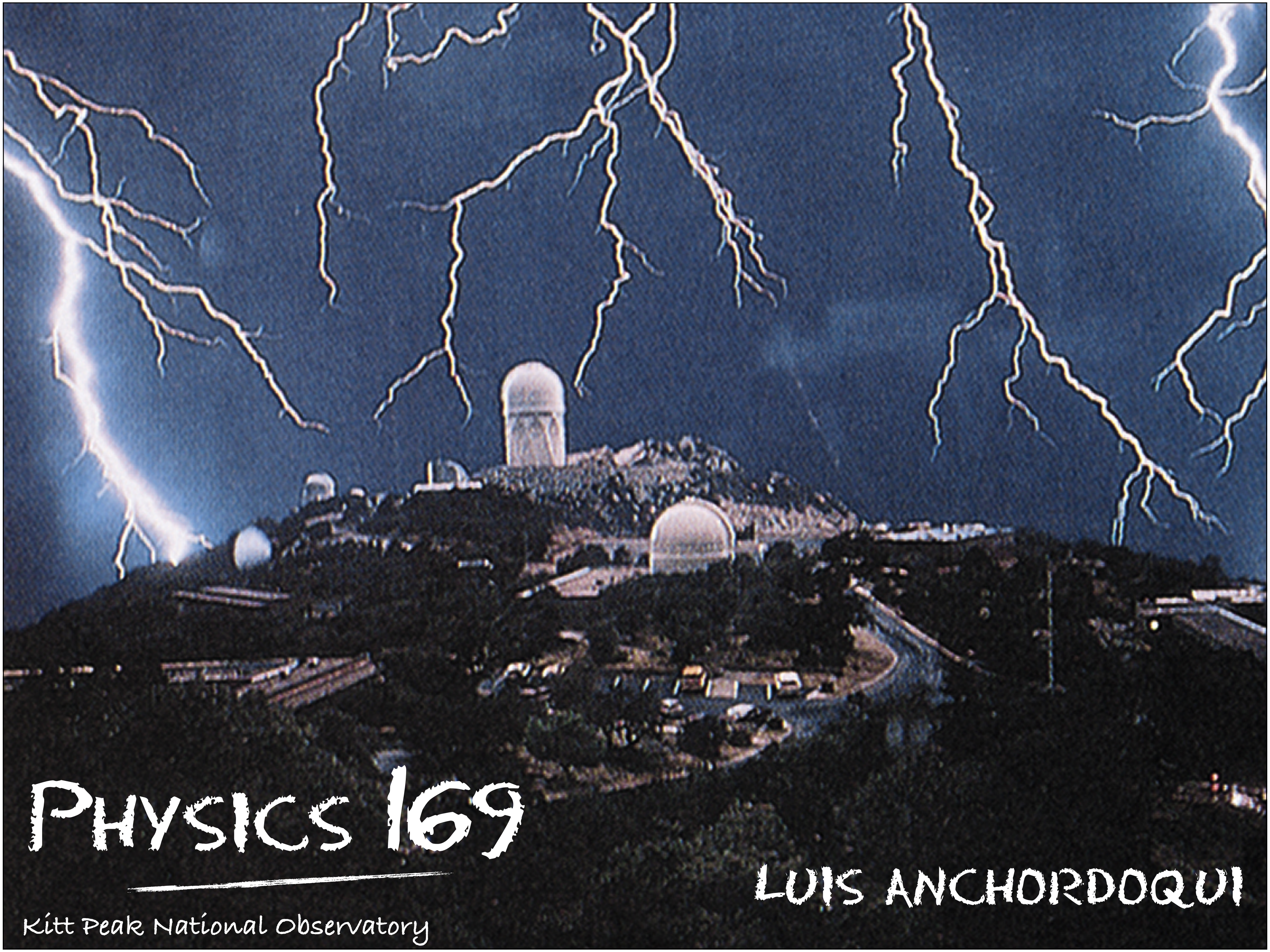


Physics 169
Luis Anchordoqui
Kitt Peak National Observatory

4.1 Capacitors

A **capacitor** is a system of **two conductors** that carries **equal and opposite charges**

A capacitor **stores charge** and **energy** in the form of electro-static field

We define **capacitance** as $$C = \frac{Q}{V}$$ **Unit ☛ Farad(F)**

$Q =$ Charge on one plate

$V =$ Potential difference between plates

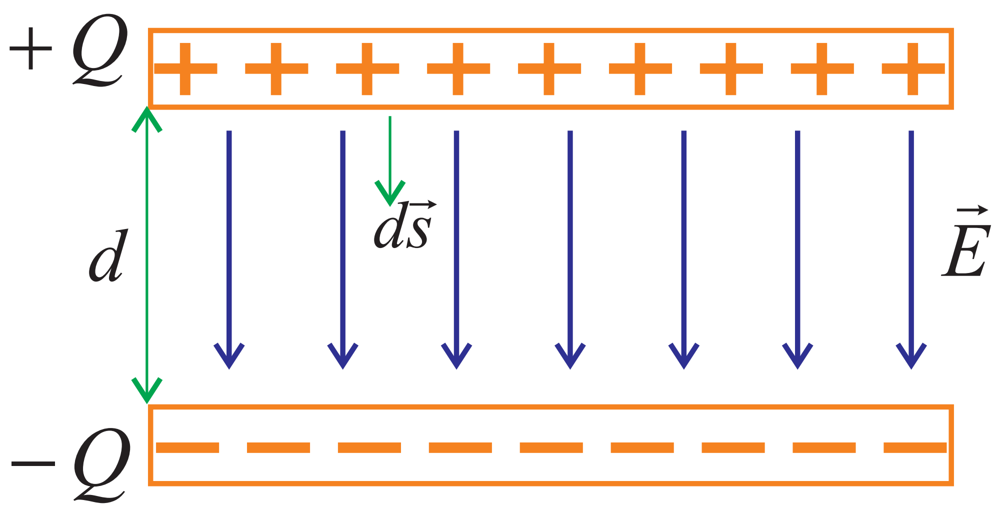
Note ☛

capacitor's C is a constant that depends only on its shape and material

i.e. If we increase V for a capacitor we increase Q stored

4.2 Calculating Capacitance

4.2.1 Parallel-Plate Capacitor



Area of conducting

① Recall ☛ $|\vec{E}| = \dfrac{\sigma}{\epsilon_0} = \dfrac{Q}{\epsilon_0 A}$

② Recall ☛ $\Delta V = V_{+} - V_{-} = -\int_{-}^{+} \vec{E} \cdot d\vec{s}$

this integral is independent of path taken

path is **parallel** to $\vec{E}$-field

$$\Delta V = -\int_{-}^{+} \vec{E} \cdot d\vec{s}$$

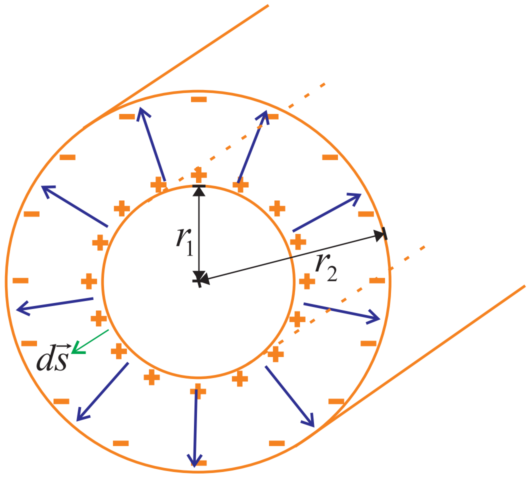
$$= \int_{+}^{-} \vec{E} \cdot d\vec{s}$$

$$= \frac{Q}{\epsilon_0 A} \underbrace{\int_{+}^{-} ds}_{\text{Length of path taken}}$$

$$= \frac{Q}{\epsilon_0 A} \cdot d$$

③ $\therefore \quad C = \frac{Q}{\Delta V} = \frac{\epsilon_0 A}{d}$

4.2.2 Cylindrical Capacitor



$\vec{E} \,//\, \hat{r}$

Consider two concentric cylindrical wires of inner and outer radii r_1 and r_2

Length of capacitor is L where $r_1 < r_2 \ll L$

① Using Gauss' law we determine that $\vec{E}$-field between conductors is

charge per unit length (λ)

$$\vec{E} = \frac{1}{2\pi\epsilon_0} \cdot \frac{\lambda}{r}\hat{r} = \frac{1}{2\pi\epsilon_0}\frac{Q}{Lr}\hat{r}$$

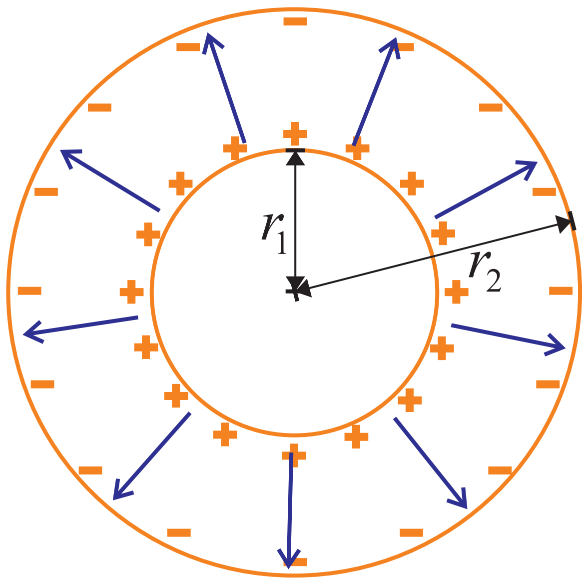
② $$\Delta V = \int_{+}^{-} \vec{E} \cdot d\vec{s}$$

Again ☛ we choose path of integration so that $d\vec{s} \parallel \hat{r} \parallel \vec{E}$

$$\therefore \quad \Delta V = \int_{r_1}^{r_2} E\,dr = \frac{Q}{2\pi\epsilon_0 L}\underbrace{\int_{r_1}^{r_2}\frac{dr}{r}}_{\ln\left(\frac{r_2}{r_1}\right)}$$

$$\therefore C = \frac{Q}{\Delta V} = 2\pi\epsilon_0 \frac{L}{\ln(r_2/r_1)}$$

4.2.3 Spherical Capacitor



$$\vec{E} \parallel \hat{r}$$

Choose $d\vec{s} \parallel \hat{r}$

For space between two conductors

$$E = \frac{1}{4\pi\epsilon_0} \cdot \frac{Q}{r^2} \qquad r_1 < r < r_2$$

$$\Delta V = \int_{+}^{-} \vec{E} \cdot d\vec{s}$$

$$= \int_{r_1}^{r_2} \frac{1}{4\pi\epsilon_0} \cdot \frac{Q}{r^2}\, dr$$

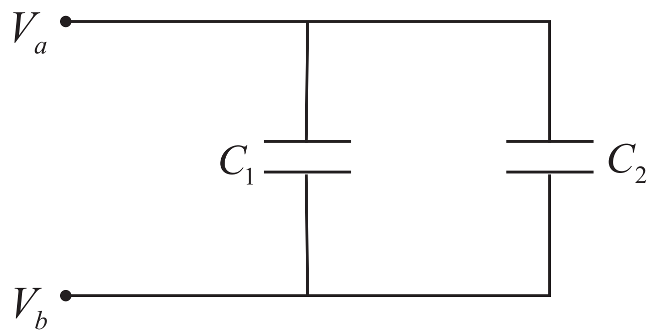
$$= \frac{Q}{4\pi\epsilon_0} \left[\frac{1}{r_1} - \frac{1}{r_2}\right]$$

$$C = 4\pi\epsilon_0 \left[\frac{r_1 r_2}{r_2 - r_1}\right]$$

4.3 Capacitors in Parallel and Series

(a) Capacitors in Parallel ☛ **potential difference** $V = V_a - V_b$

is same across capacitors



BUT ☛ **Charge** on each capacitor different

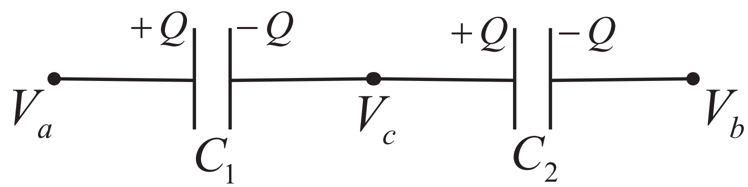
Total Charge $$Q = Q_1 + Q_2$$

$$= C_1 V + C_2 V$$

$$Q = \underbrace{(C_1 + C_2)}_{\text{Equivalent capacitance}} V$$

$\therefore$ For capacitors in parallel ☛ $C = C_1 + C_2$

(b) Capacitors in Series ☛ **charge across capacitors** are same



BUT ☛ **Potential difference** (P.D.) across capacitors different

$$\Delta V_1 = V_a - V_c = \frac{Q}{C_1}$$ P.D. across C_1

$$\Delta V_2 = V_c - V_b = \frac{Q}{C_2}$$ P.D. across C_2

$\therefore$ Potential difference

$$\Delta V = V_a - V_b$$

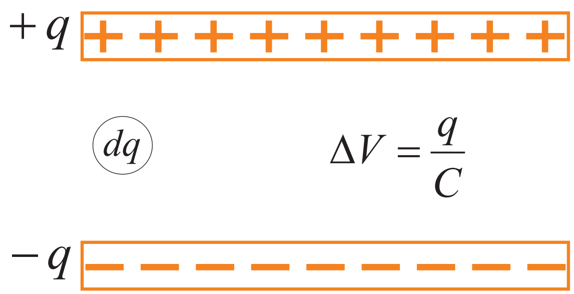
$$= \Delta V_1 + \Delta V_2$$

$$\Delta V = Q\left(\frac{1}{C_1} + \frac{1}{C_2}\right) = \frac{Q}{C}$$

C ☛ **Equivalent Capacitance**

$$\therefore \quad \frac{1}{C} = \frac{1}{C_1} + \frac{1}{C_2}$$

4.4 Energy Storage in Capacitor



While charging a capacitor

positive charge is being moved

from **negative** plate to **positive** plate

$\Rightarrow$ **NEEDS WORK DONE!**

Suppose we move charge dq from $-q$ to $+q$ plate

change in potential energy ☛ $dU = \Delta V \cdot dq = \dfrac{q}{C} dq$

Suppose we keep putting in a total charge Q to capacitor

total potential energy ☛ $U = \int dU = \int_0^Q \dfrac{q}{C} dq$

$$\therefore \ U = \frac{Q^2}{2C} = \frac{1}{2} C \Delta V^2 \qquad (\because Q = C\Delta V)$$

Energy stored in capacitor is stored in electric field between plates

Note: In parallel-plate capacitor ☛ $\vec{E}$**-field is constant between plates**

$$\text{density } u = \frac{\text{total energy stored}}{\text{total volume with } \vec{E}\text{−field}}$$

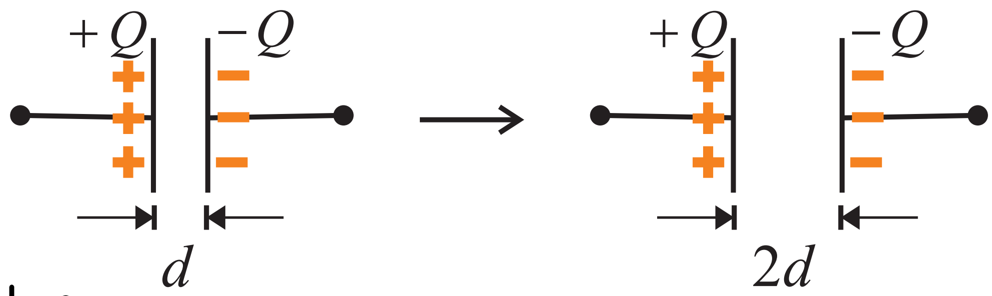
$$\therefore\ u = \underbrace{\frac{U}{Ad}}_{\text{Rectangular volume}}$$

$$\text{Recall ☛} \begin{cases} C = \dfrac{\epsilon_0 A}{d} \\[2ex] E = \dfrac{\Delta V}{d} \quad \Rightarrow \quad \Delta V = Ed \end{cases}$$

$$\therefore\ u = \frac{1}{2}\,(\overbrace{\frac{\epsilon_0 A}{d}}^{C}) \cdot (\overbrace{Ed}^{(\Delta V)^2})^2 \cdot \overbrace{\frac{1}{Ad}}^{\frac{1}{\text{Volume}}}$$

Energy per unit volume of electrostatic field ☛ $u = \frac{1}{2}\,\epsilon_0 E^2$

Example Changing capacitance by pulling plates apart



① Isolated Capacitor

Charge on capacitor plates remains **constant**

BUT ☛ $$C_{new} = \frac{\epsilon_0 A}{2d} = \frac{1}{2} C_{old}$$

$$\therefore \quad U_{new} = \frac{Q^2}{2C_{new}} = \frac{Q^2}{2C_{old}/2} = 2U_{old}$$

$\therefore$ In pulling plates apart work done $W > 0$

Summary ☛

$$Q \rightarrow Q \qquad C \rightarrow C/2$$

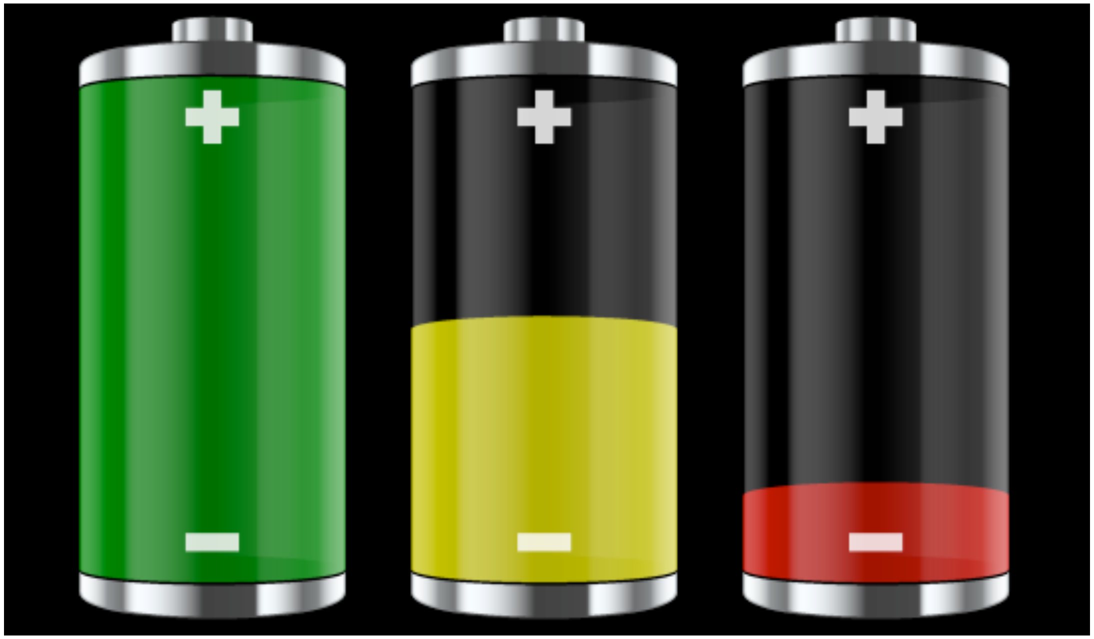
$$(V = \frac{Q}{C}) \Rightarrow V \rightarrow 2V \qquad E \rightarrow E \qquad (E = \frac{V}{d})$$

$$\frac{1}{2}\epsilon_0 E^2 = u \rightarrow u \qquad U \rightarrow 2U \qquad (U = u \cdot \text{volume})$$

Electric **battery** ☛ device consisting of 2 or more electrochemical cells that convert stored chemical energy into electrical energy

Each cell has:

- positive terminal (or cathode)
- negative terminal (or anode)



② Capacitor connected to a battery

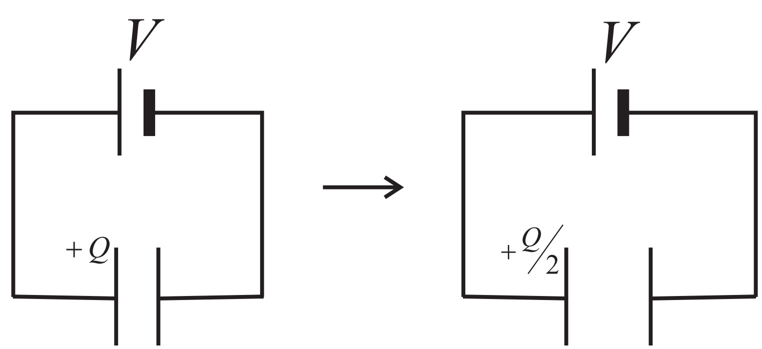
Potential difference between capacitor plates remains **constant**

$$U_{new} = \frac{1}{2} C_{new} \Delta V^2 = \frac{1}{2} \cdot \frac{1}{2} C_{old} \Delta V^2 = \frac{1}{2} U_{old}$$

$\therefore$ In pulling plates apart work done by battery < 0

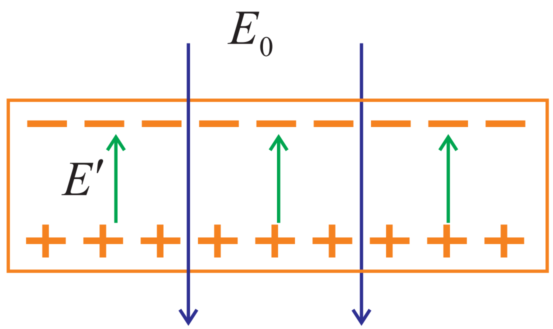
Summary ☛

Q	$\rightarrow$	$Q/2$	C	$\rightarrow$	$C/2$
V	$\rightarrow$	V	E	$\rightarrow$	$E/2$
u	$\rightarrow$	$u/4$	U	$\rightarrow$	$U/2$



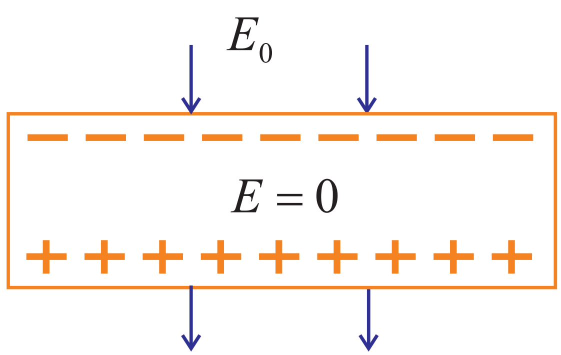
4.5 Dielectrics

Consider **conductor** being placed in an **external** E_0-field



In a conductor charges are free to move inside internal E'-field set up by these charges satisfies

$$E' = -E_0$$

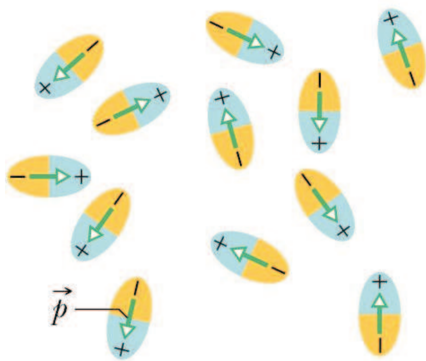


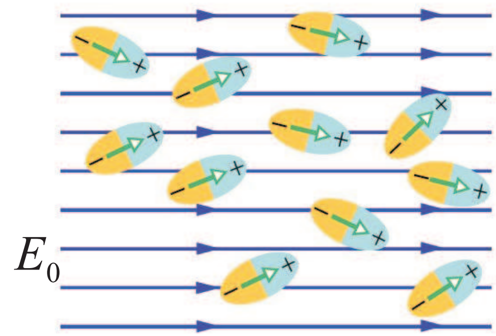
so that E-field inside conductor $= 0$

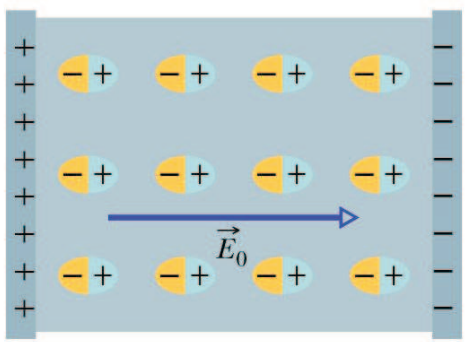
For **dielectric** ☛ atoms and molecules behave like **dipole** in $\vec{E}$-field

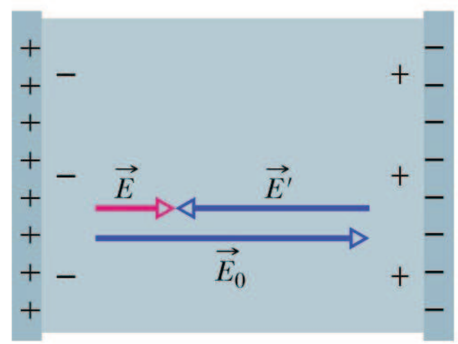
we can envision this so that in absence of $\vec{E}$-field

direction of dipole in dielectric are randomly distributed









Aligned dipoles will generate an induced E'-field satisfying $|E'| < |E_0|$

We can observe aligned dipoles in form of **induced surface charge**

Dielectric Constant

When a dielectric is placed in an external E_0-field

$\vec{E}$ -field inside a dielectric is **induced**

$$\vec{E} = \frac{1}{\kappa}\vec{E}_0$$

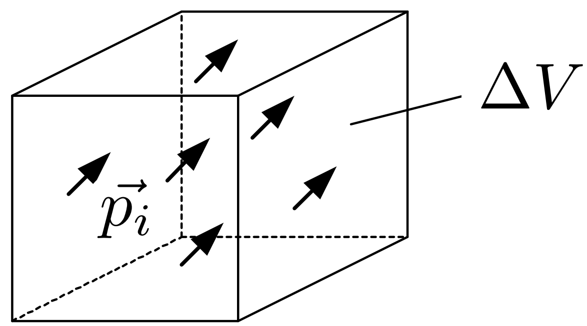
$\kappa \geq 1$ ☛ **dielectric constant**

Example

Vacuum	$\kappa = 1$
Porcelain	$\kappa = 6.5$
Water	$\kappa \sim 80$
Perfect conductor	$\kappa \to \infty$
Air	$\kappa = 1.00059$

Polarization Vector

Consider polarized volume with density of $\vec{p}\,'\mathrm{s}$ ☛



Polarization vector $\vec{P}$ defined as

$$\vec{P} = \lim_{\Delta V \to 0} \frac{\sum_{i=1}^{N} \vec{p}_i}{\Delta V} \qquad [\mathrm{C/m^2}]$$

N ☛ number of molecules in ΔV

Macroscopic effects of polarized dielectric material are modeled by $\vec{P}$ which really is average **dipole moment per unit volume** of material

Electric susceptibility and permittivity

It is customary in electromagnetism to **bury** effects of bound polarization in materials through electric flux displacement

Polarization effects of a dielectric can be accounted for by defining $\vec{D}$ as

$$\vec{D} = \epsilon_0 \vec{E} + \vec{P} \qquad [\mathrm{C/m^2}] \qquad ❶$$

What we desire now is to know $\vec{P}$ in terms of $\vec{E}$

Basically ☛ without knowing $\vec{P}$ this theory is not very useful

It has been found through experimentation that for many materials with **small** $\vec{E}$

$$\vec{P} = \epsilon_0 \chi_e \vec{E} \qquad ❷$$

χ_e ☛ **electric susceptibility** of material (dimensionless)

Substituting ❷ into ❶ gives

$$\vec{D} = \epsilon_0 \vec{E} + \epsilon_0 \chi_e \vec{E} = \epsilon_0 (1 + \chi_e) \vec{E}$$

We can rewrite this as

$$\vec{D} = \epsilon \vec{E} \qquad [\mathrm{C/m^2}]$$

Constant ϵ ☛ called **permittivity** of material

$$\epsilon = \kappa \epsilon_0 = (1 + \chi_e) \epsilon_0 \qquad [\mathrm{F/m}]$$

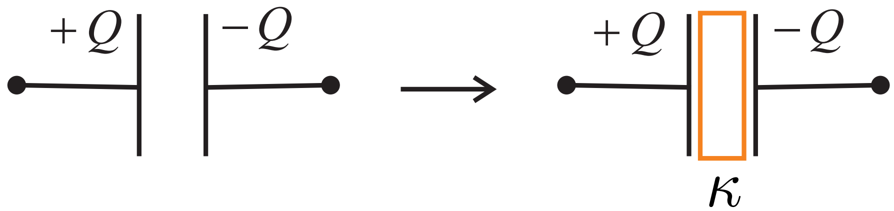
with

$$\kappa = 1 + \chi_e$$

κ ☛ called **relative permittivity** of material (dimensionless) **or dielectric constant**

4.6 Capacitor with Dielectric

Case I



Charge remains constant after dielectric is inserted

BUT ☛ $$E_{\text{new}} = \frac{1}{\kappa} E_{\text{old}}$$

$$\therefore \;\; \Delta V = Ed \Rightarrow \Delta V_{\text{new}} = \frac{1}{\kappa} \Delta V_{\text{old}}$$

$$\therefore \;\; C = \frac{Q}{\Delta V} \Rightarrow C_{\text{new}} = \kappa \, C_{\text{old}}$$

For a parallel-plate capacitor with dielectric

$$C = \frac{\kappa \;\epsilon_0 A}{d}$$

We can also write $C = \frac{\epsilon A}{d}$ in general with

$\epsilon = \kappa\,\epsilon_0$ ☛ **permittivity of dielectric**

Recall ϵ_0 ☛ **permittivity of vacuum**

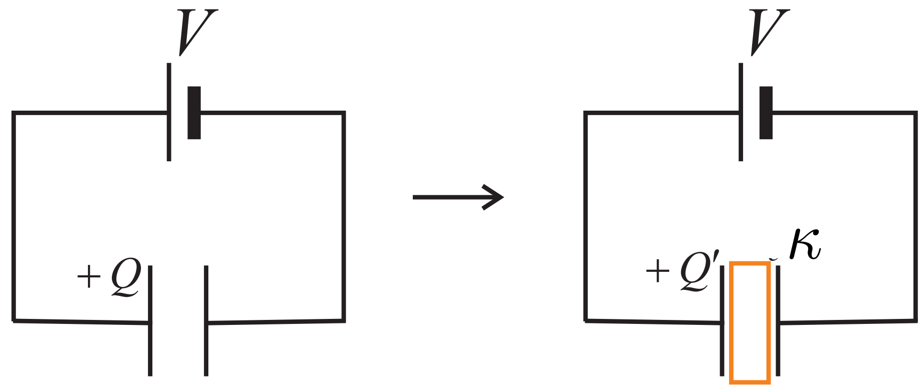
Energy stored

$$U = \frac{Q^2}{2C}$$

$$\therefore U_{\text{new}} = \frac{1}{\kappa} U_{\text{old}} < U_{\text{old}}$$

$\therefore$ Work done in inserting dielectric < 0

Case II Capacitor connected to battery



Voltage across capacitor plates **remains constant** after insertion of dielectric

$\vec{E}$-field inside capacitor remains constant

$(\because\ E = V/d)$

BUT ☛ How can E-field remain constant?

ANSWER ☛ By having extra charge on capacitor plates

Recall

For conductors

$$E = \frac{\sigma}{\epsilon_0}$$

$$\Rightarrow \quad E = \frac{Q}{\epsilon_0 A}$$ (σ charge per unit area $= Q/A$)

After insertion of dielectric

$$E' = \frac{Q'}{\kappa \epsilon_0 A}$$

But $\vec{E}$-field remains constant ☛ $E' = E \Rightarrow \dfrac{Q'}{\kappa\epsilon_0 A} = \dfrac{Q}{\epsilon_0 A}$

$$\Rightarrow Q' = \kappa Q > Q$$

Capacitor $\quad C = Q/V \quad \Rightarrow \quad C' \to \kappa C$

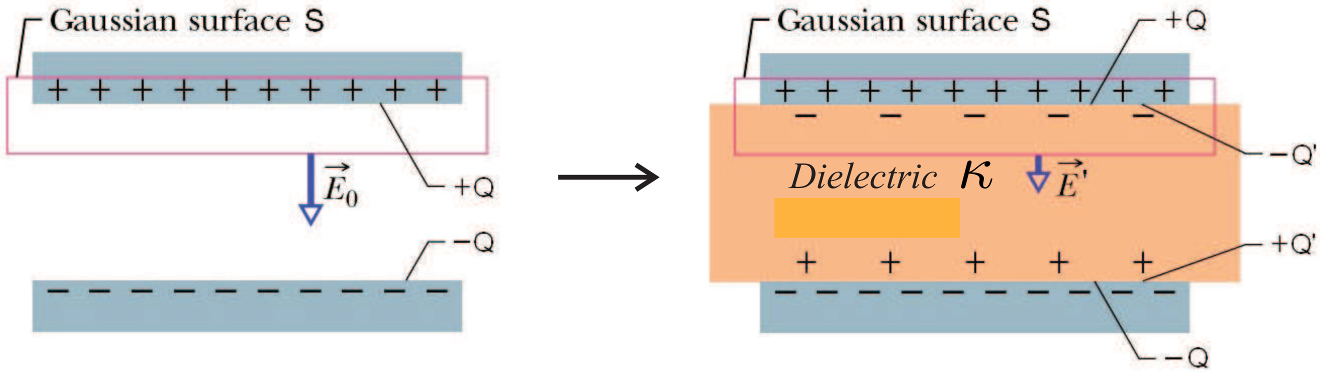
Energy stored $\quad U = \frac{1}{2}CV^2 \quad \Rightarrow \quad U' \to \kappa U$

$U_{\text{new}} > U_{\text{old}} \;\therefore$ **Work done to insert dielectric** > 0

4.7 Gauss' Law in Dielectric

Gauss' law we've learned is applicable in **vacuum only**

Let's use capacitor as an example to examine Gauss' law in dielectric



Free charge on plates $\pm Q$ $\pm Q$

Induced charge on dielectric 0 $\mp Q'$

Gauss' Law

$$\oint_S \vec{E} \cdot d\vec{A} = \frac{Q}{\epsilon_0}$$

$$\Rightarrow E_0 = \frac{Q}{\epsilon_0 A} \quad ①$$

Gauss' Law

$$\oint_S \vec{E}' \cdot d\vec{A} = \frac{Q - Q'}{\epsilon_0}$$

$$\therefore \quad E' = \frac{Q - Q'}{\epsilon_0 A} \quad ②$$

but $$E' = \frac{E_0}{\kappa} \quad ③$$

From ① ② ③

$$\frac{Q'}{\kappa\epsilon_0 A} = \frac{Q}{\epsilon_0 A} - \frac{Q'}{\epsilon_0 A}$$

$\therefore$ **Induced charge density** $\sigma' = \frac{Q'}{A} = \sigma\left(1 - \frac{1}{\kappa}\right) < \sigma$

where σ is free charge density

Recall Gauss' law in Dielectric

$$\epsilon_0 \oint_S \vec{E}' \cdot d\vec{A} = Q - Q'$$

- $\vec{E}'$: E-field in dielectric
- Q: free charge
- Q': induced charge, polarization charge

$$\oint_S (\epsilon_0 \vec{E} + \vec{P}) \cdot d\vec{S} = \oint_S \vec{D} \cdot d\vec{S} = Q$$

$$\Rightarrow \epsilon_0 \oint_S \vec{E}' \cdot d\vec{A} = Q - Q\left[1 - \frac{1}{\kappa}\right]$$

$$\Rightarrow \epsilon_0 \oint_S \vec{E}' \cdot d\vec{A} = \frac{Q}{\kappa}$$

$$\kappa \oint_S \vec{E}' \cdot d\vec{A} = \frac{Q}{\epsilon_0}$$ ☛ **Gauss' law in dielectric**

Note ☛

① This goes back to Gauss' law in vacuum with $E = \dfrac{E_0}{\kappa}$ for dielectric

② Only **free charges** need to be considered
even for dielectric where there are **induced charges**

③ Another way to write

$$\oint_S \vec{E} \cdot d\vec{A} = \frac{Q}{\epsilon}$$

Energy stored with dielectric

Total energy stored $$U = \frac{1}{2}CV^2$$

With dielectric recall

$$C = \frac{\kappa\epsilon_0 A}{d}$$

$$V = Ed$$

$\therefore$ Energy stored per unit volume ☛

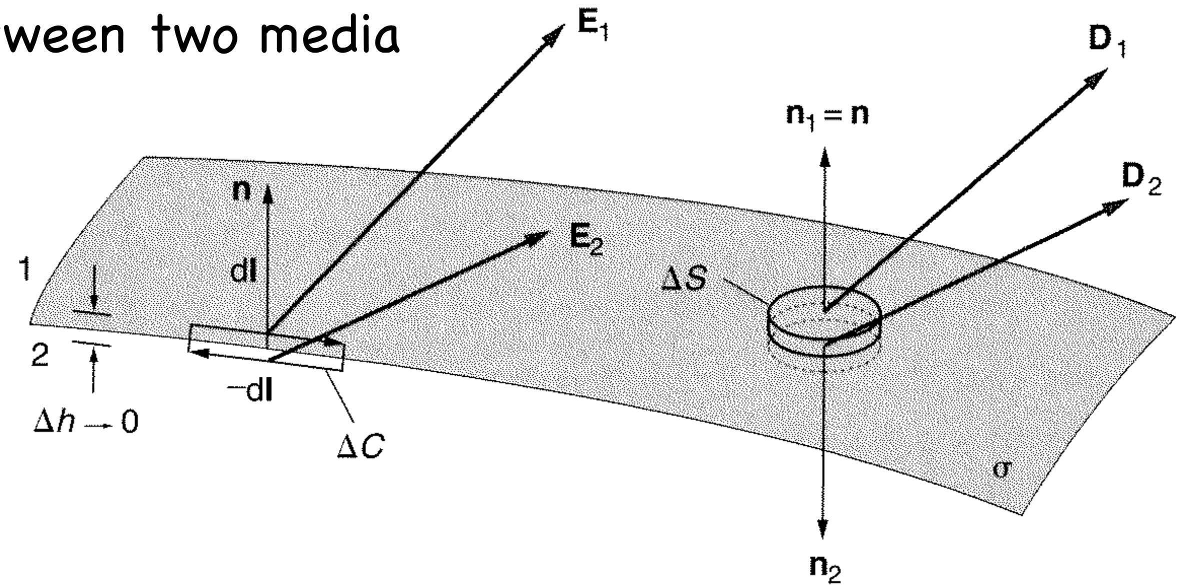
$$u = \frac{U}{Ad} = \frac{1}{2}\kappa\epsilon_0 E^2$$

and so ☛ $$u_{\mathrm{dielectric}} = \kappa\, u_{\mathrm{vacuum}}$$

$\therefore$ More energy is stored per unit volume in dielectric than in vacuum

4.7 Electrostatic Boundary Conditions

Boundary between two media



A narrow rectangular contour is used in law of conservation of energy and a coinlike closed surface is used in Gauss law for deriving boundary conditions for vectors $\vec{E}$ and $\vec{D}$ respectively

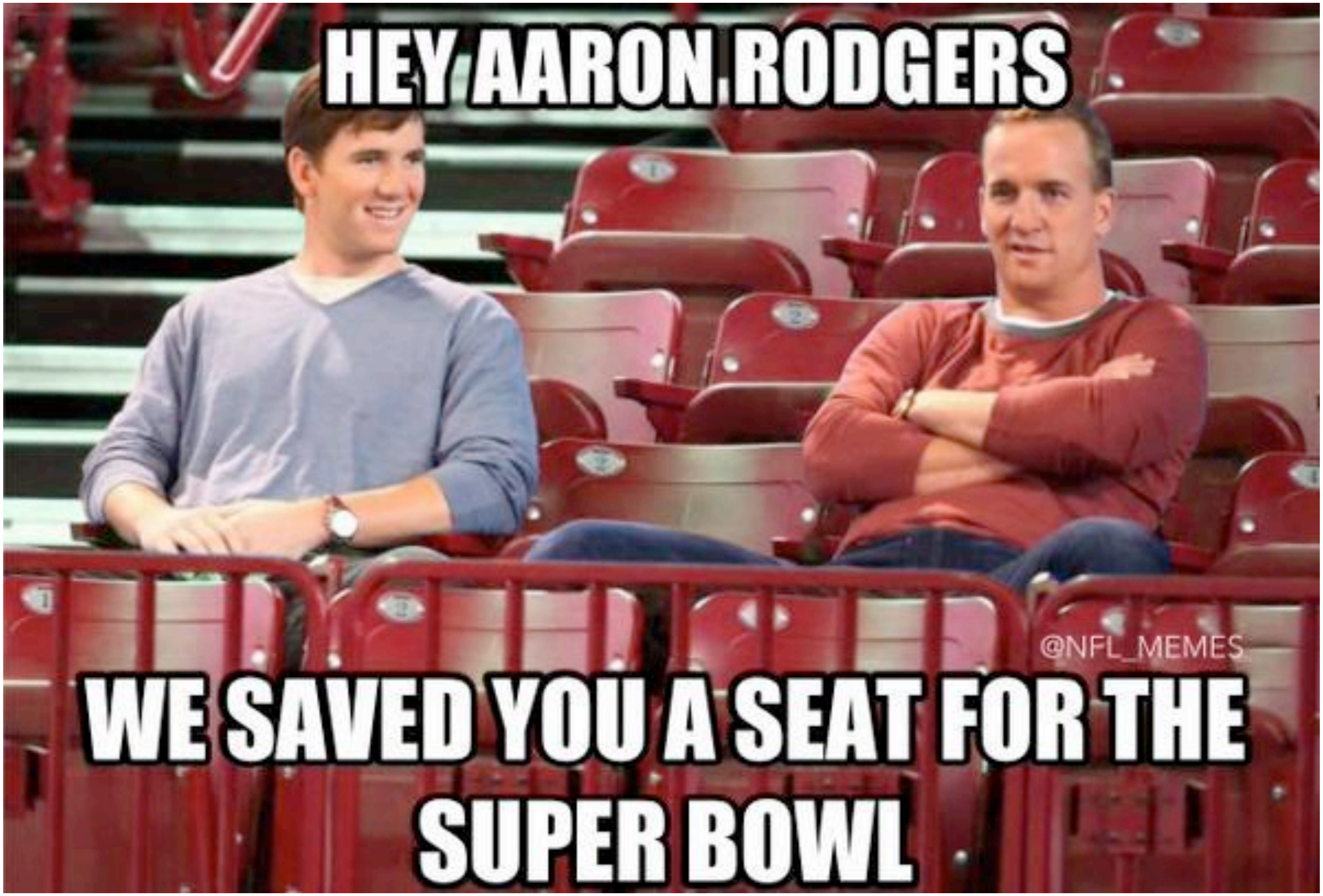
Boundary condition for tangential components of vector $\vec{E}$

$$E_{1t} = E_{2t}$$

Boundary condition for normal components of vector $\vec{D}$

(unit vector normal directed into medium 1)

$$\vec{D}_1 \cdot \hat{n} - \vec{D}_2 \cdot \hat{n} = \sigma \Rightarrow D_{1n} - D_{2n} = \sigma$$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