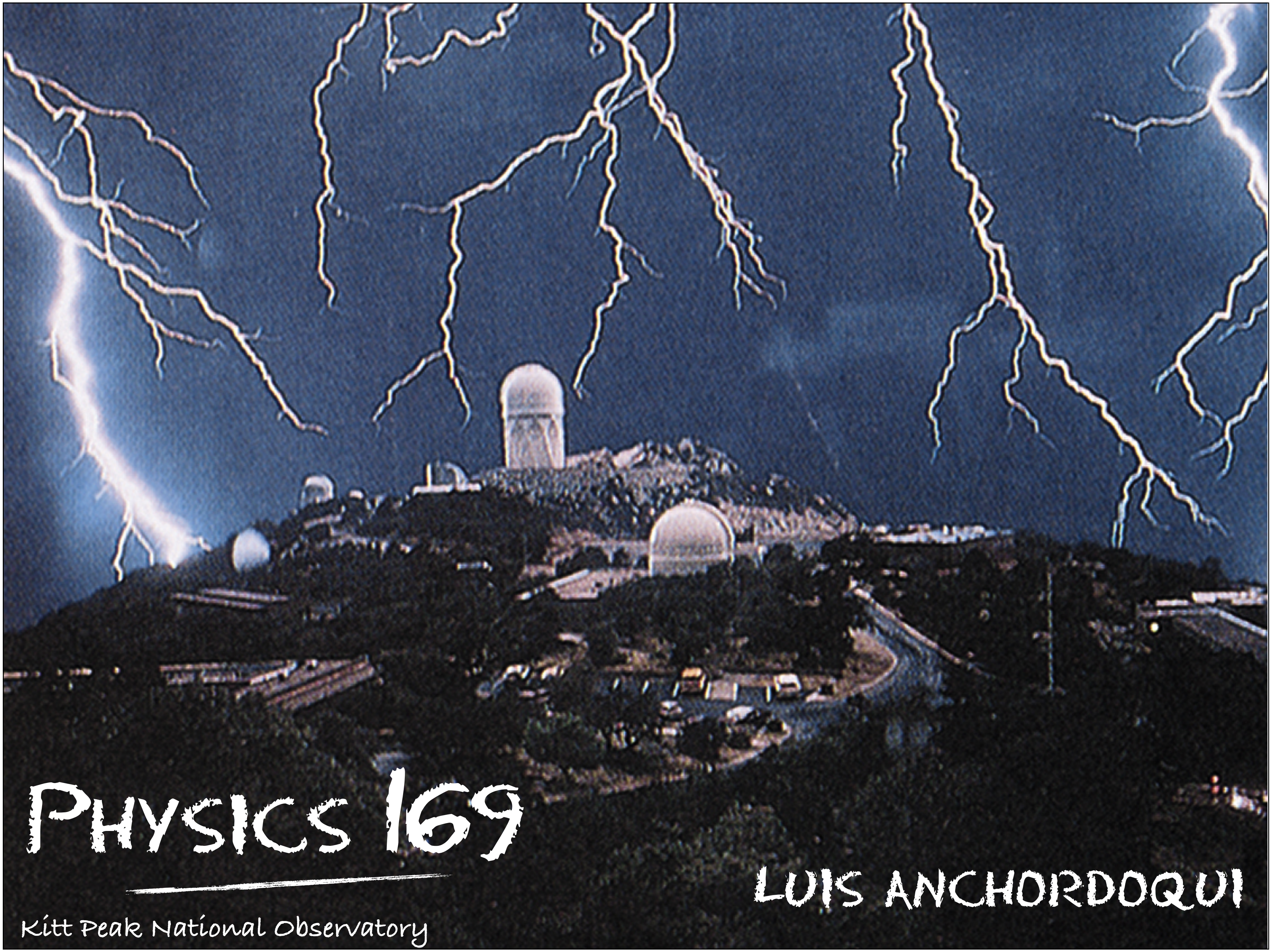

Physics 169
Luis Anchordoqui
Kitt Peak National Observatory

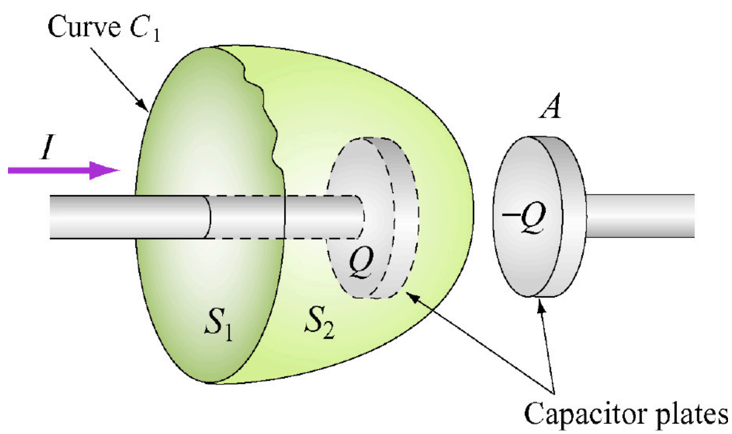
11.1 Displacement Current

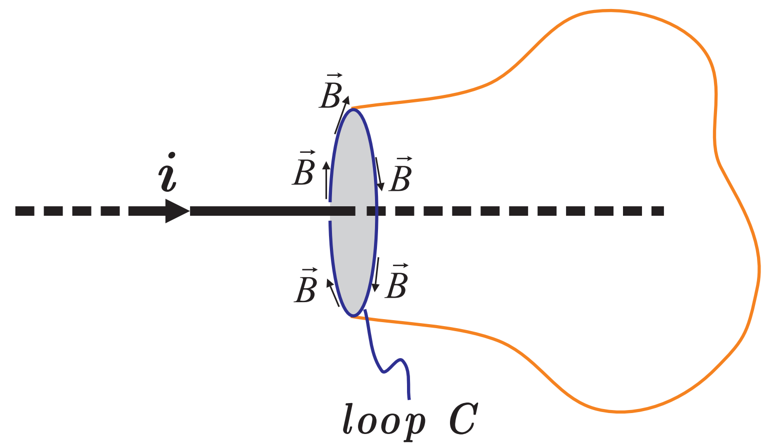
We can use **Ampere's law** to calculate magnetic fields due to currents

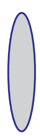
Integral $\oint_C \vec{B} \cdot d\vec{s}$ around any close loop C is equal to $\mu_0 i_{incl}$

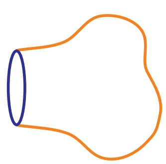
where $i_{incl} =$ **current passing an area bounded by closed curve** C

e.g.





 = Flat surface bounded by loop C

 = Curved surface bounded by loop C

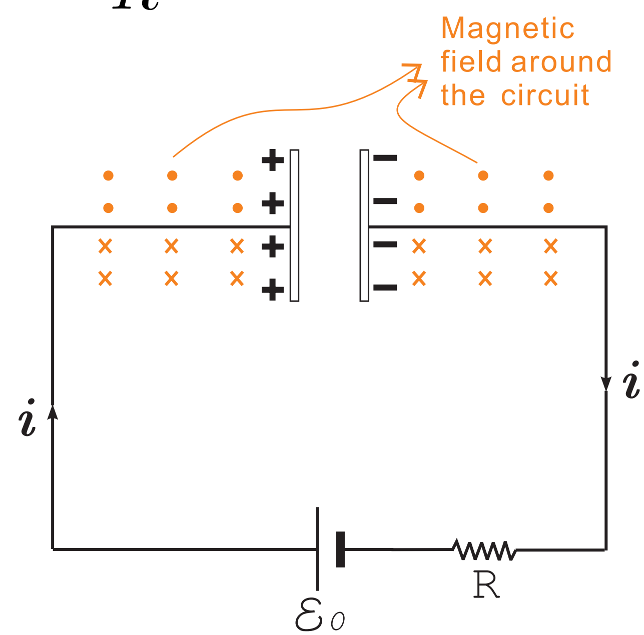
If **Ampere's law** is true all time

then i_{incl} **determined should be independent of surface chosen**

Consider a simple case: charging a capacitor

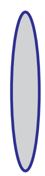
We know there is a current flowing $$i(t) = \frac{\mathcal{E}}{R} e^{-t/RC}$$

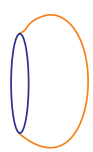
which leads to a magnetic field observed $\vec{B}$

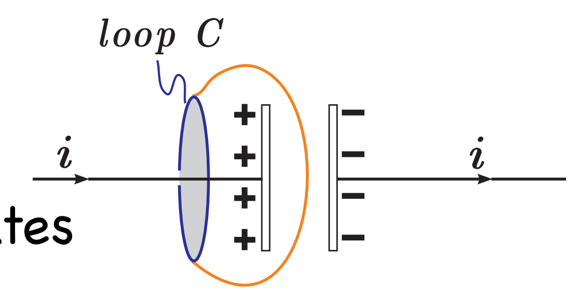


With **Ampere's law** $$\oint_C \vec{B} \cdot d\vec{s} = \mu_0 i_{\rm inc}$$

BUT WHAT IS $i_{\rm inc}$?

If we look at  $$i_{\rm inc} = i(t)$$

If we look at  $$i_{\rm inc} = 0$$



∵ There is no charge flow between capacitor plates

∴ Ampere's law is either WRONG or INCOMPLETE

Two observations

① While there is no current between capacitor's plates
there is a **time-varying electric field between plates of capacitor**

② We know **Ampere's law is mostly correct**
from measurements of B-field around circuits

$$\Downarrow$$

Can we revise Ampere's law to fix it?

Electric field between capacitor's plates $$E = \frac{\sigma}{\epsilon_0} = \frac{Q}{\epsilon_0 A}$$

$Q =$ charge on capacitor's plates
$A =$ area of capacitor's plates

$$\therefore\ Q = \epsilon_0 \underbrace{E \cdot A}_{\text{Electric flux}} = \epsilon_0 \Phi_E$$

$\therefore$ We can define

$$\frac{dQ}{dt} = \epsilon_0 \frac{d\Phi_E}{dt} = i_{\rm dis}$$

where $i_{\rm dis}$ is called **Displacement Current** (first proposed by Maxwell)

Maxwell first proposed that this is missing term for Ampere's law

$$\oint_C \vec{B} \cdot d\vec{s} = \mu_0 (i_{\rm inc} + \epsilon_0 \frac{d\Phi_E}{dt}) \qquad \textbf{Ampere-Maxwell law}$$

$i_{\rm inc} =$ current through any surface bounded by C

$\Phi_E =$ electric flux through that **same surface bounded by curve** C

$$\Phi_E = \int_S \vec{E} \cdot d\vec{a}$$

11.2 Induced Magnetic Field

Electric field can be generated by

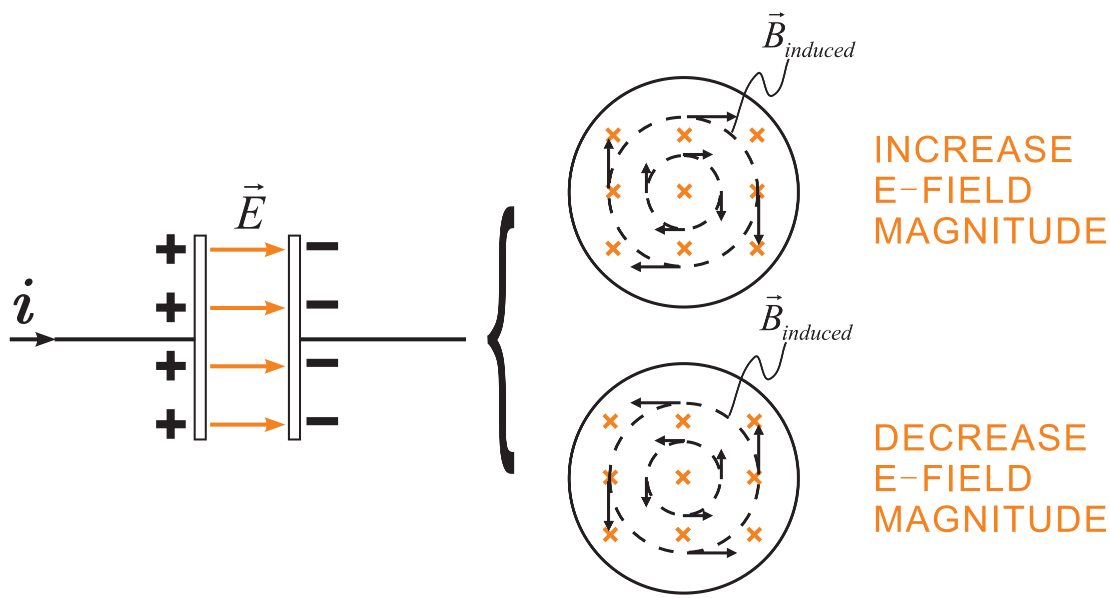
- ☛ charges
- ☛ changing magnetic flux

Ampere-Maxwell law that a magnetic field can be generated by

- ☛ moving charges (current)
- ☛ changing electric flux

A change in electric flux through a surface bounded by C

can lead to an induced magnetic field along loop C



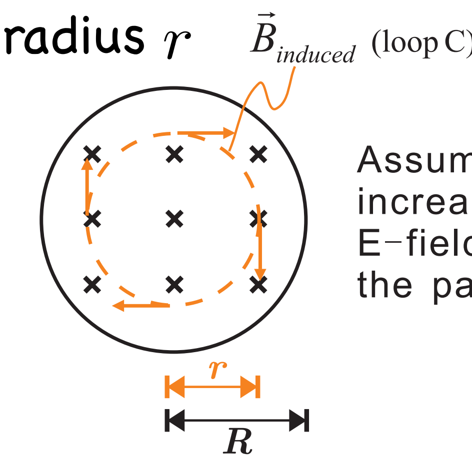
Notes Induced magnetic field is along **same direction** as caused by
☛ **changing electric flux**

Example What is magnetic field strength inside a circular plate capacitor of radius R with a current $I(t)$ charging it?

Answer Electric field of capacitor

$$E = \frac{Q}{\epsilon_0 A} = \frac{Q}{\epsilon_0 \pi R^2}$$

Electric flux inside capacitor through a loop C of radius r



Assume an increasing E-field into the page

Ampere-Maxwell Law inside capacitor:

$$\underbrace{\oint_C \vec{B} \cdot d\vec{s}}_{\because \vec{B}_{induced} || d\vec{s}} = \mu_0 (i_{incl} + \epsilon_0 \frac{d\Phi_E}{dt})$$

$$\underbrace{2\pi r}_{\text{Length of loop } C} B_{\text{induced}} = \mu_0 \epsilon_0 \frac{d}{dt} \left(\frac{Q r^2}{\epsilon_0 R^2} \right)$$

$$= \mu_0 \frac{r^2}{R^2} \underbrace{\frac{dQ}{dt}}_{I(t)}$$

$$\therefore \quad B_{\text{induced}} = \frac{\mu_0 r}{2\pi R^2} I(t) \qquad \text{for } r < R$$

Outside capacitor plate ☛

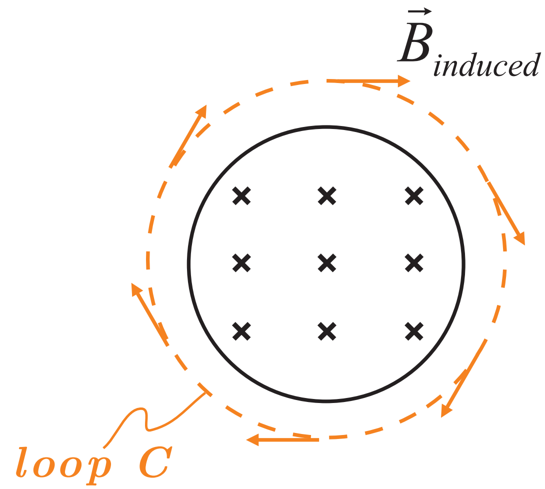
Electric flux through loop C

$$\Phi_E = E \cdot \pi R^2 = \frac{Q}{\epsilon_0}$$

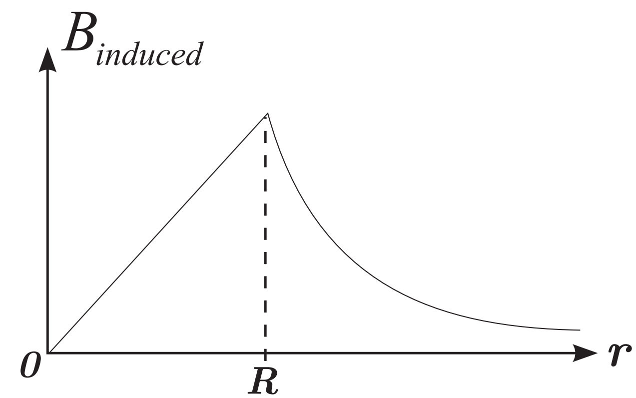
$$\oint_C \vec{B} \cdot d\vec{s} = \mu_0 \left(i_{\text{inc}} + \epsilon_0 \frac{d\Phi_E}{dt} \right)$$

$$2\pi r B_{\text{induced}} = \mu_0 \epsilon_0 \left(\frac{1}{\epsilon_0} \cdot \frac{dQ}{dt} \right)$$

$$\therefore \quad B_{\text{induced}} = \frac{\mu_0 I(t)}{2\pi r}$$



Assume an increasing E–field into the page



11.3 Maxwell's Equations

Four equations that **completely** describe ☛

behaviors of electric and magnetic fields

$$\oint_S \vec{E} \cdot d\vec{A} = \frac{Q_{\rm inc}}{\epsilon_0} = \frac{1}{\epsilon_0} \int_V \rho \, dV$$

$$\oint_S \vec{B} \cdot d\vec{A} = 0$$

$$\oint_C \vec{E} \cdot d\vec{s} = -\frac{d}{dt} \int_S \vec{B} \cdot d\vec{A}$$

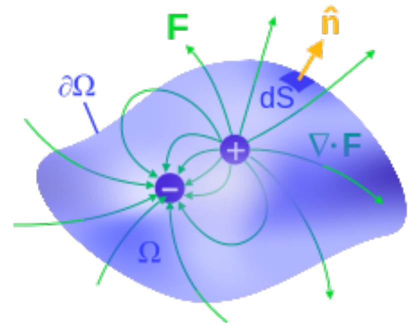
$$\oint_C \vec{B} \cdot d\vec{s} = \mu_0 i_{\rm inc} + \mu_0 \epsilon_0 \frac{d}{dt} \int_S \vec{E} \cdot d\vec{A}$$

One equation that describes ☛

how matter reacts to electric and magnetic fields

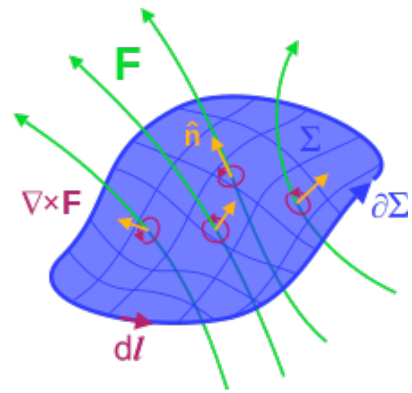
$$\vec{F} = q(\vec{E} + \vec{v} \times \vec{B})$$

Maxwell's equations in matter



$$\oint_S \vec{D}\cdot d\vec{A} = \int_V \rho \, dV$$

$$\oint_{\partial\Omega} \vec{B}\cdot d\vec{A} = 0$$



$$\oint_C \vec{E}\cdot d\vec{s} = -\frac{d}{dt}\int_S \vec{B}\cdot d\vec{A}$$

$$\oint_C \vec{H}\cdot d\vec{s} = \int_S \vec{J}\cdot d\vec{A} + \frac{d}{dt}\int_S \vec{D}\cdot d\vec{A}$$

Electric displacement field ☛ $\vec{D} = \epsilon_0\vec{E} + \vec{P}$

vacuum ☛ $\vec{D} = \epsilon_0\vec{E}$

isotropic linear dielectric ☛ $\vec{D} = \epsilon\vec{E}$

Boundary conditions ☛

	Component	General materials	Linear materials
Electric displacement	Perpendicular	$D_{2,\perp} - D_{1,\perp} = \sigma$	$D_{2,\perp} - D_{1,\perp} = \sigma$
	Parallel	$D_{2,\parallel} - D_{1,\parallel} = ?$	$\frac{\mathbf{D_{2,\parallel}}}{\epsilon_2} = \frac{\mathbf{D_{1,\parallel}}}{\epsilon_1}$
Electric field	Perpendicular	$\epsilon_2 E_{2,\perp} - \epsilon_1 E_{1,\perp} = ?$	$\epsilon_2 E_{2,\perp} - \epsilon_1 E_{1,\perp} = \sigma$
	Parallel	$E_{2,\parallel} = E_{1,\parallel}$	$E_{2,\parallel} = E_{1,\parallel}$

Maxwell's equations may also be written in differential form

$$\nabla \cdot \mathbf{D} = \rho$$
$$\nabla \cdot \mathbf{B} = 0$$
$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$
$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}$$

Important consequence of Maxwell's equations

prediction electromagnetic waves that travel @ speed of light

Reason is due to the fact that

changing electric field produces a magnetic field and vice versa

Coupling between 2 fields leads to generation of electromagnetic waves

Prediction was confirmed by Hertz in 1887

11.4 Plane Traveling Electromagnetic Waves

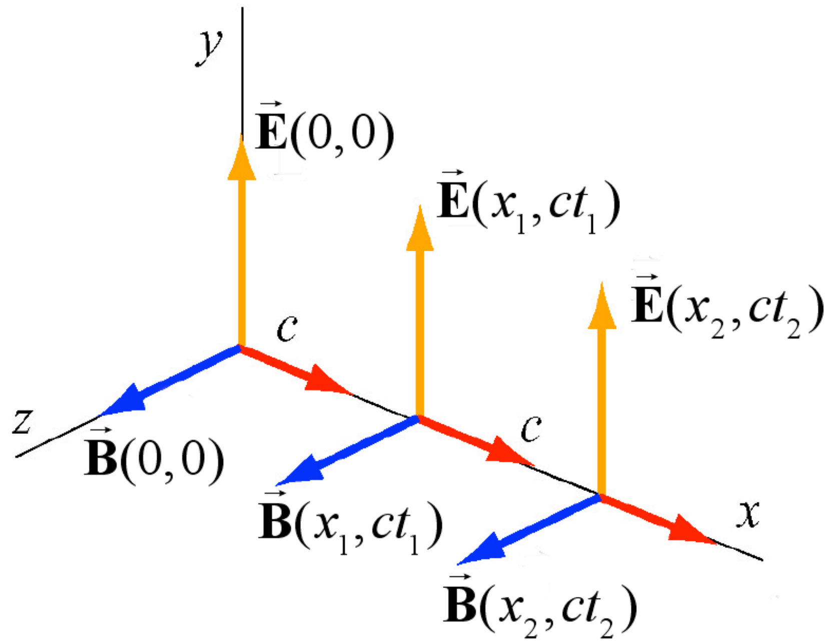
Consider electromagnetic wave propagating in $+x$ direction

with uniform $\vec{E}$ pointing in $+y$-direction and $\vec{B}$ in $+z$-direction

At any instant both $\vec{E}$ and $\vec{B}$ ☛ independent of (y, z) coordinates

$$\vec{E}(x,t) = E_y(x,t)\hat{\jmath}$$

$$\vec{B}(x,t) = B_z(x,t)\hat{k}$$



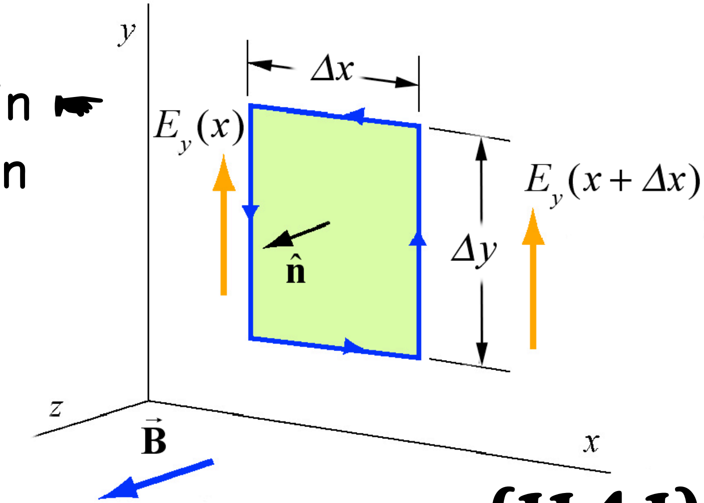
This (non-physical) electric and magnetic field yield a **plane wave**

because at any instant both $\vec{E}$ and $\vec{B}$

are uniform over any plane perpendicular to direction of propagation

Consider a rectangular loop that lies in xy-plane
left side of loop at x and right at $x + \Delta x$
bottom at y and top at $y + \Delta y$ as shown in ☛
Unit vector normal to loop positive z-direction

$$\hat{n} = \hat{k}$$



Recall Faraday's law

$$\oint \vec{E} \cdot d\vec{s} = -\frac{d}{dt} \int\int \vec{B} \cdot d\vec{A} \quad \textbf{(11.4.1)}$$

To evaluate LHS of (11.4.1) ☛ integrate around closed path

$$\oint \vec{E} \cdot d\vec{s} = E_y(x + \Delta x)\,\Delta y - E_y(x)\,\Delta y \quad \textbf{(11.4.2)}$$

use Taylor expansion to approximate

$$E_y(x + \Delta x) = E_y(x) + \frac{\partial E_y}{\partial x}\Delta x + \cdots \quad \textbf{(11.4.3)}$$

Left-hand-side of Faraday's law becomes

$$\oint \vec{E} \cdot d\vec{s} = \frac{\partial E_y}{\partial x}\,\Delta x \Delta y \quad \textbf{(11.4.4)}$$

Assume that Δx and Δy are very small such that time derivative of z-component of magnetic field is nearly uniform over area element

Rate of change of magnetic flux on right-hand-side of Eq. (11.4.1) is

$$-\frac{d}{dt}\int\int \vec{B}\cdot d\vec{A} = -\frac{\partial B_z}{\partial t}\,\Delta x\,\Delta y \qquad \textbf{(11.4.5)}$$

Equating two sides of Faraday's Law and dividing through by area $\Delta x \Delta y$

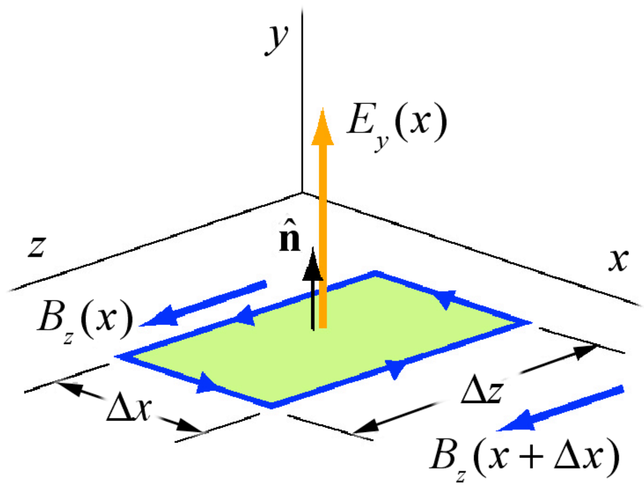
$$\frac{\partial E_y}{\partial x} = -\frac{\partial B_z}{\partial t} \qquad \textbf{(11.4.6)}$$

Eq. (11.4.6) indicates that at each point in space time-varying B-field is associated with spatially varying E-field

Second condition on relationship between electric and magnetic fields may be deduced by using Ampere-Maxwell equation

$$\oint_C \vec{B} \cdot d\vec{s} = \mu_0 i_{\text{inc}} + \mu_0 \epsilon_0 \frac{d}{dt} \int_S \vec{E} \cdot \vec{A} \qquad \textbf{(11.4.7)}$$

Consider a rectangular loop in xy- plane depicted



Evaluating line integral of magnetic field around closed path

$$\oint \vec{B} \cdot d\vec{s} = B_z(x)\Delta z - B_z(x + \Delta x)\,\Delta z \qquad \textbf{(11.4.8)}$$

Use Taylor expansion to approximate

$$B_z(x + \Delta x) = B_z(x) + \frac{\partial B_z}{\partial x}\Delta x + \cdots \qquad \textbf{(11.4.9)}$$

Left-hand-side of Maxwell-Ampere law becomes

$$\oint \vec{B} \cdot d\vec{s} = -\frac{\partial B_z}{\partial x}\Delta x \Delta z \qquad \textbf{(11.4.10)}$$

Assuming that Δx and Δz are very small such that time derivative of y-component of electric field is nearly uniform over area element

Rate of change of electric flux on right-hand-side of Eq.(11.4.7) is

$$\mu_0\, \epsilon_0 \frac{d}{dt} \int\int \vec{E} \cdot d\vec{A} = \mu_0\, \epsilon_0 \frac{\partial E_y}{\partial t}\Delta x \Delta z \qquad \textbf{(11.4.11)}$$

Equating two sides of Maxwell-Ampere law and dividing by $\Delta x \Delta z$ yields

$$-\frac{\partial B_z}{\partial x} = \mu_0\, \epsilon_0 \frac{\partial E_y}{\partial t} \qquad \textbf{(11.4.12)}$$

Eq. (11.4.12) indicates that at each point in space

time-varying E-field is associated to spatially varying B-field

Eq. (11.4.6) and **Eq. (11.4.12)** are coupled differential equations

To uncouple them

first take another partial derivative of **Eq. (11.4.6)** with respect to x

$$\frac{\partial^2 E_y}{\partial x^2} = -\frac{\partial}{\partial x}\left(\frac{\partial B_z}{\partial t}\right) = -\frac{\partial}{\partial t}\left(\frac{\partial B_z}{\partial x}\right)$$ **Eq. (11.4.13)**

Assumed that field B_z is sufficiently well behaved

such that partial derivatives are interchangeable

$$\frac{\partial}{\partial x}\left(\frac{\partial B_z}{\partial t}\right) = \frac{\partial}{\partial t}\left(\frac{\partial B_z}{\partial x}\right)$$ **Eq. (11.4.14)**

Substitute **Eq. (11.4.12)** into **Eq. (11.4.13)**

One-dimensional wave equation

$$\frac{\partial^2 E_y}{\partial x^2} = \mu_0 \epsilon_0 \frac{\partial^2 E_y}{\partial t^2}$$ **Eq. (11.4.15)**

By a dimensional analysis

quantity $\frac{1}{\mu_0 \epsilon_0}$ has same dimensions as speed squared

Repeat argument to find a one-dimensional wave equation satisfied by z-component of magnetic field ☛ taking $\partial/\partial x$ of **Eq. (11.4.12)**

$$-\frac{\partial^2 B_z}{\partial x^2} = \mu_0 \epsilon_0 \frac{\partial}{\partial x}\frac{\partial E_y}{\partial t} = \mu_0 \epsilon_0 \frac{\partial}{\partial t}\left(\frac{\partial E_y}{\partial x}\right)$$ **Eq. (11.4.16)**

Substitute **Eq. (11.4.6)** into **Eq. (11.4.16)** yielding a one-dimensional wave equation satisfied by z-component of magnetic field

$$\frac{\partial^2 B_z}{\partial x^2} = \mu_0 \epsilon_0 \frac{\partial^2 B_z}{\partial t^2}$$ **Eq. (11.4.17)**

General form of a one-dimensional wave equation is given by

$$\frac{\partial^2 \Psi(x,t)}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 \Psi(x,t)}{\partial t^2}$$ **Eq. (11.4.18)**

where v is **speed of propagation** and $\Psi(x,t)$ is **wave function**

E_y and B_z satisfy wave equation and propagate with speed

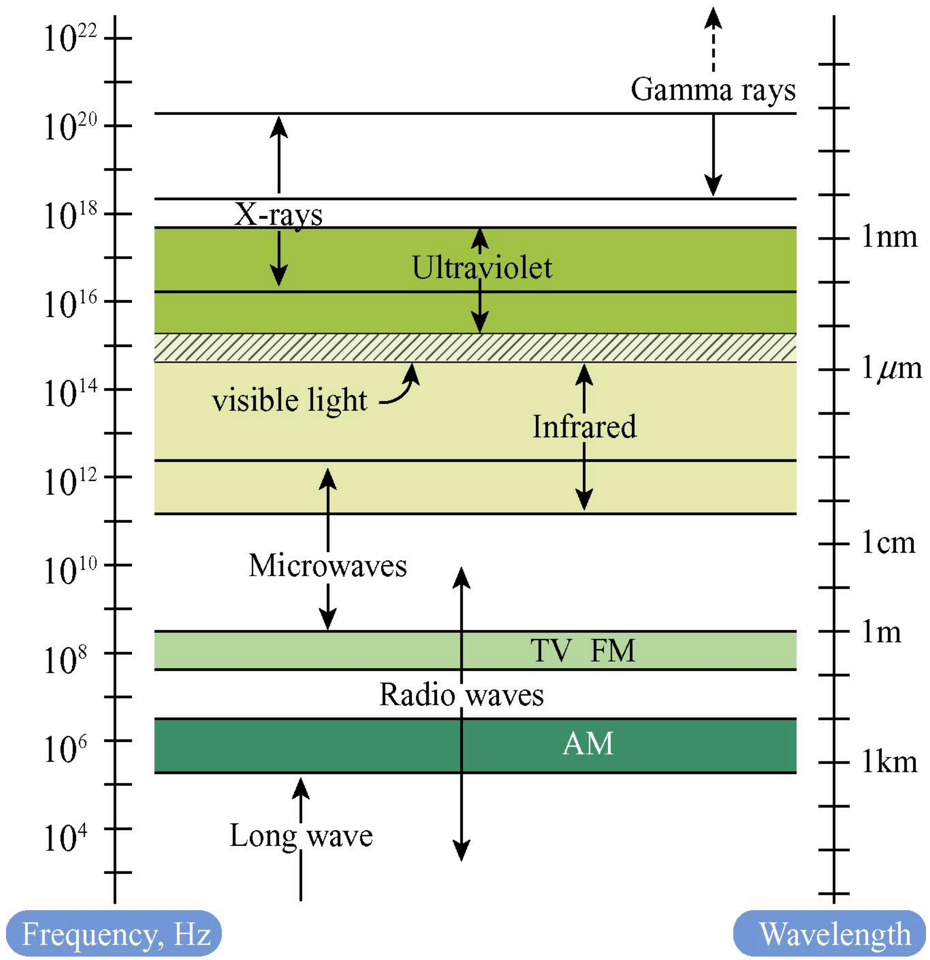
$$v = \frac{1}{\sqrt{\mu_0 \epsilon_0}}$$ **Eq. (11.4.19)**

Taking $\mu_0 = 4\pi \times 10^{-7}$ Tm/A and $\epsilon_0 = 8.854 \times 10^{-12}$ C/N/m^2

$$v = c = 2.997 \times 10^8 \text{ m/s}$$

Maxwell's Equations predict that E- and B-fields propagate through space at speed of light

Electromagnetic spectrum



And God Said

$$\nabla \cdot E = \frac{\rho}{\varepsilon_0}$$

$$\nabla \cdot B = 0$$

$$\nabla \times E = -\frac{\partial B}{\partial t}$$

$$\nabla \times B = \mu_0 J + \mu_0 \varepsilon_0 \frac{\partial E}{\partial t}$$

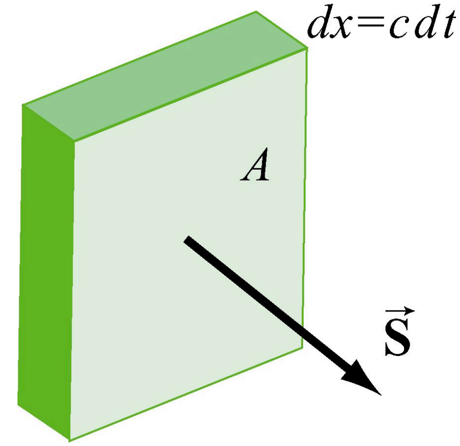
and then there was

"Light"

11.5 Poynting Vector

Energy can also be transported by electromagnetic waves

Consider plane EM wave passing through small volume element of area A and thickness dx as shown in ☛



Total energy stored in electromagnetic fields in volume element is

$$dU = uA\,dx = (u_E + u_B)\,A\,dx = \frac{1}{2}\left(\epsilon_0\,E^2 + \frac{B^2}{\mu_0}\right)A\,dx$$ **Eq. (11.5.1)**

$$u_E = \frac{1}{2}\,\epsilon_0\,E^2 \qquad u_B = \frac{B^2}{2\mu_0}$$ **Eq. (11.5.2)**

Because electromagnetic wave propagates with speed of light c

time it takes for wave to move through volume element is $dt = dx/c$

One may obtain rate of change of energy per unit area

$$S = \frac{dU}{A\,dt} = \frac{c}{2}\left(\epsilon_0\,E^2 + \frac{B^2}{\mu_0}\right)$$ **Eq. (11.5.3)**

SI unit of S is $[W \cdot \mathrm{m}^{-2}]$

Recall that magnitude of fields satisfy $E = cB$ and $c = 1/\sqrt{\mu_0 \epsilon_0}$

Therefore **Eq. (11.5.3)** may be rewritten as

$$S = \frac{c}{2}\left(\epsilon_0 E^2 + \frac{B^2}{\mu_0}\right) = \frac{cB^2}{\mu_0} = c\epsilon_0 E^2 = \frac{EB}{\mu_0}$$ **Eq. (11.5.4)**

Turn this energy flow into a vector

by assigning direction as direction of propagation

Rate of energy flow per unit area is called **Poynting vector** $\vec{S}$

(after British physicist John Poynting) and defined by vector product

$$\vec{S} = \frac{1}{\mu_0} \vec{E} \times \vec{B}$$ **Eq. (11.5.5)**

Plane transverse electromagnetic waves

fields $\vec{E}$ and $\vec{B}$ are perpendicular and magnitude of $\vec{S}$

$$|\vec{S}| = \frac{\vec{E} \times \vec{B}}{\mu_0} = \frac{EB}{\mu_0} = S$$ **Eq. (11.5.6)**

As an example, suppose electric field associated with a plane sinusoidal electromagnetic wave is $\vec{E} = E_0 \cos(kx - \omega t)\,\hat{j}$

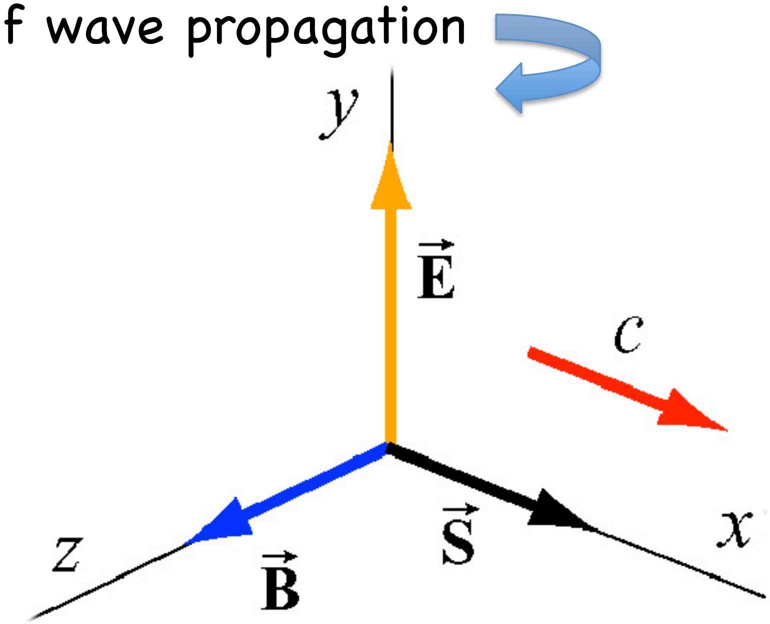
Corresponding magnetic field is $\vec{B} = B_0 \cos(kx - \omega t)\,\hat{k}$

and direction of propagation is positive x-direction

Poynting vector is then ☛ **Eq. (11.5.7)**

$$\vec{S} = \frac{1}{\mu_0}\left(E_0 \cos(kx - \omega t)\,\hat{j}\right) \times \left(B_0 \cos(kx - \omega t)\,\hat{k}\right) = \frac{E_0 B_0}{\mu_0} \cos^2(kx - \omega t)\hat{\imath}$$

As expected ☛ $\vec{S}$ points in direction of wave propagation



Intensity of wave I is defined as time-average of S

$$I = \langle S \rangle = \frac{E_0 B_0}{\mu_0} \langle \cos^2(kx - \omega t) \rangle = \frac{E_0 B_0}{2\mu_0} = \frac{E_0^2}{2c\mu_0} = \frac{cB_0^2}{2\mu_0}$$ **Eq. (11.5.8)**

recall ☛ $\langle \cos^2(kx - \omega t) \rangle = \frac{1}{2}$ **Eq. (11.5.9)**

To relate intensity to energy density

we first note equality between electric and magnetic energy densities

$$u_B = \frac{B^2}{2\mu_0} = \frac{(E/c)^2}{2\mu_0} = \frac{E^2}{2c^2\mu_0} = \frac{\epsilon_0 E^2}{2} = u_E$$ **Eq. (11.5.10)**

Time-averaged energy density of wave is then

$$\langle u \rangle = \langle u_E + u_B \rangle = \epsilon_0 \langle E^2 \rangle = \frac{\epsilon_0}{2} E_0^2 = \frac{1}{\mu_0} \langle B^2 \rangle = \frac{B_0^2}{2\mu_0}$$ **Eq. (11.5.11)**

Comparing **Eqs. (11.5.8)** and **Eq. (11.5.11)**

we can conclude that intensity is related to average energy density by

$$I = \langle S \rangle = c \langle u \rangle$$ **Eq. (11.5.12)**

11.6 Momentum and Radiation Pressure

An electromagnetic wave transports not only energy but also momentum and hence can exert a radiation pressure on a surface

due to absorption and reflection of momentum

When a plane electromagnetic wave is completely absorbed by a surface momentum transferred is related to energy absorbed by

$$\Delta p = \frac{\Delta U}{c} \quad \textbf{complete absorption} \qquad \textbf{Eq. (11.6.1)}$$

(We shall not prove this result as it involves a more complicated description of energy and momentum stored in electromagnetic fields)

If EM wave is completely reflected by a surface such as a mirror

$$\Delta p = \frac{2\Delta U}{c} \quad \textbf{complete reflection} \qquad \textbf{Eq. (11.6.2)}$$

For a wave that is completely absorbed
time-averaged radiation pressure (force per unit area) is given by

$$P = \frac{\langle F \rangle}{A} = \frac{1}{A}\left\langle \frac{dp}{dt} \right\rangle = \frac{1}{Ac}\left\langle \frac{dU}{dt} \right\rangle$$ **Eq. (11.6.3)**

Because time-averaged rate that energy delivered to surface is

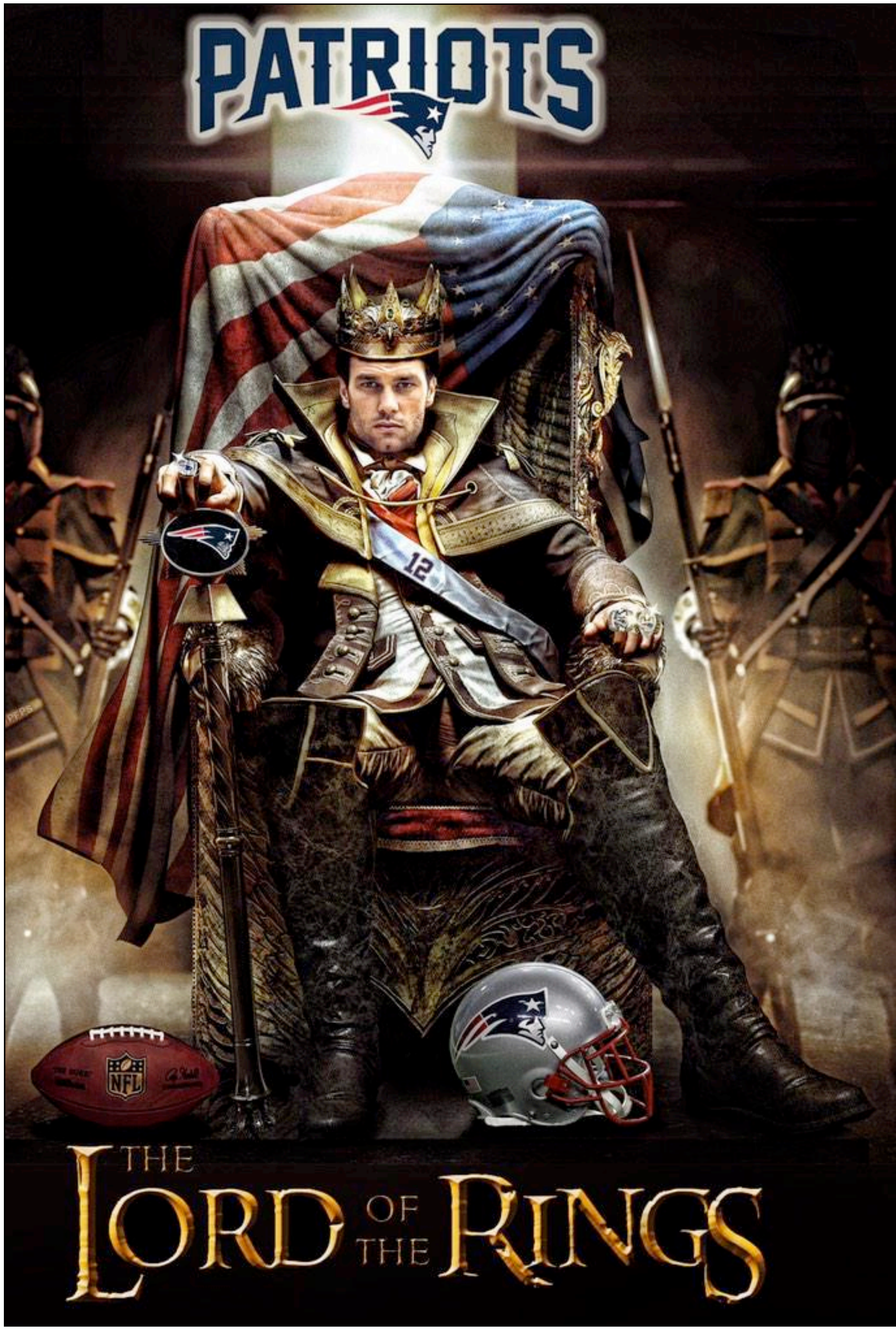
$$\left\langle \frac{dU}{dt} \right\rangle = \langle S \rangle A$$ **Eq. (11.6.4)**

Substitute **Eq. (11.6.4)** into **Eq. (11.6.3)** yielding

$$P = \frac{\langle S \rangle}{c}$$ **complete absorption** **Eq. (11.6.5)**

If radiation is completely reflected
radiation pressure is twice as great as case of complete absorption

$$P = \frac{2\langle S \rangle}{c}$$ **complete reflection** **Eq. (11.6.6)**


PATRIOTS
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THE LORD OF THE RINGS