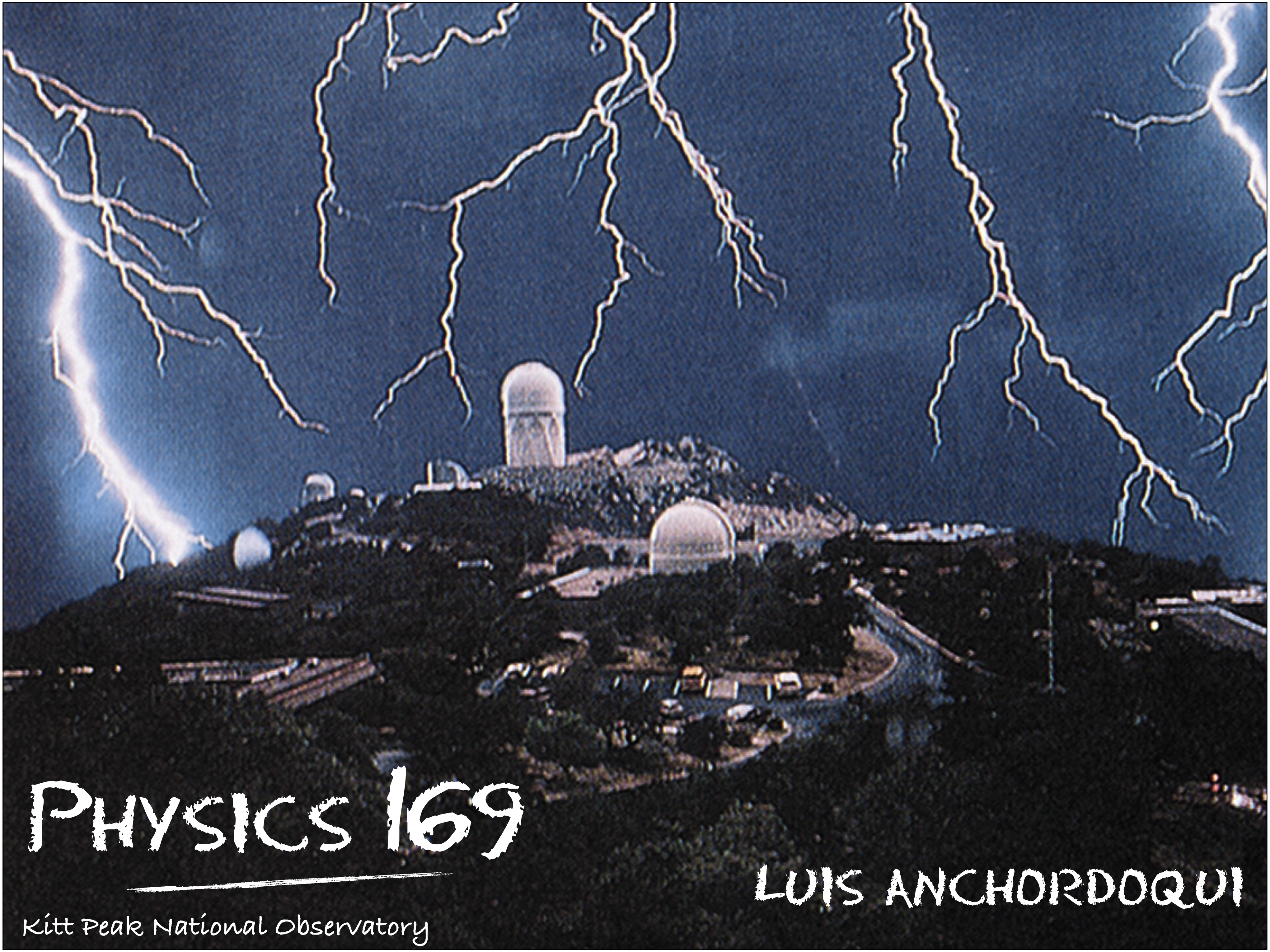
PHYSICS 169
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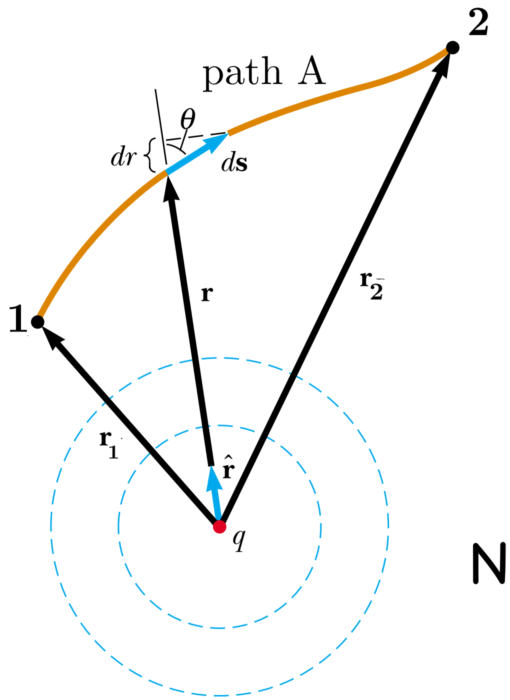
3.1 Potential Energy and Conservative Forces

Electric force is a **conservative force**



Work done by electric force $\vec{F}$ as charge moves infinitesimal distance $d\vec{s}$ along $\text{Path}\, A \;=\; dW$

Note ☛ $d\vec{s}$ is in tangent direction of curve of $\text{Path}\, A$

$$dW \;=\; \vec{F} \,\cdot\, d\vec{s}$$

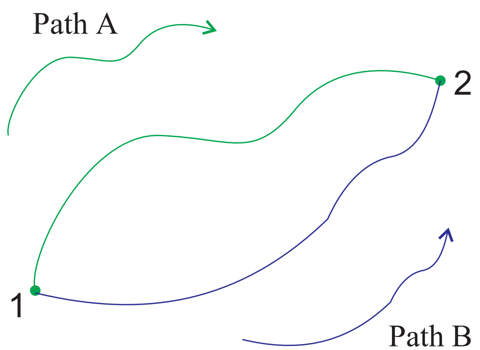
$\therefore$ Total work done W by force $\vec{F}$ in moving particle from Point 1 to Point 2

$$W \;=\; \underset{\text{Path}\, A}{\int_1^2} \vec{F} \,\cdot\, d\vec{s}$$

$$\underset{\text{Path}\, A}{\int_1^2} \;=\; \text{Path Integral} = \text{Integration over Path A from point 1 to point 2}$$

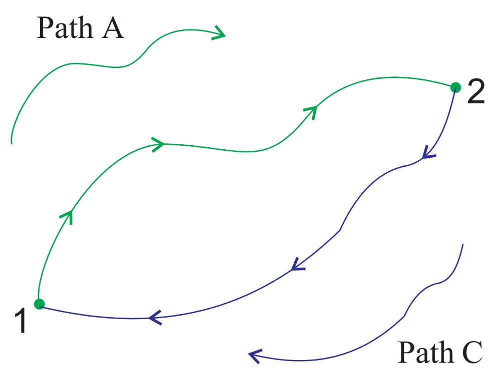
DEFINITION

A force is **conservative** if work done on a particle by force is **independent of path taken**



$\therefore$ For conservative forces

$$\underset{\text{Path A}}{\int_1^2 \vec{F}\cdot d\vec{s}} = \underset{\text{Path B}}{\int_1^2 \vec{F}\cdot d\vec{s}}$$



Let's consider a path starting at point 1 to 2 through Path A and from 2 to 1 through Path C

$$\text{Work done} = \underset{\text{Path A}}{\int_1^2 \vec{F}\cdot d\vec{s}} + \underset{\text{Path C}}{\int_2^1 \vec{F}\cdot d\vec{s}}$$

$$= \underset{\text{Path A}}{\int_1^2 \vec{F}\cdot d\vec{s}} - \underset{\text{Path B}}{\int_1^2 \vec{F}\cdot d\vec{s}}$$

Alternative
DEFINITION

Work done by a **conservative force** on a particle **when it moves around a closed path returning to its initial position** is **zero**

Conclusion

Since work done by a conservative force $\vec{F}$ is **path-independent**
we can define a quantity: **potential energy**
that depends only on **position** of particle

Convention We define potential energy U such that

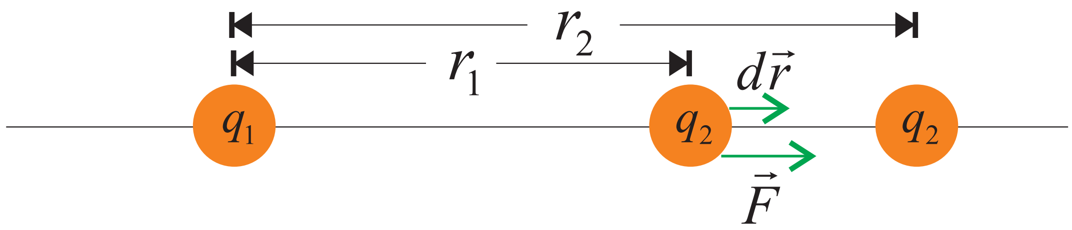
$$dU = -W = -\int \vec{F} \cdot d\vec{s}$$

$\therefore$ For particle moving from 1 to 2

$$\int_1^2 dU = U_2 - U_1 = -\int_1^2 \vec{F} \cdot d\vec{s}$$

where U_1, U_2 are potential energy at position 1, 2

Example



Suppose charge q_2 moves from point 1 to 2

From definition ☛ $$U_2 - U_1 = -\int_1^2 \vec{F}\cdot d\vec{r}$$

$$= -\int_{r_1}^{r_2} F\,dr \qquad (\because\ \vec{F} \parallel d\vec{r})$$

$$= -\int_{r_1}^{r_2} \frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{r^2}\,dr$$

$$= \frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{r}\Bigg|_{r_1}^{r_2}$$

$$-\Delta W = \Delta U = \frac{1}{4\pi\epsilon_0} q_1 q_2 \left(\frac{1}{r_2} - \frac{1}{r_1}\right)$$

Note

① This result is generally true for 2-D and/or 3-D motion

② If q_2 moves away from q_1 then $r_2 > r_1$ we have

- If q_1, q_2 are of **same** sign then $\Delta U < 0, \quad \Delta W > 0$

  ($\Delta W =$ Work done by electric **repulsive** force)

- If q_1, q_2 are of **different** sign then $\Delta U > 0, \quad \Delta W < 0$

  ($\Delta W =$ Work done by electric **attractive** force)

③ If q_2 moves towards q_1 then $r_2 < r_1$ we have

- If q_1, q_2 are of **same** sign then $\Delta U > 0, \quad \Delta W < 0$

- If q_1, q_2 are of **different** sign then $\Delta U < 0, \quad \Delta W > 0$

④ Note: It is the **difference** in potential energy that is important

REFERENCE POINT $U(r = \infty) = 0$

$$\therefore U_\infty - U_1 = \frac{1}{4\pi\epsilon_0} q_1 q_2 \Big(\underset{\downarrow \atop \infty}{\frac{1}{r_2}} - \frac{1}{r_1} \Big)$$

$$U(r) = \frac{1}{4\pi\epsilon_0} \cdot \frac{q_1 q_2}{r}$$

If q_1, q_2 **same** sign then $U(r) > 0$ for all r

If q_1, q_2 **opposite** sign then $U(r) < 0$for all r

⑤ Conservation of Mechanical Energy

For a system of charges with no external force,

$$E = \underbrace{K}_{\textbf{Kinetic Energy}} + \underbrace{U}_{\textbf{Potential Energy}} = \textbf{Constant}$$

or $\Delta E = \Delta K + \Delta U = 0$

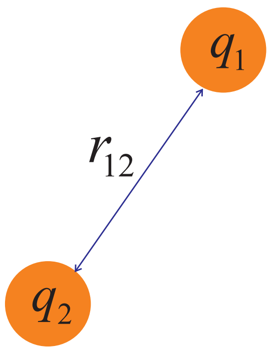
Potential Energy of a System of Charges

Example P.E. of 3 charges q_1, q_2, q_3

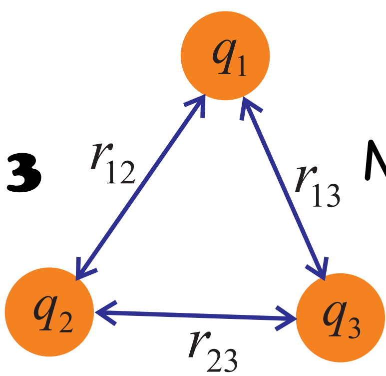
Start q_1, q_2, q_3 all at $r = \infty$, $U = 0$

Step 1 

Move q_1 from ∞ to its position $\Rightarrow U = 0$

Step 2 

Move q_2 from ∞ to new position $\Rightarrow U = \dfrac{1}{4\pi\epsilon_0}\dfrac{q_1 q_2}{r_{12}}$

Step 3 

Move q_3 from ∞ to new position $\Rightarrow$ **Total P.E**

$$U = \frac{1}{4\pi\epsilon_0}\left[\frac{q_1 q_2}{r_{12}} + \frac{q_1 q_3}{r_{13}} + \frac{q_2 q_3}{r_{23}}\right]$$

Step 4 What if there are 4 charges?

3.2 Electric Potential

Let q be charge at the center and consider its effect on test charge q_0

DEFINITION We define electric potential V so that

$$\Delta V = \frac{\Delta U}{q_0} = \frac{-\Delta W}{q_0}$$

($\therefore V$ is P.E. per unit charge)

❍ Similarly ☛ we take $V(r = \infty) = 0$

❍ Electric Potential is a **scalar**

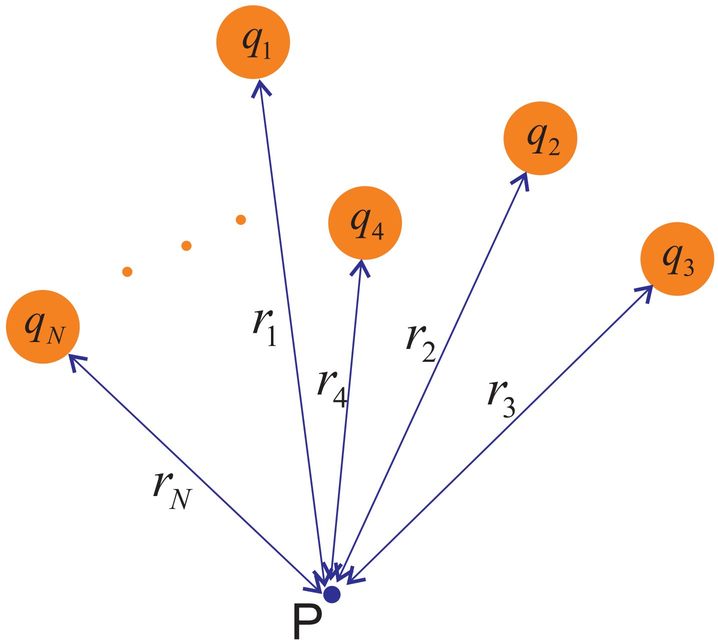
❍ Unit ☛ $\mathrm{Volt}(V) =$ **Joules/Coulomb**

❍ For a single point charge $V(r) = \frac{1}{4\pi\epsilon_0} \cdot \frac{q}{r}$

❍ Energy Unit ☛ $\Delta U = q\Delta V$

$$\mathrm{elecrton-volt\ (eV)} = \underbrace{1.6 \times 10^{-19}\ \mathrm{J}}_{\text{charge of electron/C}}$$

Potential For A System of Charges



For a total of N point charges potential V at any point P can be derived from

principle of superposition

Recall that potential due to q_1 at point P

$$V_1 = \frac{1}{4\pi\epsilon_0} \cdot \frac{q_1}{r_1}$$

$\therefore$ Total potential at point P due to N charges

principle of superposition

$$V = V_1 + V_2 + \cdots + V_N = \frac{1}{4\pi\epsilon_0}\left[\frac{q_1}{r_1} + \frac{q_2}{r_2} + \cdots \frac{q_N}{r_N}\right]$$

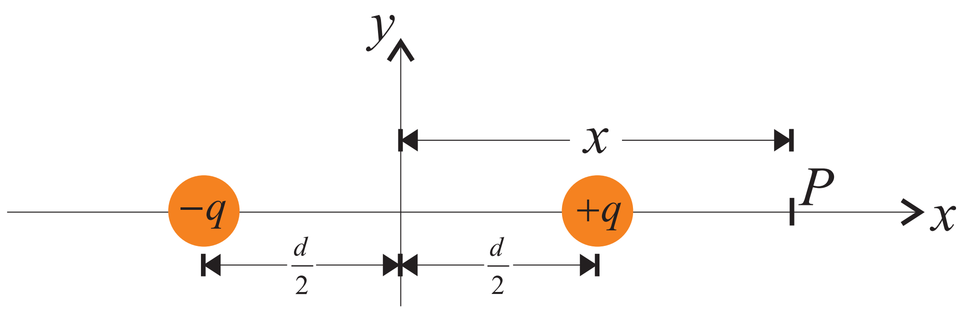
$$V = \frac{1}{4\pi\epsilon_0} \sum_{i=1}^{N} \frac{q_i}{r_i}$$

Note For $\vec{E}, \vec{F}$ we have a sum of vectors

For V, U we have a sum of scalars

Example

Potential of an electric dipole



$$\vec{p} = q\vec{d}$$

Consider potential of point at distance $x > \frac{d}{2}$ from dipole P

$$V = \frac{1}{4\pi\epsilon_0}\left[\frac{+q}{x - \frac{d}{2}} + \frac{-q}{x + \frac{d}{2}}\right]$$

Special Limiting Case ☛ $x \gg d$

$$\frac{1}{x \mp \frac{d}{2}} = \frac{1}{x} \cdot \frac{1}{1 \mp \frac{d}{2x}} \simeq \frac{1}{x}\left[1 \pm \frac{d}{2x}\right]$$

$$\therefore V = \frac{1}{4\pi\epsilon_0} \cdot \frac{q}{x}\left[1 + \frac{d}{2x} - (1 - \frac{d}{2x})\right]$$

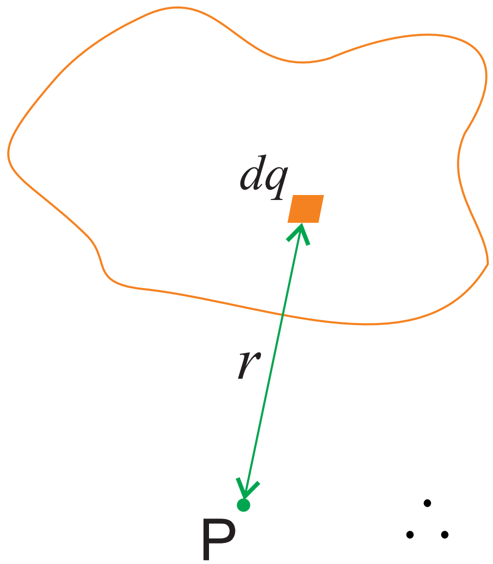
$$V = \frac{p}{4\pi\epsilon_0 x^2}$$ **Recall** ☛ $p = qd$

For a point charge $E \propto \frac{1}{r^2}$ $V \propto \frac{1}{r}$

For a dipole $E \propto \frac{1}{r^3}$ $V \propto \frac{1}{r^2}$

For a quadrupole $E \propto \frac{1}{r^4}$ $V \propto \frac{1}{r^3}$

Electric Potential of Continuous Charge Distribution



For any charge distribution
we write electrical potential dV
due to infinitesimal charge dq

$$dV = \frac{1}{4\pi\epsilon_0} \cdot \frac{dq}{r}$$

$$\therefore \quad V = \int_{\text{charge distribution}} \frac{1}{4\pi\epsilon_0} \cdot \frac{dq}{r}$$

Similar to previous examples on E-field for case of uniform charge distribution

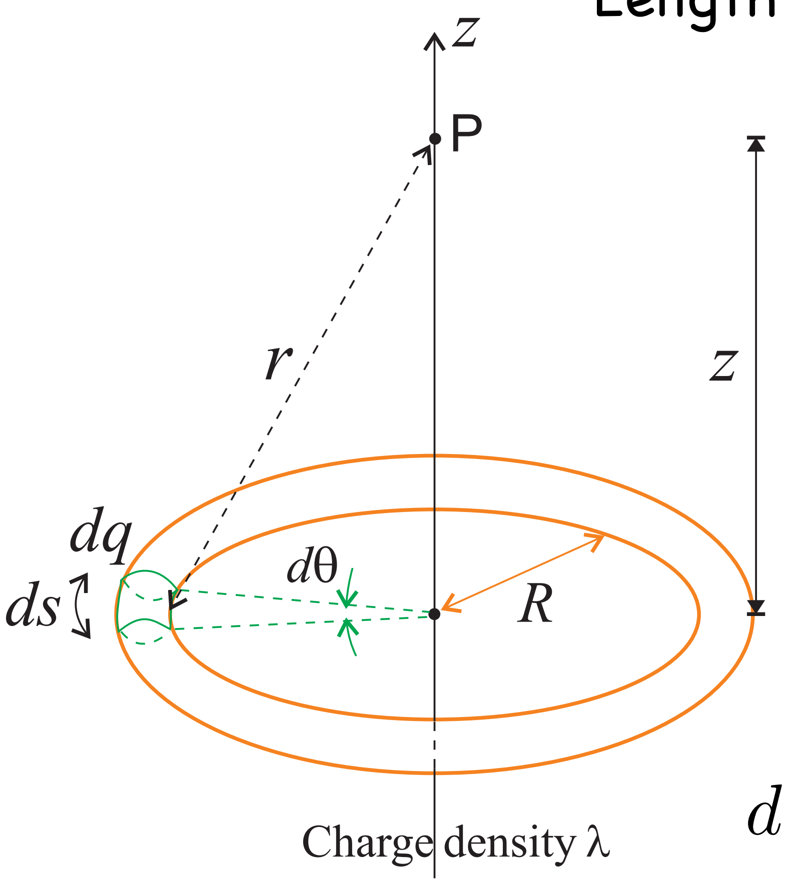
1-D $\Rightarrow$ long road $\quad \Rightarrow dq = \lambda\, dx$

2-D $\Rightarrow$ charge sheet $\quad \Rightarrow dq = \sigma\, dA$

3-D $\Rightarrow$ uniformly charge body $\quad \Rightarrow dq = \rho\, dV$

Example (1)

Uniformly-charged ring



Length of infinitesimal ring element $= ds = R d\theta$

$\therefore$ charge $dq = \lambda\, ds$

$$= \lambda R\, d\theta$$

$$dV = \frac{1}{4\pi\epsilon_0} \cdot \frac{dq}{r} = \frac{1}{4\pi\epsilon_0} \cdot \frac{\lambda R\, d\theta}{\sqrt{R^2 + z^2}}$$

Integration is around entire ring

$$\therefore \quad V = \int_{\text{ring}} dV$$

$$= \int_0^{2\pi} \frac{1}{4\pi\epsilon_0} \cdot \frac{\lambda R\, d\theta}{\sqrt{R^2 + z^2}}$$

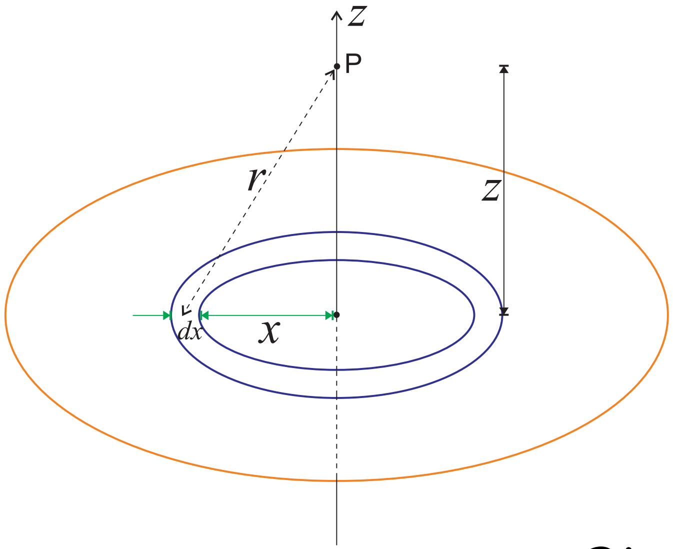
$$= \frac{\lambda R}{4\pi\epsilon_0 \sqrt{R^2 + z^2}} \underbrace{\int_0^{2\pi} d\theta}_{2\pi}$$

$$V = \frac{Q}{4\pi\epsilon_0 \sqrt{R^2 + z^2}}$$

Total charge
on ring $= \lambda \cdot (2\pi R)$

LIMITING CASE ☛ $z \gg R \Rightarrow V = \dfrac{Q}{4\pi\epsilon_0 \sqrt{z^2}} = \dfrac{Q}{4\pi\epsilon_0 |z|}$

Example (2) Uniformly-charged disk



Total charge = Q

Charge density = σ

Using **principle of superposition**

we will find potential of disk

of uniform charge density

by integrating potential of concentric rings

$$\therefore \quad dV = \frac{1}{4\pi\epsilon_0}\int_{\text{disk}} \frac{dq}{r}$$

Ring of radius x ☛ $dq = \sigma\, dA = \sigma(2\pi x dx)$

$$\therefore \quad V = \int_0^R \frac{1}{4\pi\epsilon_0}\cdot\frac{\sigma 2\pi x dx}{\sqrt{x^2+z^2}}$$

$$= \frac{\sigma}{4\epsilon_0}\int_0^R \frac{d(x^2+z^2)}{(x^2+z^2)^{1/2}}$$

$$V = \frac{\sigma}{2\epsilon_0}\left(\sqrt{z^2+R^2} - \sqrt{z^2}\right)$$

$$= \frac{\sigma}{2\epsilon_0}\left(\sqrt{z^2+R^2} - |z|\right)$$

Recall $|x| = \begin{cases} +x; & x \geq 0 \\ -x; & x < 0 \end{cases}$

LIMITING CASE ☛

① If $|z| \gg R$

$$\sqrt{z^2 + R^2} = \sqrt{z^2(1 + \frac{R^2}{z^2})}$$

$$= |z| \cdot (1 + \frac{R^2}{z^2})^{\frac{1}{2}} \qquad ((1+x)^n \approx 1 + nx \text{ if } x \ll 1)$$

$$\simeq |z| \cdot (1 + \frac{R^2}{2z^2}) \qquad (\frac{|z|}{z^2} = \frac{1}{|z|})$$

$\therefore$ At large z, $V \simeq \dfrac{\sigma}{2\epsilon_0} \cdot \dfrac{R^2}{2|z|} = \dfrac{Q}{4\pi\epsilon_0 |z|}$ (like a point charge)

where $Q = \text{total charge on disk} = \sigma \cdot \pi R^2$

② If $|z| \ll R$

$$\sqrt{z^2 + R^2} = R \cdot \left(1 + \frac{z^2}{R^2}\right)^{\frac{1}{2}}$$

$$\simeq R\left(1 + \frac{z^2}{2R^2}\right)$$

$$\therefore \quad V \simeq \frac{\sigma}{2\epsilon_0}\left[R - |z| + \frac{z^2}{2R}\right]$$

At $z = 0$, $V = \dfrac{\sigma R}{2\epsilon_0}$ Let's call this V_0

$$\therefore \quad V(z) = \frac{\sigma R}{2\epsilon_0}\left[1 - \frac{|z|}{R} + \frac{z^2}{2R^2}\right]$$

$$V(z) = V_0\left[1 - \frac{|z|}{R} + \frac{z^2}{2R^2}\right]$$

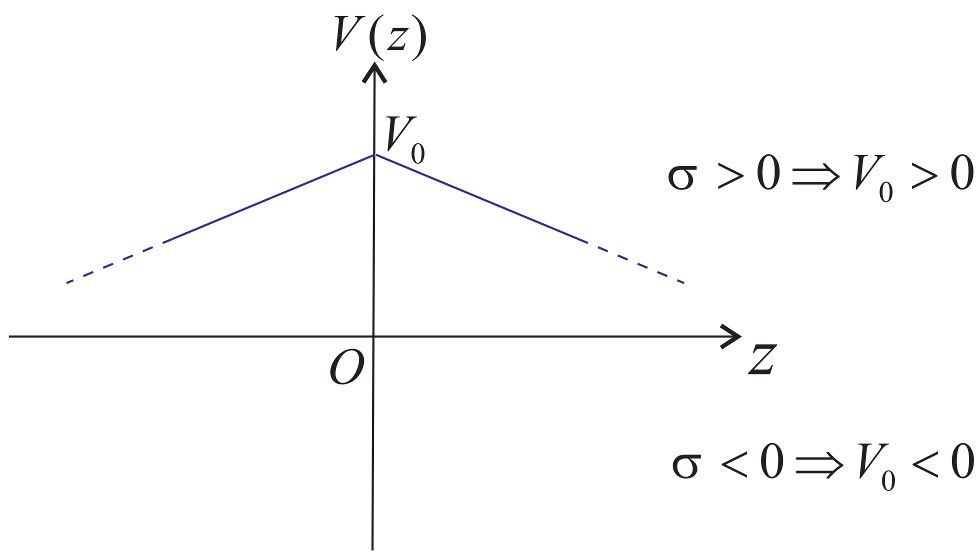
$\Rightarrow$ A convenience reference point to compare in this example is potential of charged disk

$\therefore$ Important quantity here is

$$V(z) - V_0 = -\frac{|z|}{R} V_0 + \cancel{\frac{z^2}{2R^2}} V_0$$

neglected as $z \ll R$

$$V(z) - V_0 = -\frac{V_0}{R} |z|$$



3.3 Relation Between Electric Field E and Electric Potential V

(A) To get V from $\vec{E}$

Recall our definition of potential V

$$\Delta V = \frac{\Delta U}{q_0} = -\frac{W_{12}}{q_0}$$

ΔU ☛ is change in P.E.

ΔW ☛ work done in bringing charge q_0 from point 1 to 2

$$\therefore \quad \Delta V = V_2 - V_1 = \frac{-\int_1^2 \vec{F} \cdot d\vec{s}}{q_0}$$

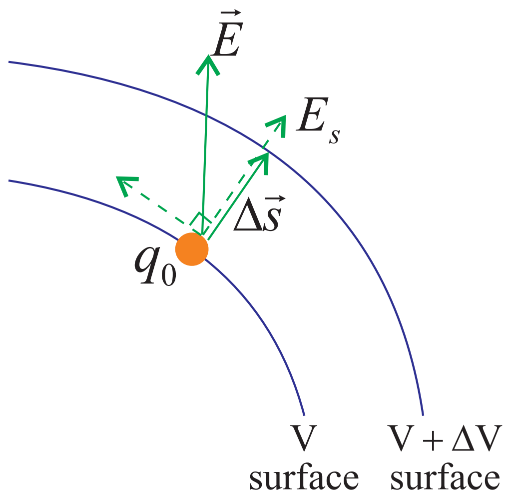
Using definition of $\vec{E}$-field

$$\therefore \quad \Delta V = V_2 - V_1 = -\int_1^2 \vec{E} \cdot d\vec{s}$$

Note ☛ Integral on right hand side of above can be calculated **along any path from point 1 to 2 (Path-Independent)**

Convention $V_\infty = 0 \quad \Rightarrow \quad V_P = -\int_\infty^P \vec{E} \cdot d\vec{s}$

(B) To get $\vec{E}$ from V ☛ use definition of V



(i.e. Potential = V on the surface)

$$\Delta U = q_0 \Delta V = \underbrace{-\Delta W}_{\text{work done}}$$

$$\Delta W = \underbrace{q_0 \vec{E}}_{\text{electric force}} \cdot \Delta\vec{s}$$

E_s ☛ $\vec{E}$ -field component along path $\Delta\vec{s}$

$$\therefore \quad q_0 \Delta V = -q_0 \, E_s \, \Delta s$$

$$\therefore \quad E_s = -\frac{\Delta V}{\Delta s}$$

For infinitesimal Δs

$$\therefore \quad E_s = -\frac{dV}{ds}$$

All in all...

(1) $\vec{E}$-field component along **any direction** is negative derivative of potential **along same direction**

(2) If $d\vec{s} \perp \vec{E}$ then $\Delta V = 0$

(3) ΔV is biggest/smallest if $d\vec{s} \parallel \vec{E}$

Generally for a potential $V(x, y, z)$ relation between $\vec{E}(x, y, z)$ and V is

$$E_x = -\frac{\partial V}{\partial x} \qquad E_y = -\frac{\partial V}{\partial y} \qquad E_z = -\frac{\partial V}{\partial z}$$

$\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}$ are **partial derivatives**

For $\frac{\partial}{\partial x} V(x, y, z)$ everything y, z are treated like a **constant** and we only take derivative with respect to x

Example If $V(x,y,z) = x^2y - z$

$$\frac{\partial V}{\partial x} = 2xy$$

$$\frac{\partial V}{\partial y} = x^2$$

$$\frac{\partial V}{\partial z} = -1$$

For other co-ordinate systems

① Cylindrical:

$$V(r,\theta,z) \begin{cases} E_r = -\dfrac{\partial V}{\partial r} \\ E_\theta = -\dfrac{1}{r}\cdot\dfrac{\partial V}{\partial \theta} \\ E_z = -\dfrac{\partial V}{\partial z} \end{cases}$$

② Spherical:

$$V(r,\theta,\phi) \quad \begin{cases} E_r = -\dfrac{\partial V}{\partial r} \\[2ex] E_\theta = -\dfrac{1}{r} \cdot -\dfrac{\partial V}{\partial \theta} \\[2ex] E_\phi = -\dfrac{1}{r\sin\theta} \cdot \dfrac{\partial V}{\partial \phi} \end{cases}$$

Note ☛
Calculating V involves summation of **scalars**
which is easier than adding **vectors** for calculating E-field

$\therefore$ To find $\vec{E}$-field of a general charge system

we first calculate V and then derive $\vec{E}$ from partial derivative

Example Uniformly charged disk From potential calculations

From potential calculations

$$V = \frac{\sigma}{2\epsilon_0}\left(\sqrt{R^2 + z^2} - |z|\right)$$ for a point along z-axis

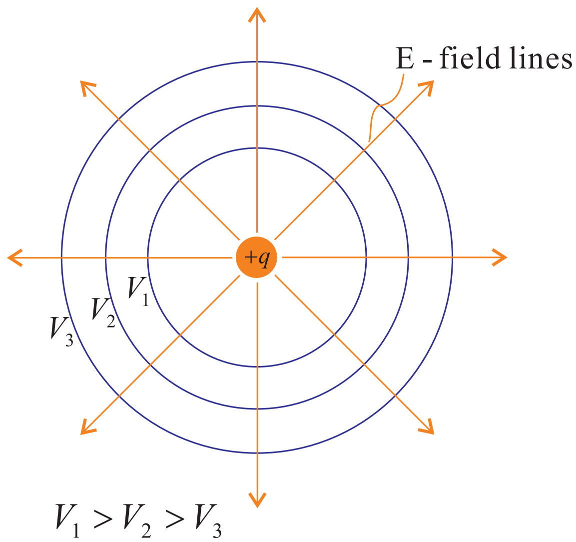
For $z > 0, \quad |z| = z$

$$\therefore \; E_z = -\frac{\partial V}{\partial z} = \frac{\sigma}{2\epsilon_0}\left[1 - \frac{z}{\sqrt{R^2 + z^2}}\right]$$

3.4 Equipotential Surfaces

Equipotential surface is a surface on which **potential** is constant

$$\Rightarrow \quad (\Delta V = 0)$$



$$V(r) = \frac{1}{4\pi\epsilon_0} \cdot \frac{+q}{r} = const$$

$$\Rightarrow \quad r = const$$

$\Rightarrow$ Equipotential surface are $circles\,/\,spherical\,surface$

Note ☛

① A charge can move freely on an equipotential surface without any work done

② **Electric field lines** must be perpendicular to **equipotential surfaces**

(Why?)

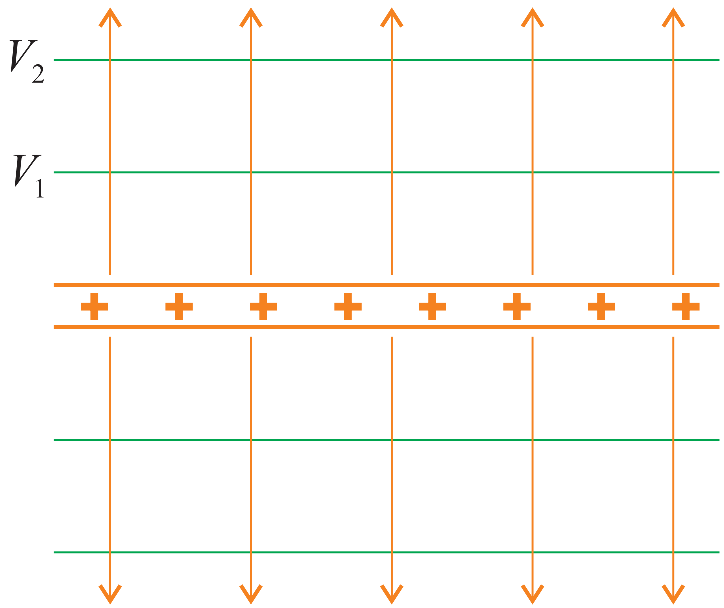
On an equipotential surface $V = constant$

$\Rightarrow \Delta V = 0 \Rightarrow \vec{E} \cdot d\vec{l} = 0$ where $d\vec{l}$ is **tangent** to equipotential surface

$\therefore$ $\vec{E}$ must be **perpendicular** to equipotential surfaces

Example

Uniformly charged surface (infinite)



$V_1 > V_2$

Recall

$$V = V_0 - \frac{\sigma}{2\epsilon_0}|z|$$

Potential at $z = 0$

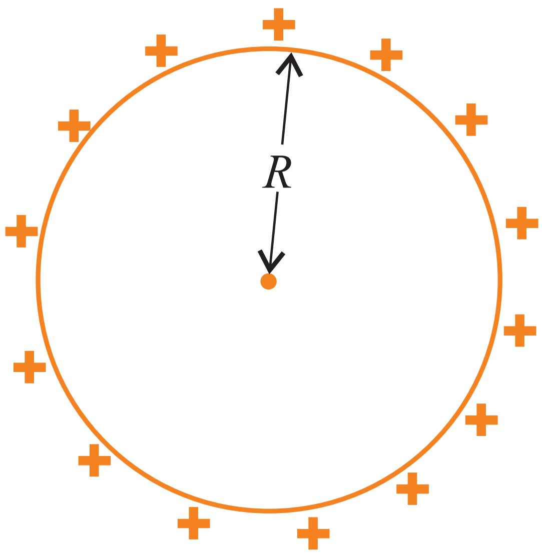
Equipotential surface means

$$V = const \Rightarrow V_0 - \frac{\sigma}{2\epsilon_0}|z| = C$$

$$\Rightarrow |z| = constant$$

Example

Isolated spherical charged conductors



Recall

① E-field inside $= 0$

② charge distributed on **outside** of conductors

(i) Inside conductor

$E = 0 \Rightarrow \Delta V = 0$ everywhere in conductor

$\Rightarrow V = constant$ everywhere in conductor

$\Rightarrow$ entire conductor is at same potential

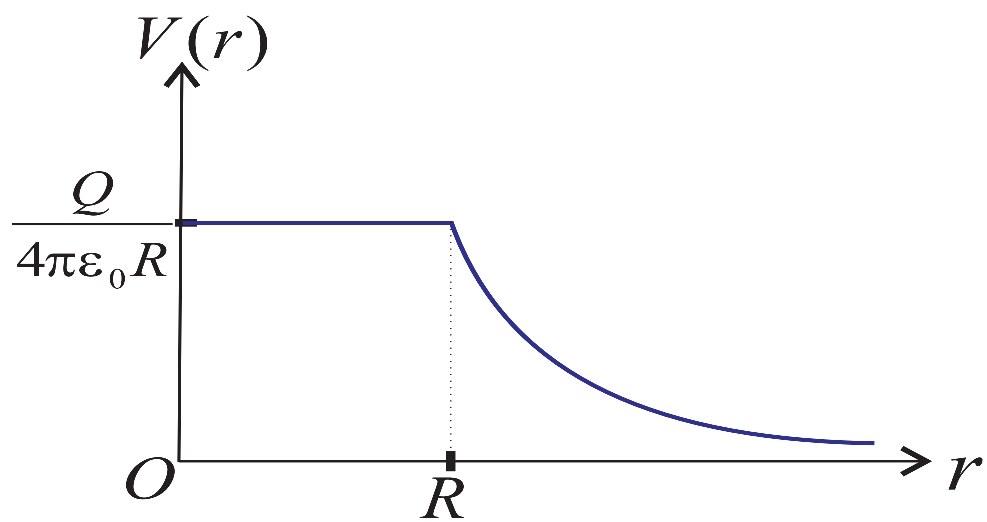
(ii) Outside conductor

$$V = \frac{Q}{4\pi\epsilon_0 r}$$

$\because$ Spherically symmetric

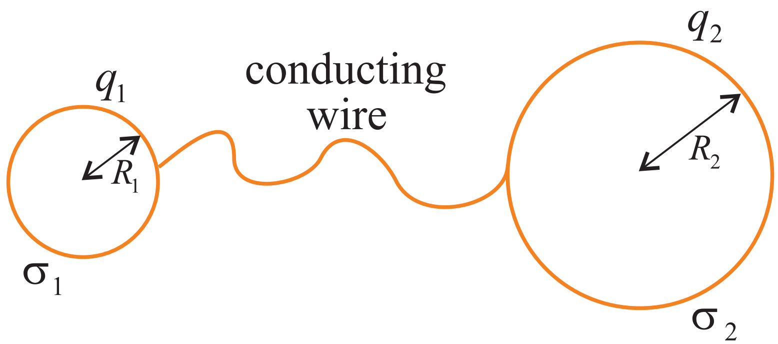
(Just like a point charge)

BUT not true for conductors of arbitrary shape



Example Connected conducting spheres

Two conductors connected can be seen as a **single conductor**



$\therefore$ **Potential everywhere is identical**

Potential of radius R_1 sphere ☛ $$V_1 = \frac{q_1}{4\pi\epsilon_0 R_1}$$

Potential of radius R_2 sphere ☛ $$V_2 = \frac{q_2}{4\pi\epsilon_0 R_2}$$

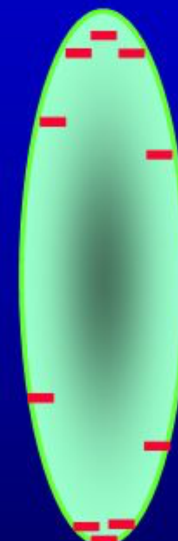
$$V_1 = V_2$$

$$\Rightarrow \frac{q_1}{R_1} = \frac{q_2}{R_2} \quad \Rightarrow \quad \frac{q_1}{q_2} = \frac{R_1}{R_2}$$

$$U_R = \frac{1}{4\pi\varepsilon_0}\frac{Q}{R} = \frac{4\pi R^2\sigma_R}{4\pi\varepsilon_0 R} = \frac{R\sigma_R}{\varepsilon_0}$$

$$U_r = \frac{1}{4\pi\varepsilon_0}\frac{q}{r} = \frac{4\pi r^2\sigma_r}{4\pi\varepsilon_0 r} = \frac{r\sigma_r}{\varepsilon_0}$$

$$U_R = U_r \qquad \boxed{\frac{Q}{q} = \frac{R}{r}} \qquad \boxed{\frac{\sigma_r}{\sigma_R} = \frac{R}{r}}$$



In a qualitative way, for a conductor of arbitrary shape, the charge density distribution on its surface is inverse proportional with its radius of curvature.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