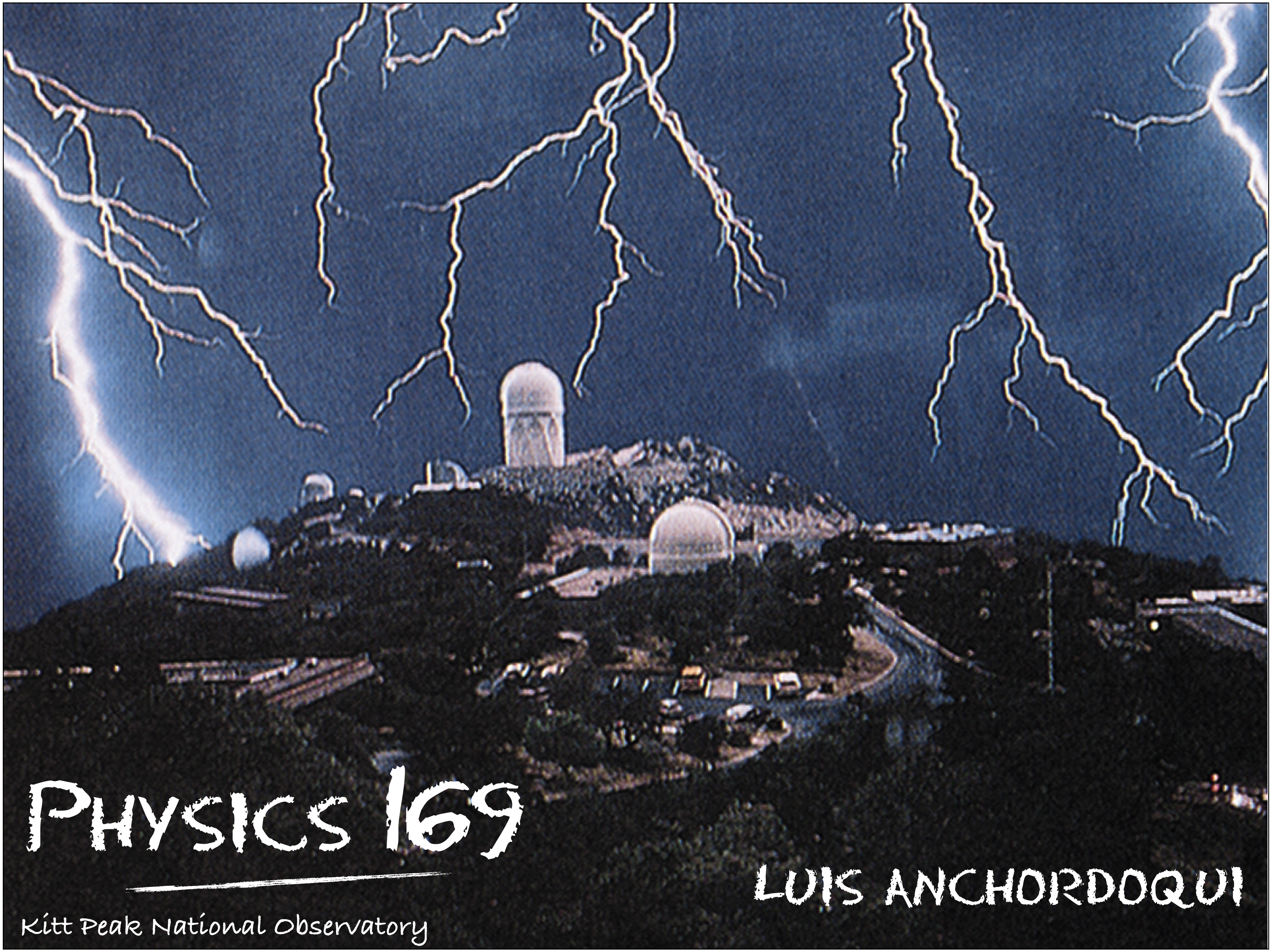
PHYSICS 169
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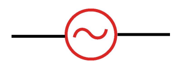
10.1 Alternating Current (AC) Voltage

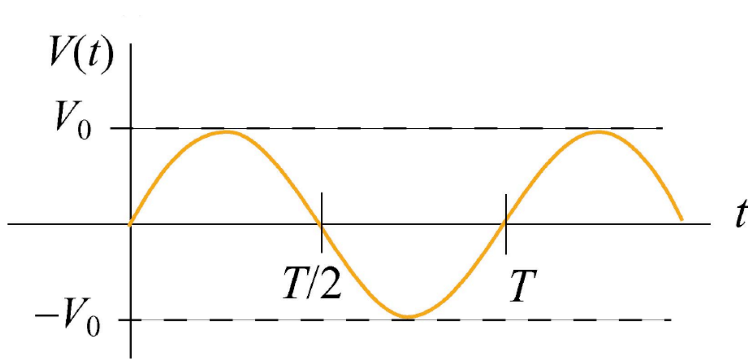
We learned that changing magnetic flux could induce an emf according to Faraday's law

If coil rotates in presence of magnetic field

induced emf varies sinusoidally with time and leads to AC

Symbol for AC voltage source ☛ 

$$V(t) = V_0 \sin(\omega t)$$

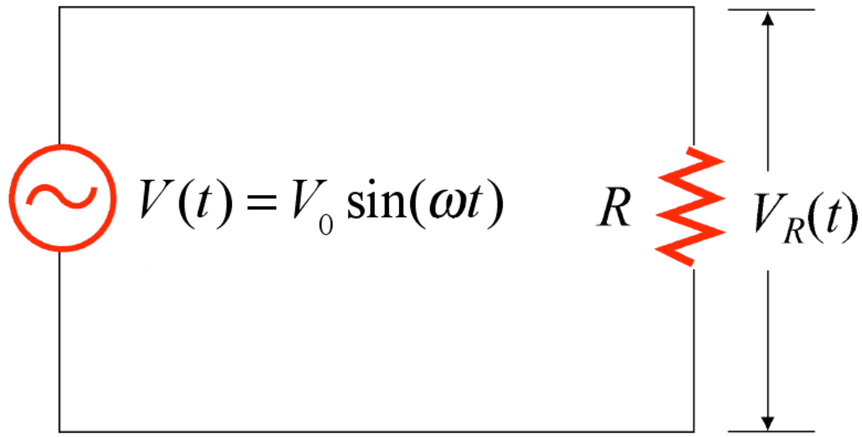


$$\omega = 2\pi f = 2\pi / T$$

unit ☛ $\mathrm{s}^{-1} \equiv \mathrm{Hz}$

10.2 AC circuits with a source and one circuit element

Purely Resistive Load



We'd like to find current through resistor

$$I_R(t) = I_{0,R} \sin(\omega t + \phi_R)$$

Applying Kirchhoff's loop rule yields

$$V(t) - I_R(t)R = 0$$

$$I_R(t) = \frac{V(t)}{R} = \frac{V_0 \sin(\omega t)}{R} = I_{0,R} \sin(\omega t)$$

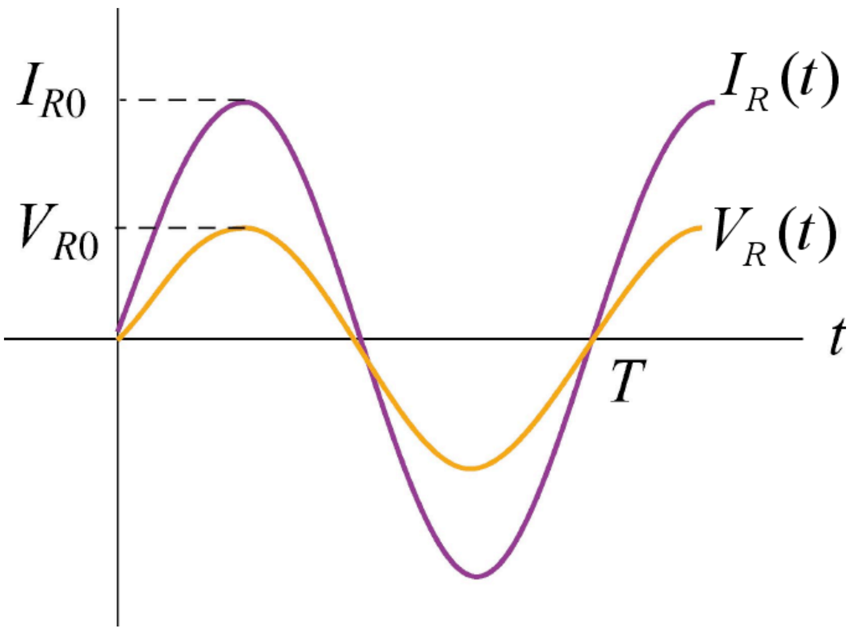
$V_R(t) = I_R(t)R$ ☛ instantaneous voltage drop across the resistor

$$I_{0,R} = \frac{V_{0,R}}{R} = \frac{V_{0,R}}{X_R}$$

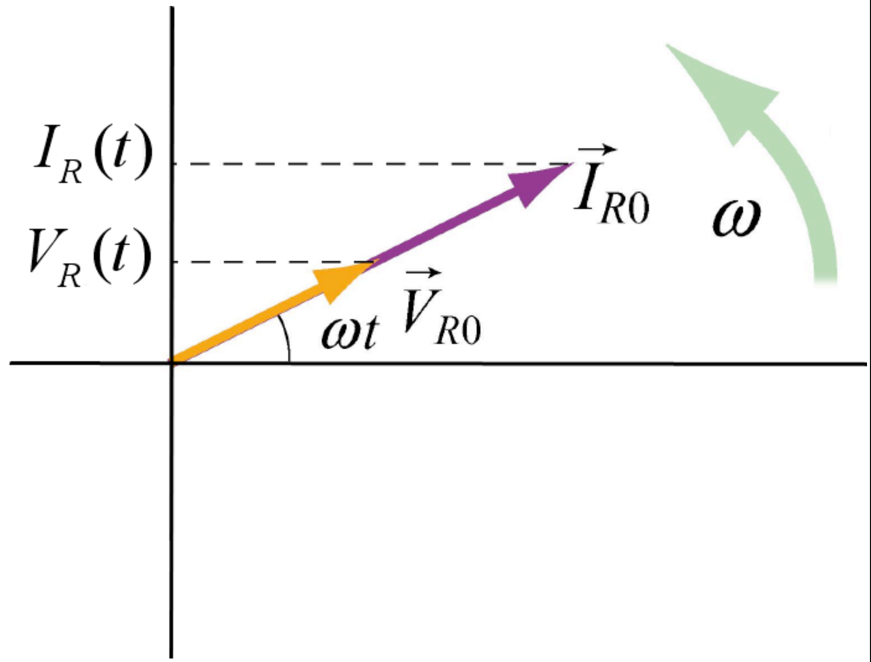
$X_R = R$ ☛ resistive reactance

$\phi_R = 0$ ☛ $I_R(t)$ and $V_R(t)$ are in phase with each other

Time dependence of $V_R(t)$ and $I_R(t)$



Phasor diagram for resistive circuit



Phasor ☛ rotating vector having following properties:

(i) length: the length corresponds to the amplitude

(ii) angular speed: the vector rotates counterclockwise with angular speed ω

(iii) projection: the projection of vector along vertical axis corresponds to value of the alternating quantity at time t

Average value of current over one period

$$\langle I_R(t)\rangle = \frac{1}{T}\int_0^T I_R(t)\ dt = \frac{1}{T}\int_0^T I_{0,R}\sin(\omega t)dt = \frac{I_{0,R}}{T}\int_0^T \sin(2\pi t/T)\ dt = 0$$

Average of the square of the current is non-vanishing

$$\langle I_R^2(t)\rangle = \frac{1}{T}\int_0^T I_R^2(t)\ dt = \frac{1}{T}\int_0^T I_{0,R}^2\sin^2(\omega t)dt = \frac{I_{0,R}^2}{T}\int_0^T \sin^2(2\pi t/T)\ dt = \frac{1}{2}I_{0,R}^2$$

It is convenient to define:

root-mean-square (rms) current ☛ $I_{\rm rms} = \sqrt{\langle I_R^2(t)\rangle} = \dfrac{I_{0,R}}{\sqrt{2}}$

rms voltage ☛ $V_{\rm rms} = \sqrt{\langle V_R^2(t)\rangle} = \dfrac{V_{0,R}}{\sqrt{2}}$

rms voltage supplied to domestic wall outlets in US ☛ $V_{\rm rms} = 110\ \mathrm{V}\ @\ 60\ \mathrm{Hz}$

Power dissipated in the resistor ☛ $P_R(t) = I_R(t)V_R(t) = I_R^2(t)R$

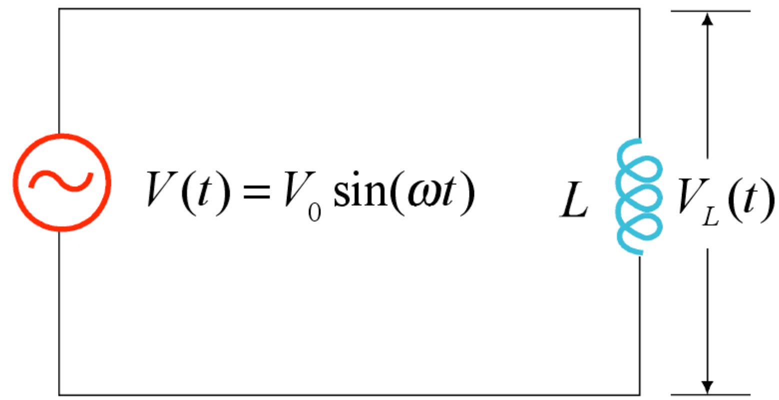
Average power over one period

$$\langle P_R(t)\rangle = \langle I_R^2(t)R\rangle = \frac{1}{2}I_{0,R}^2R = I_{\rm rms}^2R = I_{\rm rms}V_{\rm rms} = \frac{V_{\rm rms}^2}{R}$$

Purely Inductive Load

We'd like to find current in the circuit

$$I_L(t) = I_{0,L}\sin(\omega t - \phi_L)$$



Applying Kirchhoff's loop rule yields

$$V(t) - V_L(t) = V(t) - L\frac{dI_L}{dt} = 0$$

$$\frac{dI_L}{dt} = \frac{V(t)}{L} = \frac{V_{0,L}}{L}\sin(\omega t)$$

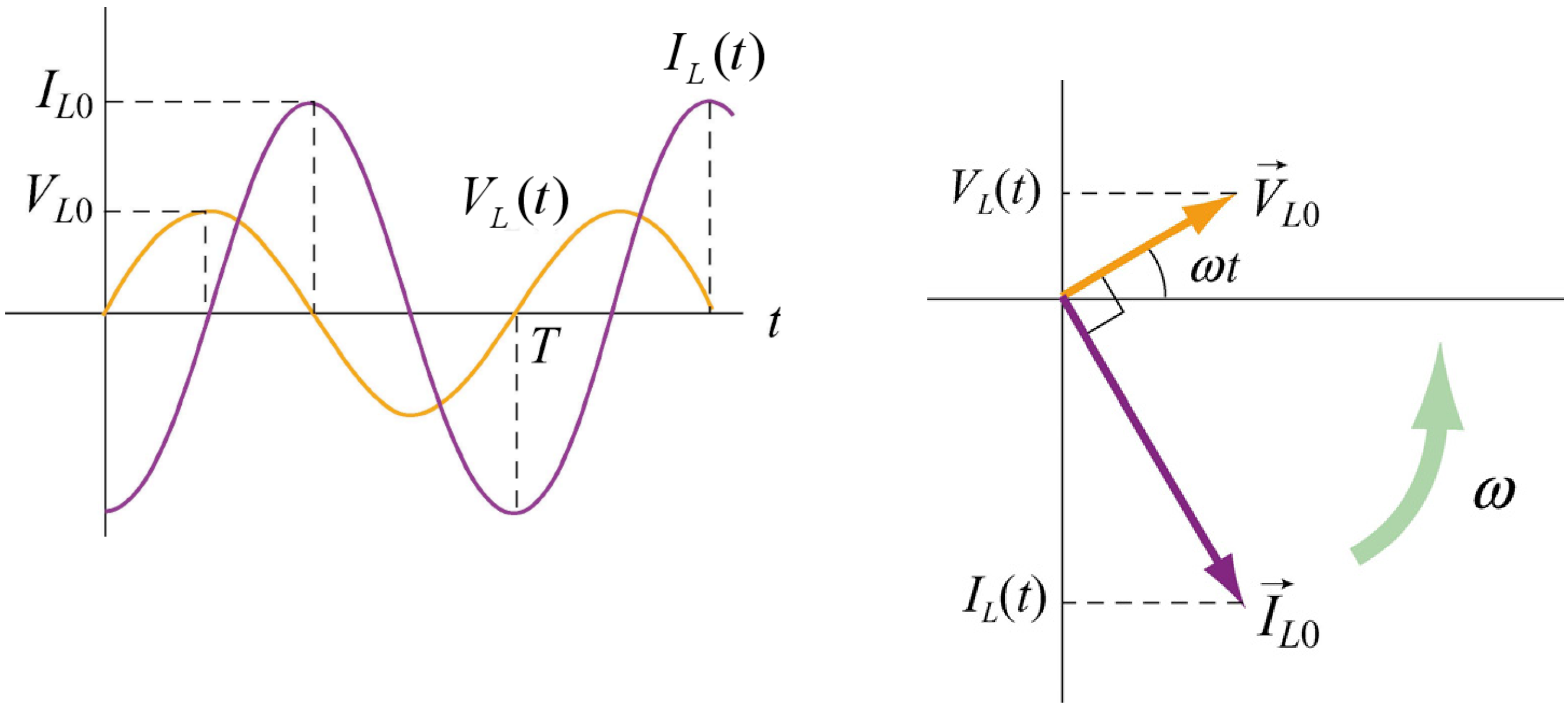
$$I_L(t) = \int dI_L = \frac{V_{0,L}}{L}\int \sin(\omega t)dt = -\frac{V_{0,L}}{\omega L}\cos(\omega t) = \frac{V_{0,L}}{\omega L}\sin(\omega t - \pi/2)$$

$$I_{0,L} = \frac{V_{0,L}}{\omega L} = \frac{V_{0,L}}{X_L}$$

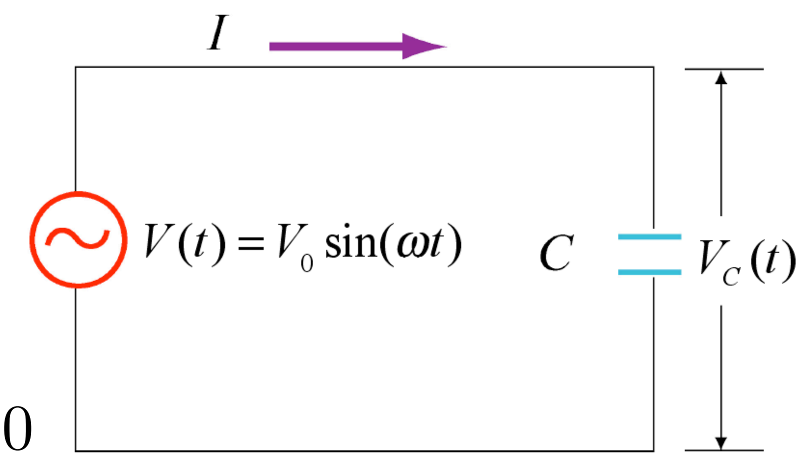
$X_L = \omega L$ ☛ inductive reactance

$\phi_L = +\pi/2$ ☛ phase constant

The current lags voltage by $\pi / 2$ in a purely inductive circuit



Purely Capacity Load



We'd like to find current in circuit

$$I_C(t) = I_{0,C} \sin(\omega t - \phi_C)$$

Again ☛ Kirchhoff's loop rule yields

$$V(t) - V_C(t) = V(t) - \frac{Q(t)}{C} = 0$$

Charge on capacitor

$$Q(t) = CV(t) = CV_C(t) = CV_{0,C} \sin(\omega t)$$

Current

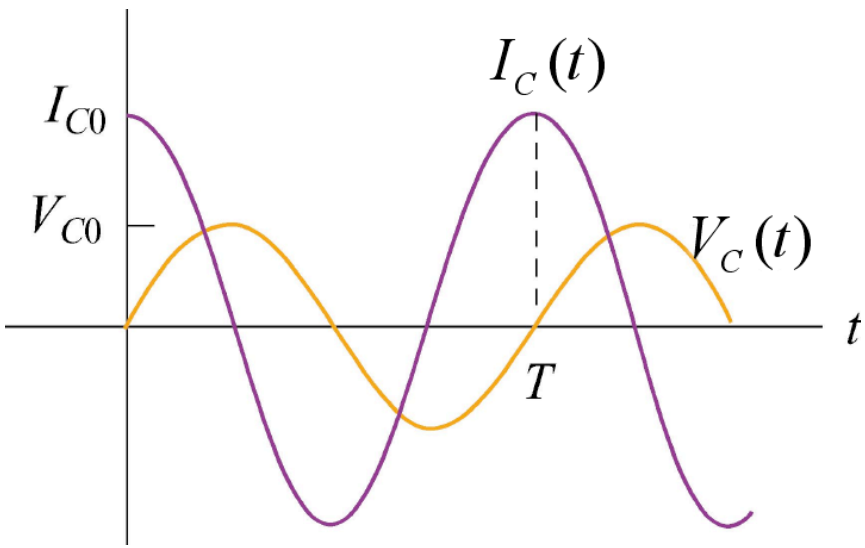
$$I_C(t) = \frac{dQ}{dt} = \omega C V_{0,C} \cos(\omega t) = \omega C V_{0,C} \sin(\omega t + \pi/2)$$

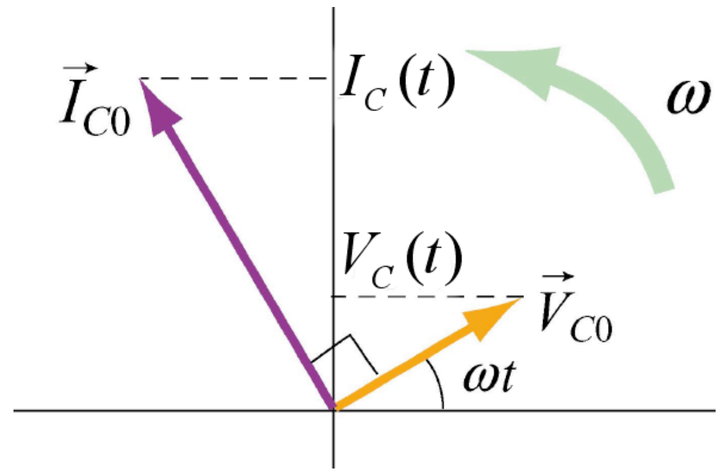
$$I_{0,C} = \omega C V_{0,C} = \frac{V_{0,C}}{X_C}$$

$$X_C = \frac{1}{\omega C}$$ ☛ capacitance reactance

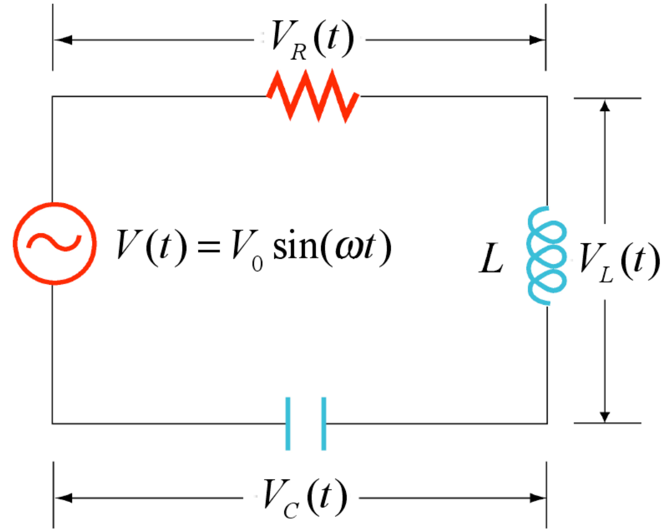
$$\phi_C = -\pi/2$$ ☛ phase constant

The current leads the voltage by $\pi/2$ in a capacitive circuit





10.3 Single Loop RLC AC Circuit



We'd like to find current in circuit

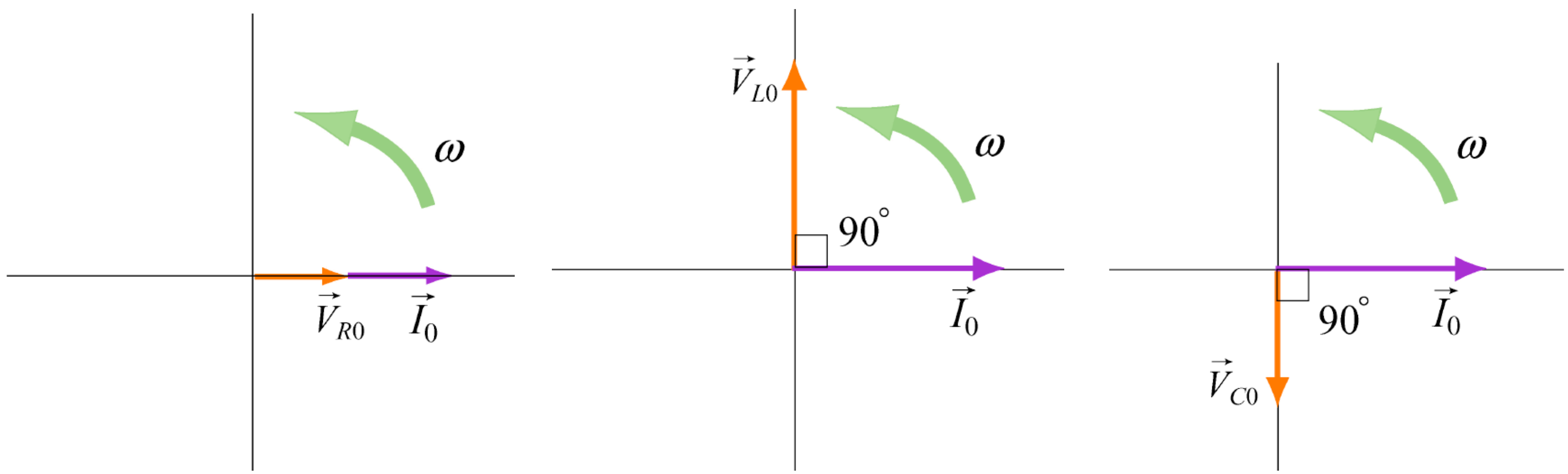
$$I(t) = I_0 \sin(\omega t - \phi)$$

Kirchhoff's loop rule yields

$$V(t) - V_R(t) - V_L(t) - V_C(t) = 0$$

$$V(t) = V_R(t) + V_L(t) + V_C(t)$$

Using phasor representation



$$\vec{V}_0 = \vec{V}_{0,R} + \vec{V}_{0,L} + \vec{V}_{0,C}$$

$$\vec{V}_0 = |\vec{V}_0| = |\vec{V}_{0,R} + \vec{V}_{0,L} + \vec{V}_{0,C}| = \sqrt{V_{0,R}^2 + (V_{0,L} - V_{0,C})^2}$$

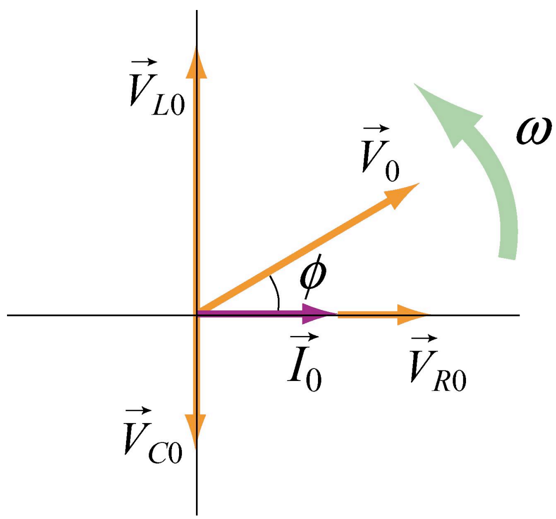
$$= \sqrt{(I_0 X_R)^2 + (I_0 X_L - I_0 X_C)^2} = I_0\sqrt{X_R^2 + (X_L - X_c)^2}$$

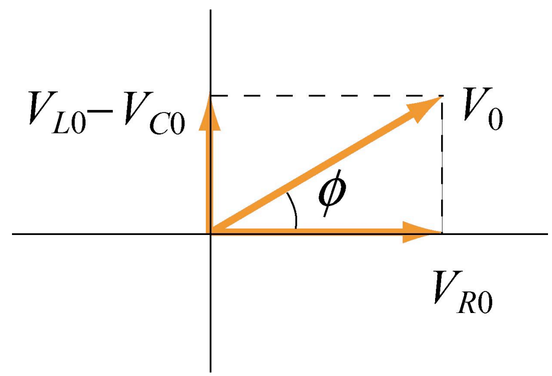
current amplitude

$$I_0 = \frac{V_0}{\sqrt{X_R^2 + (X_L - X_C)^2}} = \frac{V_0}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}}$$

phase

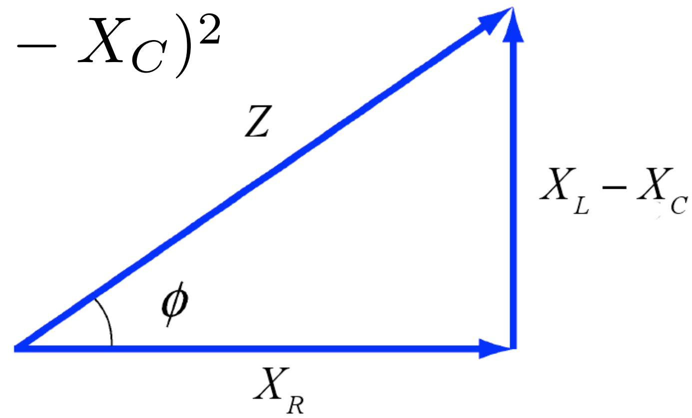
$$\tan\phi = \left(\frac{X_L - X_C}{X_R}\right) = \frac{1}{R}\left(\omega L - \frac{1}{\omega C}\right) \rightarrow \phi = \tan^{-1}\frac{1}{R}\left(\omega L - \frac{1}{\omega C}\right)$$





Note that ☛ $V_0 \neq V_{0,R} + V_{0,L} + V_{0,C}$

Impedance ☛ $$Z = \sqrt{X_R^2 + (X_l - X_C)^2}$$

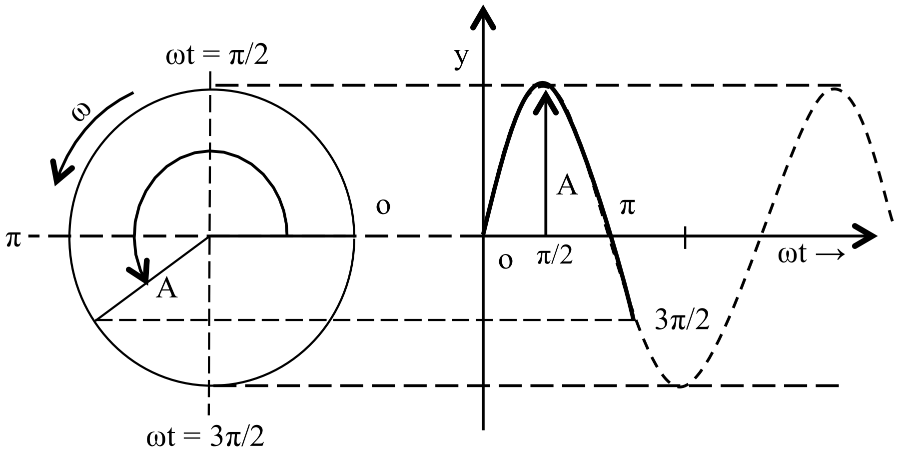


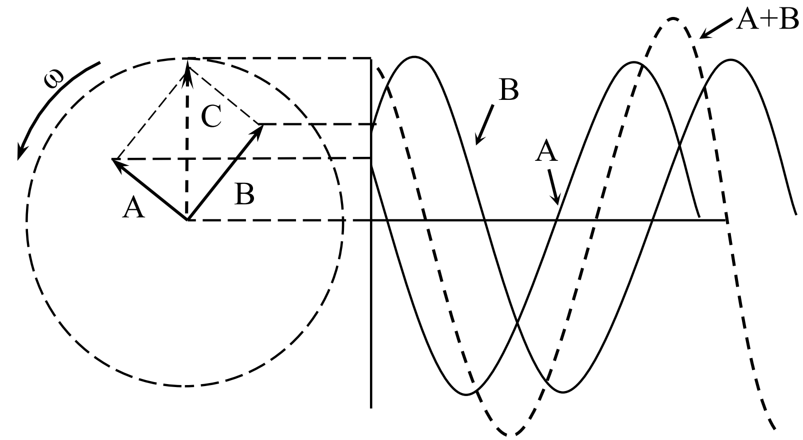
$$I(t) = \frac{V_0}{Z}\sin(\omega t)$$

Simple-circuit limits of the series RLC circuit

Simple Circuit	R	L	C	$X_L = \omega L$	$X_C = \dfrac{1}{\omega C}$	$\phi = \tan^{-1}\left(\dfrac{X_L - X_C}{X_R}\right)$	$Z = \sqrt{X_R{}^2 + (X_L - X_C)^2}$
purely resistive	R	0	∞	0	0	0	X_R
purely inductive	0	L	∞	X_L	0	$\pi / 2$	X_L
purely capacitive	0	0	C	0	X_C	$-\pi / 2$	X_C

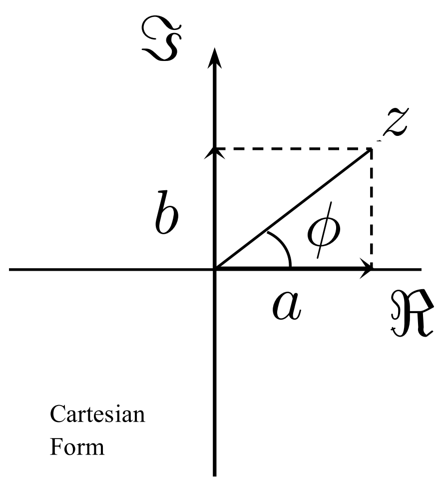
Sinusoidal functions of time

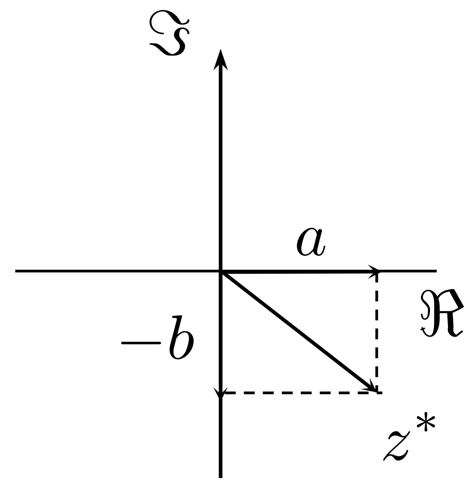


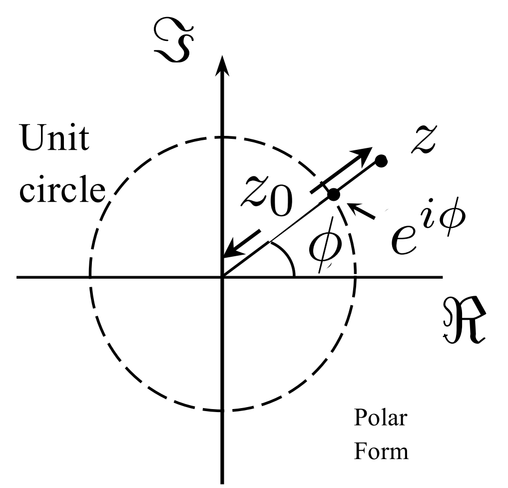


Complex numbers

$$z = a + ib \qquad z^* = a - ib \qquad zz^* = a^2 + b^2 = z_0^2$$







$$|z| = \sqrt{a^2 + b^2} \qquad \tan\phi = b/a \qquad z = z_0 e^{i\phi}$$

Complex Impedance

$$\tilde{Z} = \tilde{Z}_R + \tilde{Z}_L + \tilde{Z}_C$$

Impedance of ideal resistor is purely real

$$\tilde{Z}_R = R$$

Ideal inductors and capacitors have purely imaginary reactive impedance

$$\tilde{Z}_L = i\omega L \qquad \tilde{Z}_C = \frac{1}{i\omega C} = -\frac{i}{\omega C}$$

Substitution leads to ☛

$$\tilde{Z} = R + i\left(\omega L - \frac{1}{\omega C}\right)$$

Amplitude of impedance ☛

$$|\tilde{Z}| = \sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}$$

and its phase angle ☛

$$\tan\phi = \frac{(\omega L - \frac{1}{\omega C})}{R}$$

10.4 Resonance

In driven RLC series circuit ☛ amplitude of current has a maximum value: a resonance which occurs at the resonant angular frequency ω_0

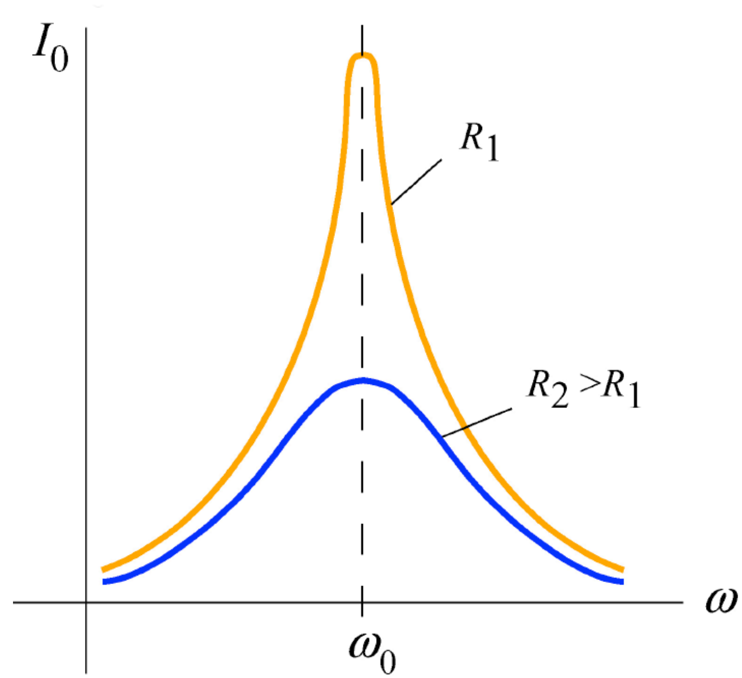
Because current amplitude is inversely proportional to impedance

I_0 maximum occurs when Z is minimum

This occurs at angular frequency ω_0 such that $X_L = X_C$

$$\omega_0 L = \frac{1}{\omega_0 C} \Rightarrow \omega_0 = \frac{1}{\sqrt{LC}}$$

At resonance ☛ $Z = R \Rightarrow I_0 = \dfrac{V_0}{R}$



10.5 Power in AC Circuits

In series RLC circuit ☛ instantaneous power delivered by AC generator is given by

$$P(t) = I(t)V(t) = \frac{V_0}{Z}\sin(\omega t - \phi)V_0\sin(\omega t) = \frac{V_0^2}{Z}\sin(\omega t)\sin(\omega t - \phi)$$
$$= \frac{V_0^2}{Z}[\sin^2(\omega t)\cos(\phi) - \sin(\omega t)\cos(\omega t)\sin(\phi)]$$

we have used trigonometric identity

$$\sin(\omega t - \phi) = \sin(\omega t)\cos(\phi) - \cos(\omega t)\sin(\phi)$$

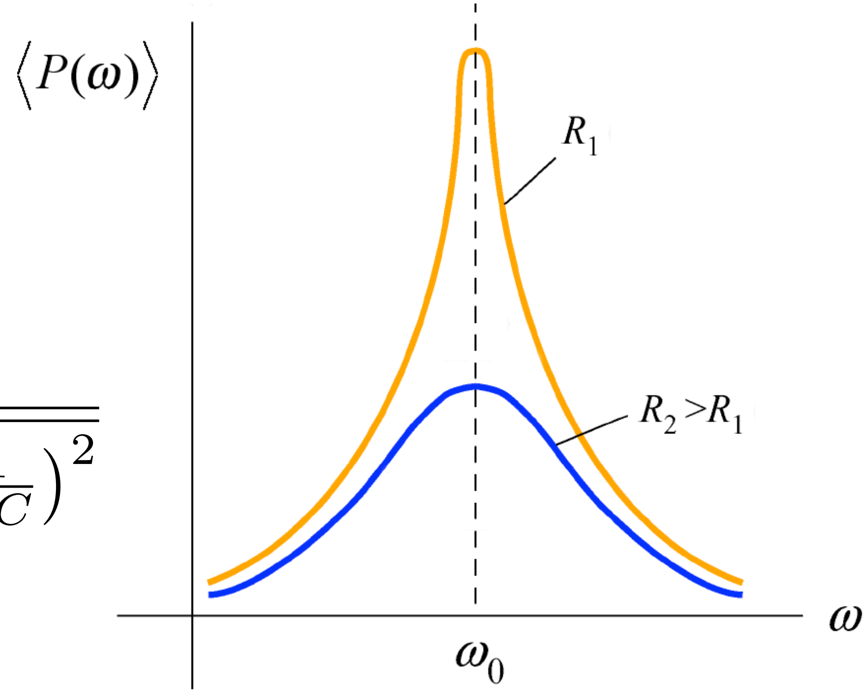
Time average of the power is

$$\langle P(\omega)\rangle = \frac{1}{T}\int_0^T \frac{V_0^2}{Z}\sin^2(\omega t)\cos(\phi)dt - \frac{1}{T}\int_0^T \frac{V_0^2}{Z}\sin(\omega t)\cos(\omega t)\sin(\phi)dt = \frac{1}{2}\frac{V_0^2}{Z}\cos\phi$$
$$= \frac{V_{\rm rms}^2}{Z}\cos\phi = I_{\rm rms}V_{\rm rms}\cos\phi$$

power factor ☛ $$\cos\phi = \frac{R}{Z}$$

$$\langle P(\omega)\rangle = I^2_{\rm rms}(\omega)R$$

$$I_{\rm rms}(\omega) = \frac{1}{\sqrt{2}}\frac{V_0}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}}$$



Maximum @ resonance condition ☛ $\cos\phi = 1$ or $Z = R$

$$\langle P(\omega_0)\rangle = I_{\rm rms}V_{\rm rms} = I^2_{\rm rms}R$$

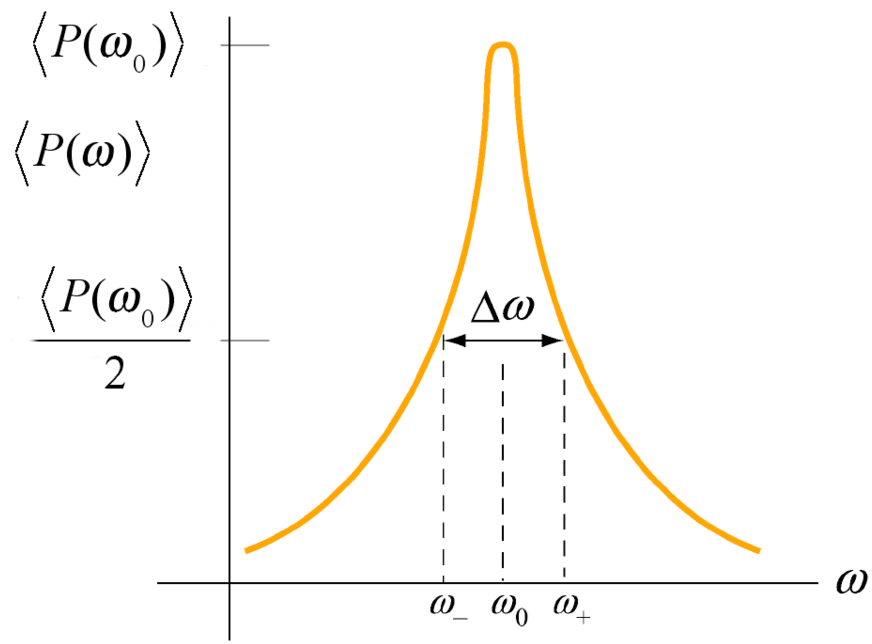
Width of the peak

The peak has a line width

One way to characterize the width is to define $\Delta\omega = \omega_+ - \omega_-$

$\omega_\pm$ ☛ values of driving angular frequency such that power is equal to half its maximum power at resonance

This is called full width at half maximum



Width $\Delta\omega$ increases with resistance R

To find $\Delta\omega$ it is instructive to first rewrite the average power as

$$\langle P(\omega)\rangle = \frac{1}{2}\frac{V_0^2 R}{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2} = \frac{1}{2}\frac{V_0^2 R\omega^2}{\omega^2 R^2 + L^2(\omega^2 - \omega_0^2)^2}$$

$$\langle P(\omega_0)\rangle = \frac{V_0^2}{2R}$$

condition for finding $\omega_\pm$ is

$$\frac{1}{2}\langle P(\omega_0)\rangle = \langle P(\omega_\pm)\rangle \Rightarrow \frac{V_0}{4R} = \frac{1}{2}\frac{V_0^2 R\omega_\pm^2}{\omega_\pm^2 R^2 + L^2(\omega_\pm^2 - \omega_0^2)^2}$$

after some algebra ☛ $(\omega_\pm^2 - \omega_0^2)^2 = (R\omega_\pm/L)^2$

Taking square roots yields two solutions which we analyze separately

Case 1: Taking the positive root leads to

$$\omega_+^2 - \omega_0^2 = +\frac{R\omega_+}{L} \Rightarrow \omega_+ = \frac{R}{2L} + \sqrt{\left(\frac{R}{4L}\right)^2 + \omega_0^2}$$

Case 2: Taking the negative yields

$$\omega_-^2 - \omega_0^2 = -\frac{R\omega_-}{L} \Rightarrow \omega_- = -\frac{R}{2L} + \sqrt{\left(\frac{R}{4L}\right)^2 + \omega_0^2}$$

width at half maximum ☛ $$\Delta\omega = \omega_+ - \omega_- = \frac{R}{L}$$

Quality factor

$$Q_{\rm qual} = \frac{\omega_0}{\Delta\omega} = \frac{\omega_0 L}{R}$$

10.6 Transformer

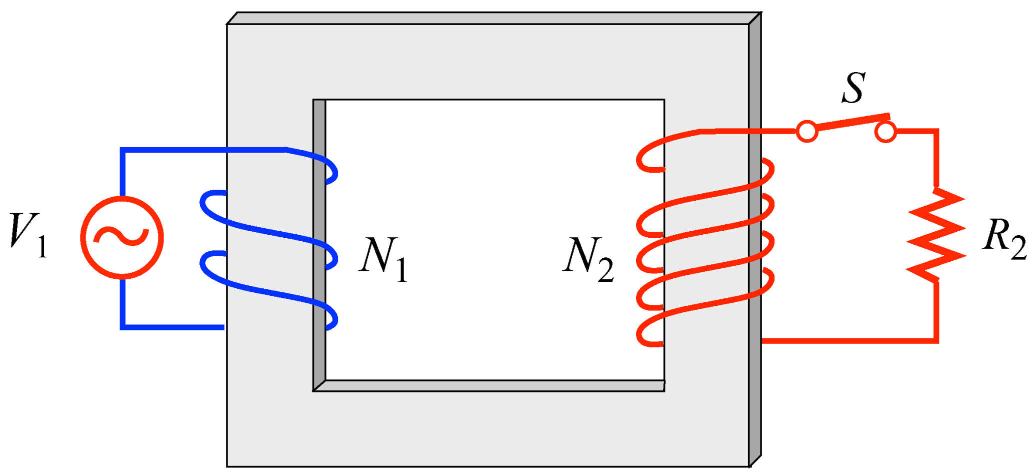
Transformer is device used to increase or decrease the AC voltage in a circuit

Typical device consists of two coils of wire

primary and secondary wound around an iron core

Primary coil with N_1 turns is connected to alternating voltage source $V(t)$

Secondary coil has N_2 turns and is connected to a load with resistance R_2



The way transformers operate is based on the principle that alternating current in primary coil will induce an alternating emf on secondary coil due to their mutual inductance

Neglecting small resistance in coil ☛ Faraday's law of induction implies

$$V_1 = -N_1 \frac{d\Phi_B}{dt}$$

Φ_B ☛ magnetic flux through one turn of primary coil

Iron core extends from primary to secondary coils

Iron core serves to increase magnetic field produced by current in primary coil

and ensures that nearly all magnetic flux through primary coil

also passes through each turn of the secondary coil

Voltage (or induced emf) across secondary coil is ☛ $V_2 = -N_2 \frac{d\Phi_B}{dt}$

Ideal transformer ☛ power loss due to Joule heating can be ignored

so that power supplied by primary coil is completely transferred to secondary coil

$$I_1 V_1 = I_2 V_2$$

In addition ☛ if no magnetic flux leaks out from iron core

through each turn is same in both primary and secondary coils

$$\frac{V_2}{V_1} = \frac{N_2}{N_1}$$

Transformation of currents in the two coils reads

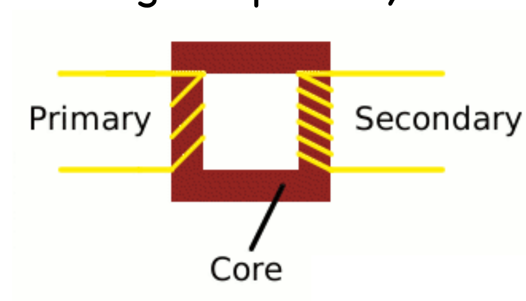
$$I_1 = \frac{V_2}{V_1} I_2 = \frac{N_2}{N_1} I_2$$

Ratio of output voltage to input voltage is determined by turn ratio $\dfrac{N_2}{N_1}$

$$N_2 > N_1 \Rightarrow V_2 > V_1$$

output voltage in secondary coil is greater than input voltage in primary coil

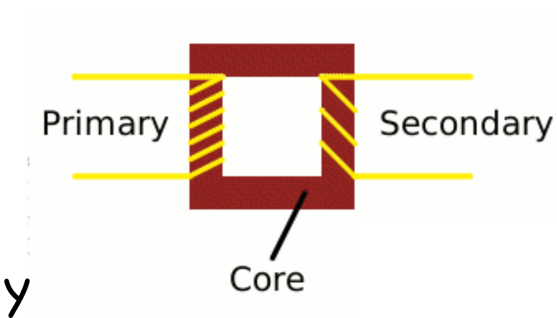
Transformer with $N_2 > N_1$ ☛ step-up transformer



$$N_1 > N_2 \Rightarrow V_1 > V_2$$

output voltage is smaller than input

Transformer with $N_1 > N_2$ ☛ step-down transformer



For home safety ☛ we would like LOW emf supply

Why do we use high voltages?

As electricity flows down a metal wire

electrons carrying its energy jiggle through the metal structure

That's why wires get hot when electricity flows through them

(useful for electric toasters and other appliances that use heating elements)

Electricity that comes from power plants is sent dow wires at extremely high voltages

to save energy

$$\langle P(\omega)\rangle = I_{\rm rms}^2 R$$

For power transmission ☛ we'd like to keep $I_{\rm rms}^2$ at **minimum**

The higher the voltage and the lower the current ☛ the less energy is wasted