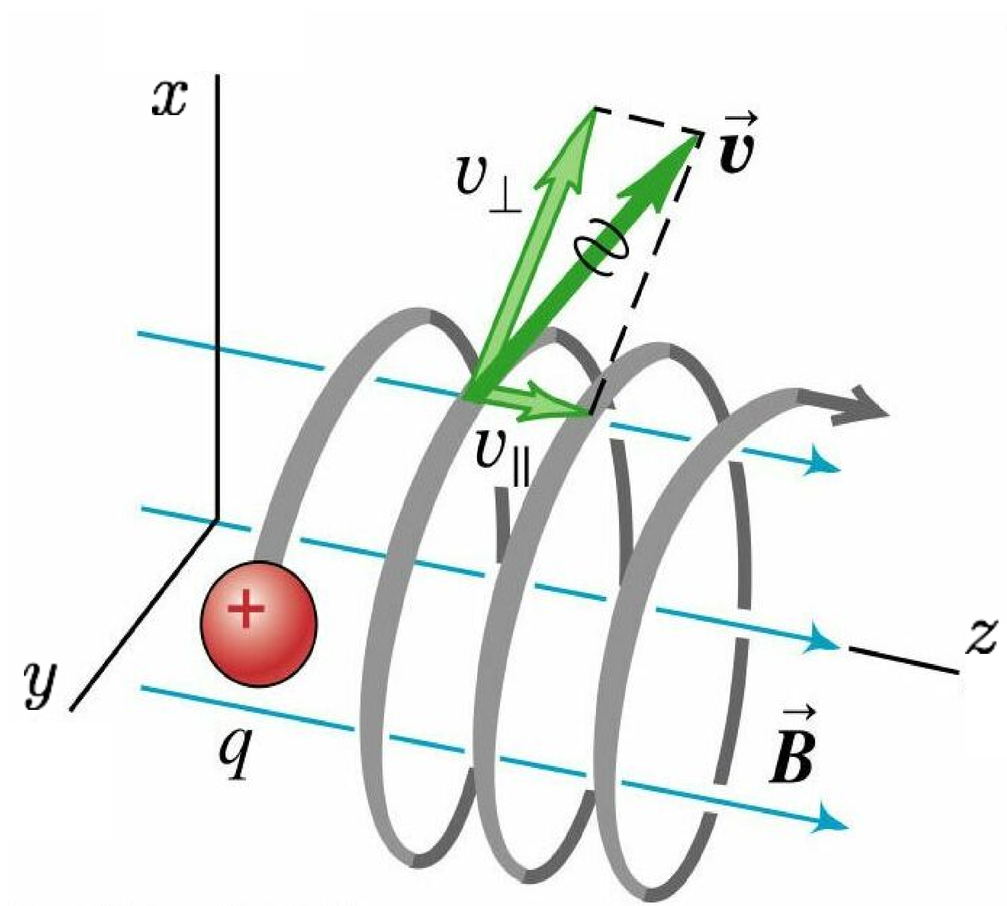
Integration leads to

$$x(t) = -R \cos(\omega_c t)$$

$$y(t) = R \sin(\omega_c t)$$

$$R = \frac{v}{\omega_c} = \frac{mv}{qB}$$

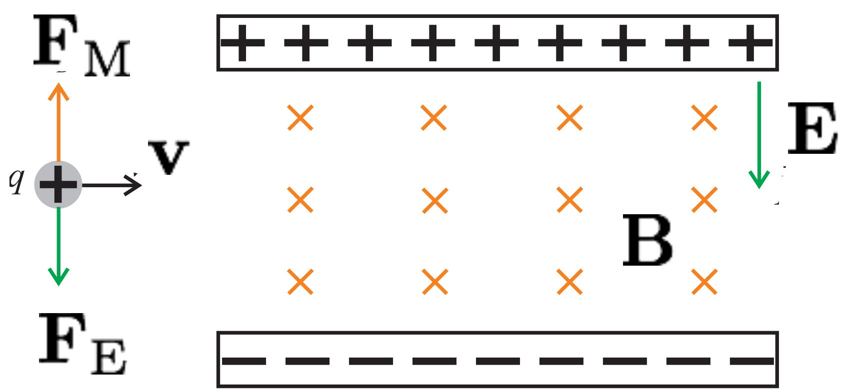
If non-zero velocity in $\hat{\mathbf{z}}$ direction



charge particle moves in a helix

Lorentz Force ➡ $\mathbf{F} = q\ [\mathbf{E} + (\mathbf{v} \times \mathbf{B})]$

Velocity Selector: special case ☛ $\mathbf{E} \perp \mathbf{B}$



$$|\mathbf{F}_{\mathrm{E}}| = |\mathbf{F}_{\mathrm{M}}|$$

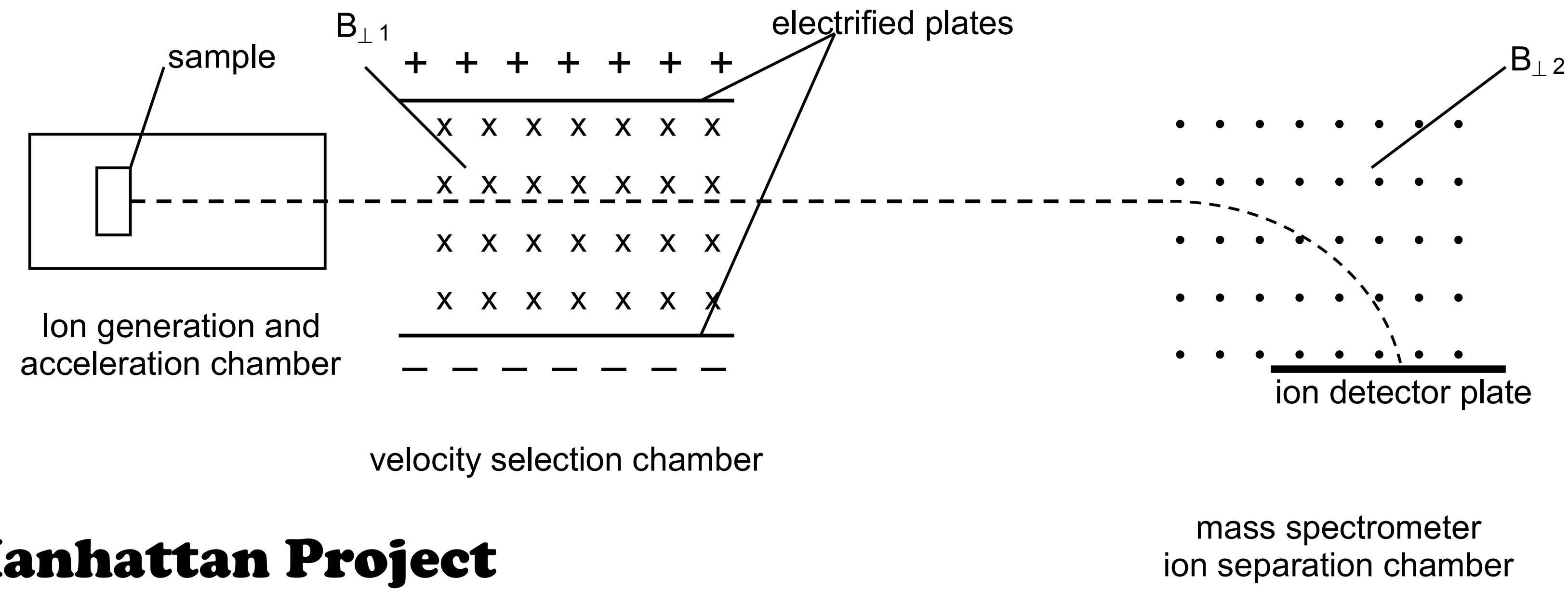
$$qE = qvB_{\perp} \Rightarrow v = E/B_{\perp}$$

Applications

- Measuring e/m for electrons (Thomson 1897)
- Cyclotron (Lawrence & Livingston 1934)
- Mass Spectrometer (Aston 1919)

Mass spectrometers have three basic parts:

1. ion source and accelerator
2. a velocity selector
3. an ion separator



Manhattan Project

Uranium isotopes, uranium–235 and uranium–239, can be separated using a mass spectrometer. The uranium–235 isotope travels through a smaller circle and can be gathered at a different point than the uranium–239. During World War II, the Manhattan project was attempting to make an atomic bomb. Uranium–235 is fissionable but it makes up only 0.70% of the uranium on Earth. A large mass spectrometer at Oakridge, Tennessee was used to separate uranium–235 from the raw uranium metal.

Uranium–235 and uranium–239 ions, each with a charge of +2, are directed into a velocity selector which has a magnetic field of 0.250 T and an electric field of 1.25×10^7 V/m perpendicular to each other. The ions then pass into a magnetic field of 2.00 mT. What is the radius of deflection for each isotope?

A charge of +2 means that each ion has lost two electrons. Therefore

$$q = 2 \times +1.60 \times 10^{-19}\text{ C} = +3.20 \times 10^{-19}\text{ C}$$

Since each proton and neutron has a mass of 1.67×10^{-27} kg, the mass of each isotope is.

$$m_{235} = 235 \times (1.67 \times 10^{-27}\text{ kg}) = 3.9245 \times 10^{-25}\text{ kg}$$
$$m_{239} = 239 \times (1.67 \times 10^{-27}\text{ kg}) = 3.9913 \times 10^{-25}\text{ kg}$$

For the velocity selector:

$$F_{\mathrm{E}} = F_{\mathrm{M}}$$

$$q\left|\vec{E}\right| = qvB_{\perp 1}$$

$$v = \frac{\left|\vec{E}\right|}{B_{\perp 1}} = \frac{1.25 \times 10^{7}\ \mathrm{V/m}}{0.250\mathrm{T}} = 5.00 \times 10^{7}\ \mathrm{m/s}$$

For the ion separator

$$F_{\mathrm{M}} = ma$$

$$qvB_{\perp} = \frac{mv^2}{r}$$

$$r = \frac{mv}{qB_{\perp}}$$

uranium – 235

$$r_{235} = \frac{(3.9245 \times 10^{-25}\mathrm{kg})(5.00 \times 10^{7}\ \mathrm{m/s})}{(3.20 \times 10^{-19}\ \mathrm{C})\ (2.00 \times 10^{-3}\mathrm{T})}$$

$$r_{235} = \mathbf{3.07 \times 10^{4}\ m}$$

uranium – 239

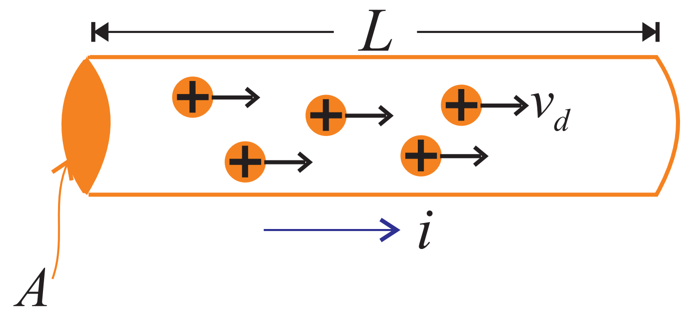
$$r_{239} = \frac{(3.9913 \times 10^{-25}\mathrm{kg})(5.00 \times 10^{7}\ \mathrm{m/s})}{(3.20 \times 10^{-19}\ \mathrm{C})\ (2.00 \times 10^{-3}\mathrm{T})}$$

$$r_{239} = \mathbf{3.12 \times 10^{4}\ m}$$

6.3 Magnetic Force on Currents

Current = many charges moving together

Consider a wire segment of length L carrying current i in a magnetic field



Total magnetic force $= \underbrace{(q\vec{v}_d \times \vec{B})}_{\text{force on one charge carrier}} \cdot \underbrace{n\,A\,L}_{\text{Total number of charge carrier}}$

Recall $i = nqv_dA$

$\therefore$ Magnetic force on current $\vec{F} = i\vec{L} \times \vec{B}$

$\vec{L}$ = vector with $\begin{cases} |\vec{L}| = \text{length of current segment} \\ \text{direction = direction of current} \end{cases}$

For infinitesimal wire segment $d\vec{l}$

$$d\vec{F} = i\,d\vec{l} \times \vec{B}$$

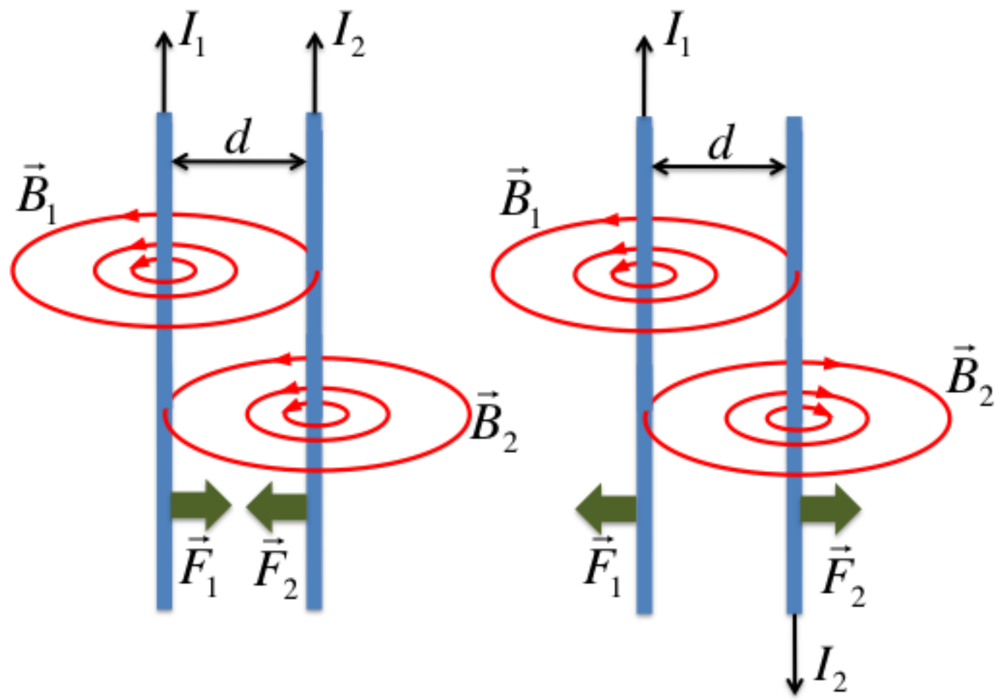
Example 1: Force between parallel currents

Consider a segment of length L on i_2

$$\vec{B}_1 = \frac{\mu_0 i_1}{2\pi d}$$

$$\vec{B}_2 = \frac{\mu_0 i_2}{2\pi d}$$

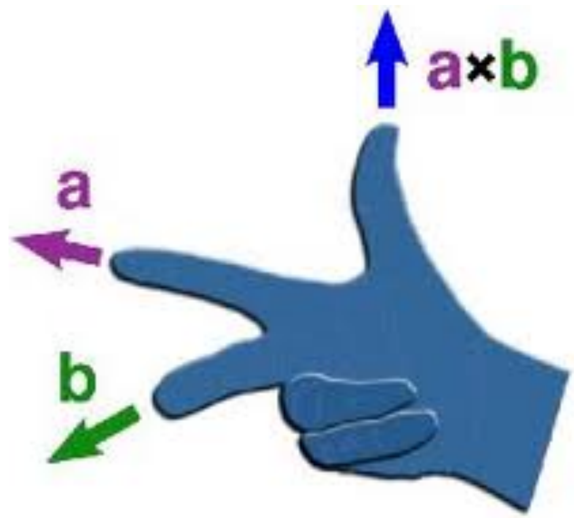
Force on i_2 coming from i_1

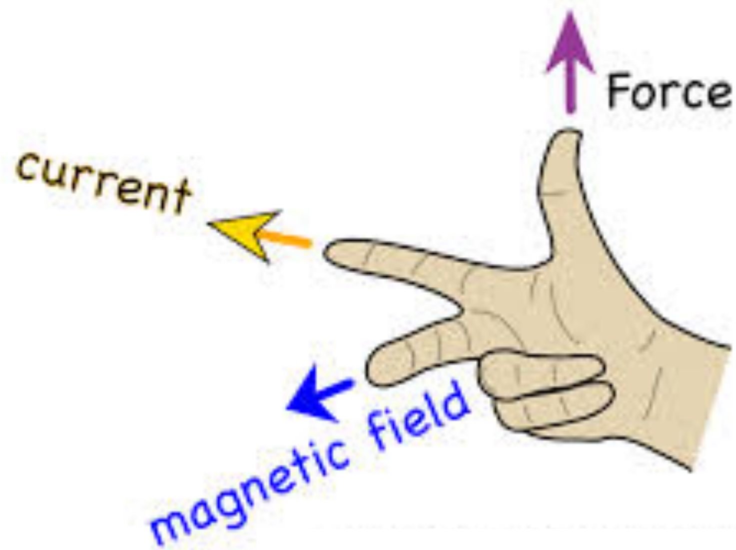


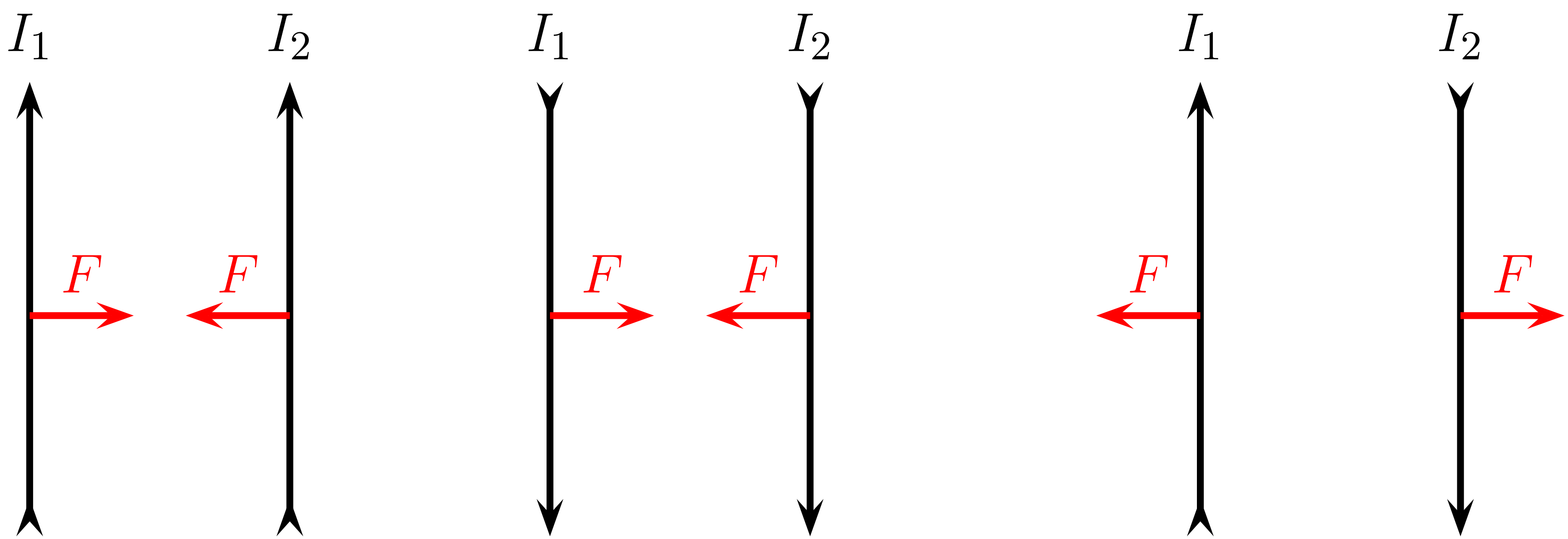
$$|\vec{F}_{21}| = i_2\vec{L} \times \vec{B}_1 = \frac{\mu_0 L\, i_1\, i_2}{2\pi d} = |\vec{F}_{12}|$$

$\therefore$ **Parallel currents attract and anti-parallel currents repel**

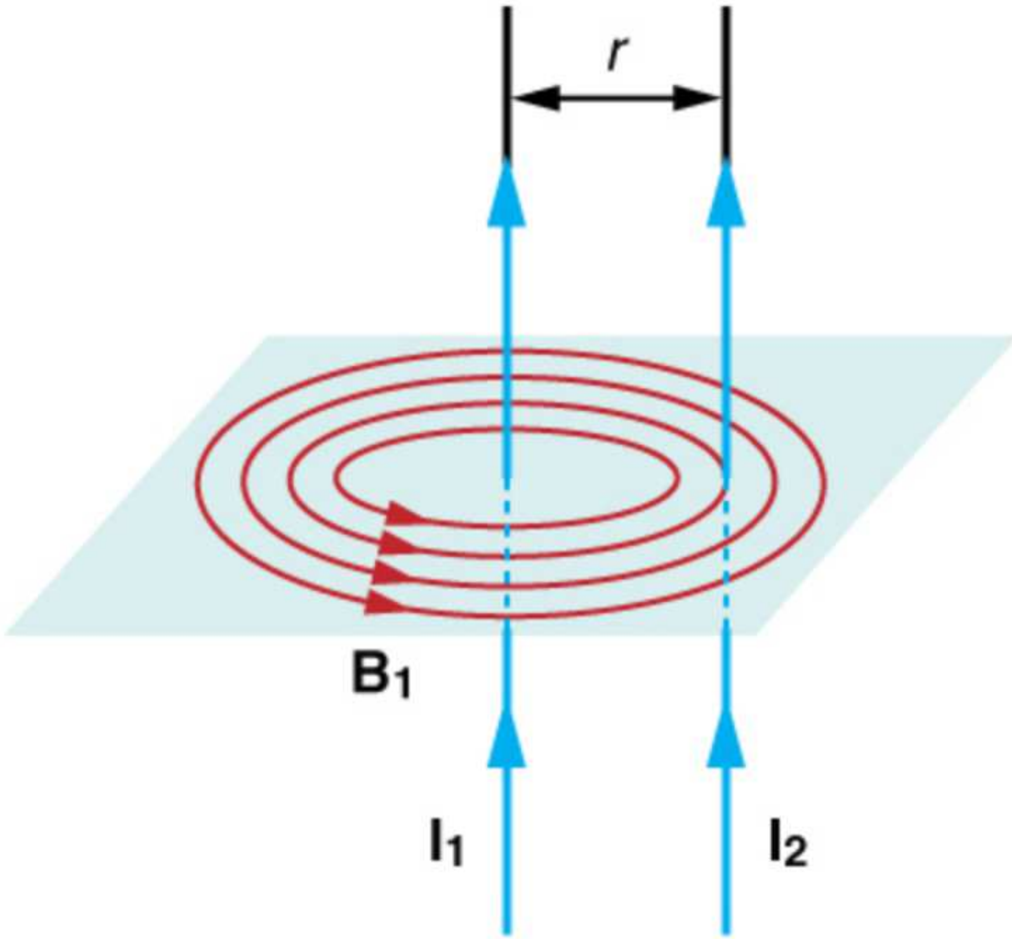
More on Right Hand Rule

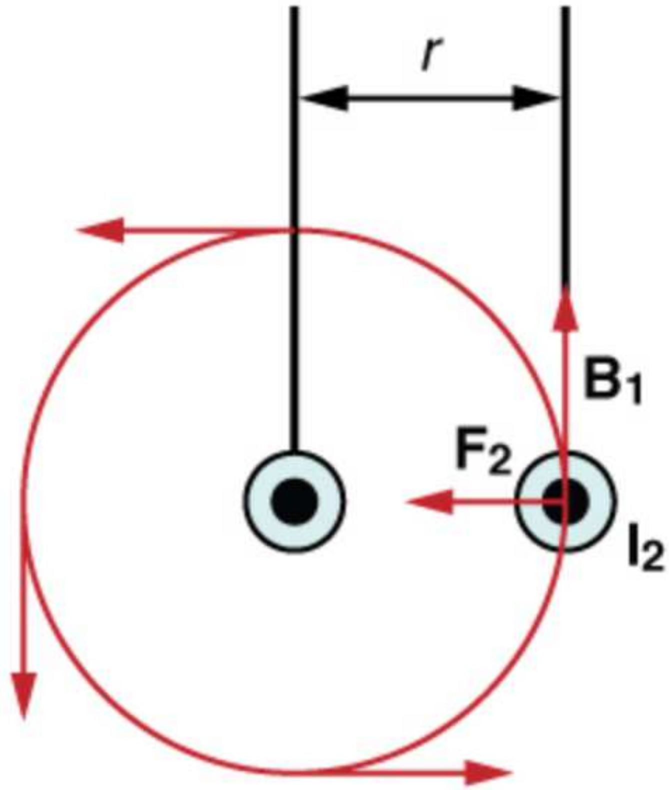




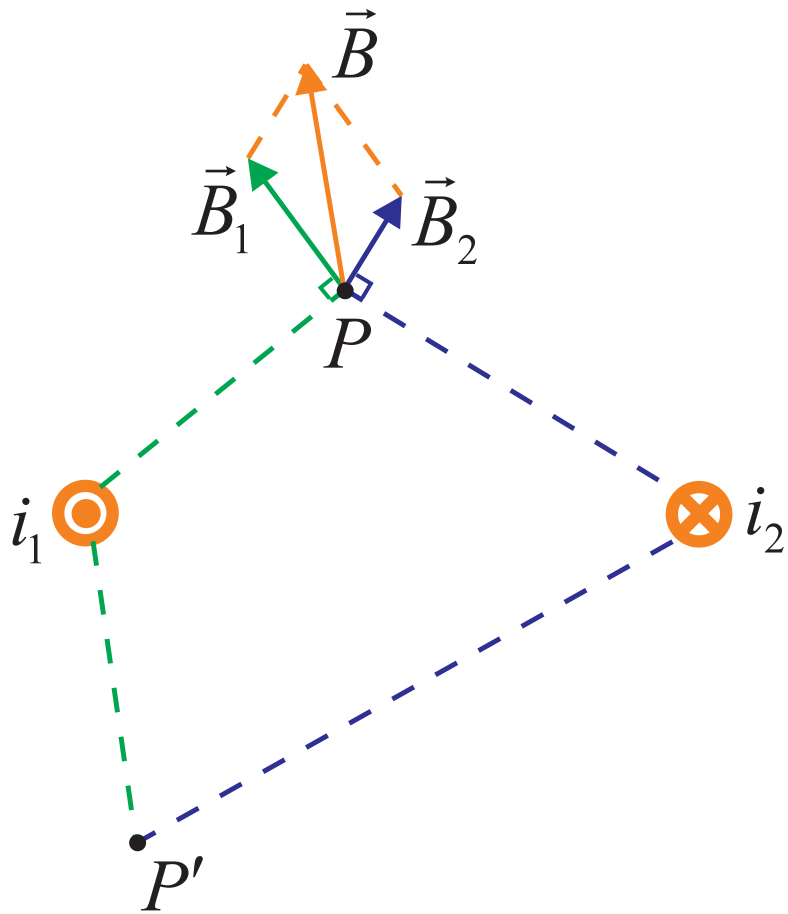


Graphical explanation of force's direction for currents in same direction



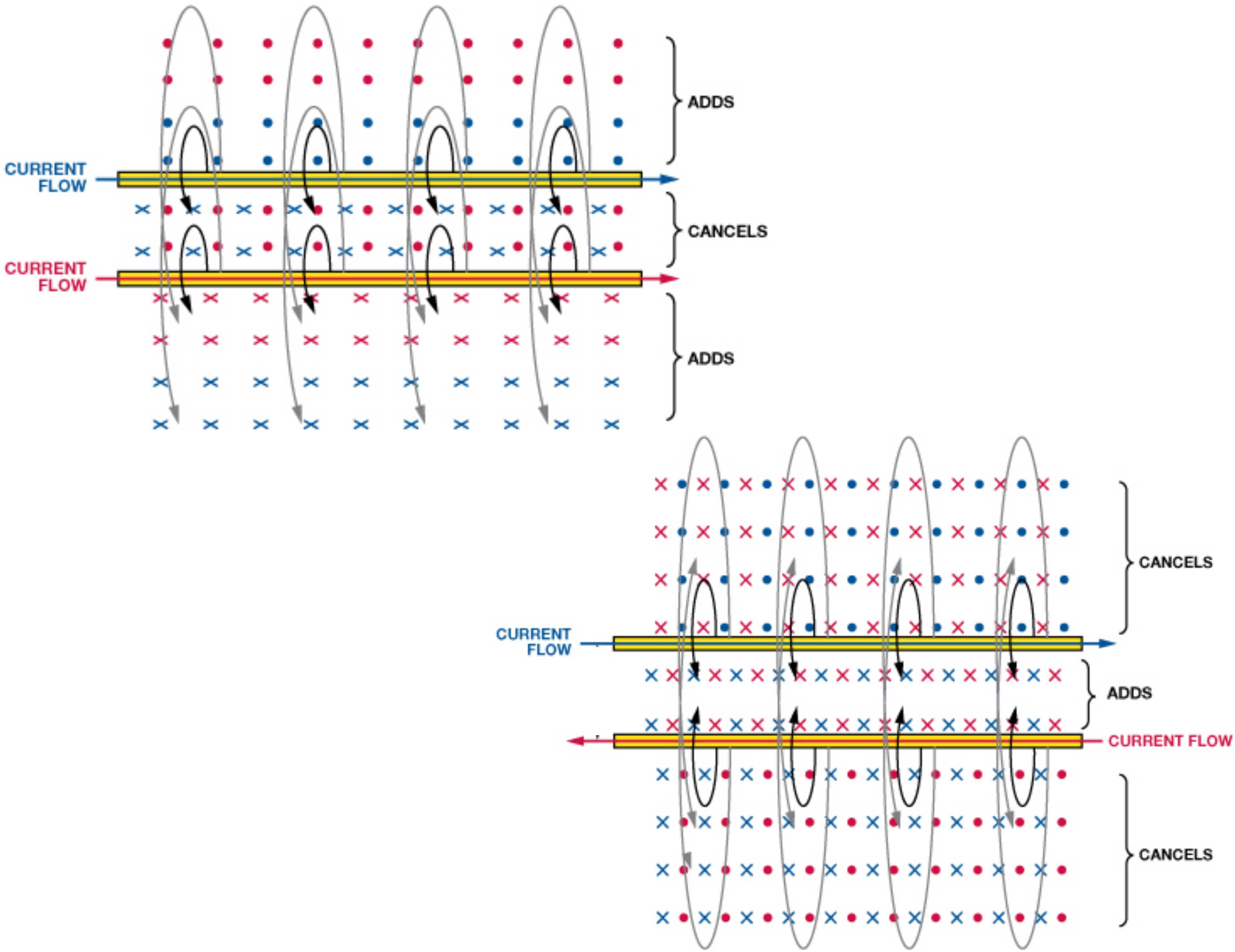


Principle of Superposition



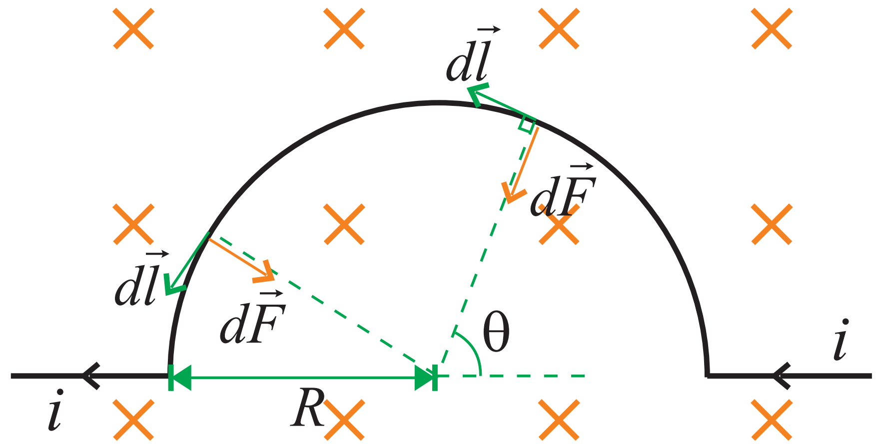
Magnetic field $\vec{B}$ at point P

due to individual currents i_1 and i_2 is **vector sum** of $\vec{B}_1, \vec{B}_2$-fields


ADDS
CURRENT FLOW
CANCELS
CURRENT FLOW
ADDS
CANCELS
CURRENT FLOW
ADDS
CURRENT FLOW
CANCELS

Example 2: Force on a semicircle current loop

$$d\vec{l} = \text{infinitesimal arc length element} \perp \vec{B}$$



$$\therefore \quad dl = R\, d\theta$$

$$\therefore \quad dF = iRB\, d\theta$$

By symmetry argument ☛ we only need to consider vertical forces

$$dF \cdot \sin\theta$$

$\therefore$ Net force ➡ $F = \int_0^\pi dF \sin\theta$

$$= iRB \int_0^\pi \sin\theta\, d\theta$$

$F = 2iRB$ (downward)

Method 2

Write $d\vec{l}$ in $\hat{\imath}, \hat{\jmath}$ components

$$d\vec{l} = -dl \ \sin\theta \ \hat{\imath} + dl \ \cos\theta \ \hat{\jmath}$$

$$= Rd\theta(-\sin\theta \ \hat{\imath} + \cos\theta \ \hat{\jmath})$$

$\vec{B} = -B\hat{k}$ (into the page)

$$\therefore \quad d\vec{F} = i\, d\vec{l} \times \vec{B}$$

$$= -iRB(\sin\theta \ d\theta \ \hat{\jmath} + \cos\theta \ d\theta \ \hat{\imath})$$

$$\therefore \quad \vec{F} = \int_0^{\pi} d\vec{F}$$

$$= -iRB\left[\int_0^{\pi} \sin\,\theta\, d\theta\hat{\jmath} + \int_0^{\pi} \cos\,\theta\, d\theta\hat{\imath}\right]$$

$$= -2iRB\hat{\jmath}$$


OFFICER, I NEED TO FILE A MISSING PERSONS REPORT
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THOUSANDS OF SEAHAWKS FANS HAVE GONE MISSING!


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