Prof. Anchordoqui

Problems set # 7 **Physics 169** March 31, 2015

1. A particle of charge $-e$ is moving with an initial velocity v when it enters midway between two plates where there exists a uniform magnetic field pointing into the page, as shown in Fig. 1. You may ignore effects of the gravitational force. *(i)* Is the trajectory of the particle deflected upward or downward? *(ii)* What is the magnitude of the velocity of the particle if it just strikes the end of the plate?

Solution: *(i)* Choose unit vectors as shown in Fig. 1. The force on the particle is given by $\vec{F} = -e(v\hat{\imath} \times B\hat{\jmath}) = -evB\hat{k}$. The direction of the force is downward. *(ii)* Remember that when a charged particle moves through a uniform magnetic field, the magnetic force on the charged particle only changes the direction of the velocity hence leaves the speed unchanged so the particle undergoes circular motion. Therefore we can use Newtons second law in the form $evB = m\frac{v^2}{R}$. The speed of the particle is then $v = eBR/m$. In order to determine the radius of the orbit we note that the particle just hits the end of the plate. From the figure above, by the Pythagorean theorem, we have that $R62 = (R - d/2)^2 + l^2$ Expanding this equation yields $R^2 = R^2 - Rd + d^2/4 + l^2$. We can now solve for the radius of the circular orbit: $R = \frac{d}{4} + \frac{l^2}{d}$. We can now substitute this value in the equation for the velocity and find the speed necessary for the particle to just hit the end of the plate: $v = \frac{eB}{m}\left(\frac{d}{4} + \frac{l^2}{d}\right)$.

2. The entire $x - y$ plane to the right of the origin O is filled with a uniform magnetic field of magnitude B pointing out of the page, as shown in Fig. 2. Two charged particles travel along the negative x axis in the positive x direction, each with velocity $\vec{v}$, and enter the magnetic field at the origin O. The two particles have the same mass m, but have different charges, q_1 and q_2. When propagate thorugh the magnetic field, their trajectories both curve in the same direction (see sketch in Fig. 2), but describe semi-circles with different radii. The radius of the semi-circle traced out by particle 2 is exactly twice as big as the radius of the semi-circle traced out by particle 1. *(i)* Are the charges of these particles positive or negative? Explain your reasoning. *(ii)* What is the ratio q_2/q_1?

Solution: *(i)* Because $\vec{F}_B = q\vec{v} \times \vec{B}$, the charges of these particles are positive. *(ii)* We first find an expression for the radius R of the semi-circle traced out by a particle with charge q in terms of q, $v \equiv |\vec{v}|$, B, and m. The magnitude of the force on the charged particle is qvB and the magnitude of the acceleration for the circular orbit is v^2/R. Therefore applying Newtons second law yields $qvB = \frac{mv^2}{R}$. We can solve this for the radius of the circular orbit $R = \frac{mv}{qB}$. Therefore the charged ratio $\frac{q_2}{q_1} = \frac{mv/(R_2B)}{mv(R_1B)} = \frac{R_1}{R_2}$.

3. Shown in Fig. 3 are the essentials of a commercial mass spectrometer. This device is used to measure the composition of gas samples, by measuring the abundance of species of different masses. An ion of mass m and charge $q = +e$ is produced in source S, a chamber in which a gas discharge is taking place. The initially stationary ion leaves S, is accelerated by a potential difference $\Delta V > 0$, and then enters a selector chamber, S_1, in which there is an adjustable magnetic field $\vec{B}_1$, pointing

out of the page and a deflecting electric field $\vec{E}$, pointing from positive to negative plate. Only particles of a uniform velocity $\vec{v}$ leave the selector. The emerging particles at S_2, enter a second magnetic field B_2, also pointing out of the page. The particle then moves in a semicircle, striking an electronic sensor at a distance x from the entry slit. Express your answers to the questions below in terms of $E \equiv |\vec{E}|$, e, x, m, $B_2 \equiv |\vec{B}_2|$, and ΔV. *(i)* What magnetic field B_1 in the selector chamber is needed to insure that the particle travels straight through? *(ii)* Find an expression for the mass of the particle after it has hit the electronic sensor at a distance x from the entry slit.

Solution: *(i)* We first find an expression for the speed of the particle after it is accelerated by the potential difference ΔV, in terms of m, e, and ΔV. The change in kinetic energy is $\Delta K = \frac{1}{2}mv^2$. The change in potential energy is $\Delta U = -e\Delta V$. From conservation of energy, $\Delta K = -\Delta U$, we have that $\frac{1}{2}mv^2 = e\Delta V$. So the speed is $v = \sqrt{\frac{2e\Delta V}{m}}$ Inside the selector the force on the charge is given by $\vec{F} = e(\vec{E} + \vec{v} \times \vec{B}_1)$. If the particle travels straight through the selector then force on the charge is zero, therefore $\vec{E} = -\vec{v} \times \vec{B}_1$. Because the velocity is to the right in Fig. 3 (define this as the $+\hat{\imath}$ direction), the electric field points up (define this as the $+\hat{\jmath}$ direction) from the positive plate to the negative plate, and the magnetic field is pointing out of the page (define this as the $+\hat{k}$ direction). Then $E\hat{\jmath} = -v\hat{\imath} \times B_1\hat{k} = vB_1\hat{\jmath}$. Thus, $\vec{B}_1 = \frac{E}{v}\hat{k} = \sqrt{\frac{m}{2e\Delta V}}E\hat{k}$. *(ii)* The force on the charge when it enters the magnetic field $\vec{B}_2$ is given by $\vec{F} = ev\hat{\imath} \times B_2\hat{k} = -evB_2\hat{\jmath}$. This force points downward and forces the charge to start circular motion. You can verify this because the magnetic field only changes the direction of the velocity of the particle and not its magnitude which is the condition for circular motion. Recall that in circular motion the acceleration is towards the center. In particular when the particle just enters the field $\vec{B}_2$ the acceleration is downward $\vec{a} = -\frac{v^2}{x/2}\hat{\jmath}$. Newtons Second Law becomes $-evB_2 = -m\frac{v^2}{x/2}$. Thus, the particle hits the electronic sensor at a distance $x = \frac{2mv}{eB_2} = \frac{2}{eB_2}\sqrt{2em\Delta V}$ from the entry slit. The mass of the particle is then $m = \frac{eB_2^2x^2}{8\Delta V}$.

4. Electrons in a beam are accelerated from rest through a potential difference V. The beam enters an experimental chamber through a small hole. As shown in Fig. 4, the electron velocity vectors lie within a narrow cone of half angle ϕ oriented along the beam axis. We wish to use a uniform magnetic field directed parallel to the axis to focus the beam, so that all of the electrons can pass through a small exit port on the opposite side of the chamber after they travel the length d of the chamber. What is the required magnitude of the magnetic field? [*Hint:* Because every electron passes through the same potential difference and the angle ϕ is small, they all require the same time interval to travel the axial distance d.

Solution The electrons are all fired from the electron gun with the same speed v. Since $U_i = K_f$, we have $(-e)(-\Delta V) = \frac{1}{2}m_ev^2$, yielding $v = \sqrt{\frac{2e\Delta V}{m_e}}$. For ϕ small, $\cos\phi$ is nearly equal to 1. The time T of passage of each electron in the chamber is given by $d = vT$, and so $T = d\sqrt{\frac{m_2}{2e\Delta V}}$. Each electron moves in a different helix, around a different axis. If each completes just one revolution within the chamber, it will be in the right place to pass through the exit port. Its transverse velocity component $v = v\sin\phi$ swings around according to $F_\perp = ma_\perp$. Explicitly, $qv_\perp B\sin(\pi/2) = \frac{mv_\perp^2}{r}$, or equivalently $eB = \frac{m_ev_\perp}{r} = m_e\omega = m_e\frac{2\pi}{T}$, yielding $T = \frac{m_e2\pi}{eB} = d\sqrt{\frac{m_e}{2e\Delta V}}$.

Therefore, $\frac{2\pi}{B}\sqrt{\frac{m_e}{e}} = \frac{d}{\sqrt{2\Delta V}}$, which leads to $B = \frac{2\pi}{d}\sqrt{\frac{2m_e\Delta V}{e}}$.

5. Find the magnetic field at point P due to the current distribution shown in Fig. 5.

Solution: The fields due to the straight wire segments are zero at P because $d\vec{s}$ and $\hat{r}$ are parallel or anti-parallel. For the field due to the arc segment, the magnitude of the magnetic field due to a differential current carrying element is given in this case by $d\vec{B} = \frac{\mu_0 I}{4\pi}\frac{d\vec{s}\times\hat{r}}{R^2} = \frac{\mu_0}{4\pi}\frac{IRd\theta}{R^2}(\sin\theta\hat{\imath} - \cos\theta\hat{\jmath}) \times (-\cos\theta\hat{\imath} - \sin\theta\hat{\jmath}) = -\frac{\mu_0}{4\pi}\frac{I(\sin^2\theta+\cos^2\theta)d\theta}{R}\hat{k} = -\frac{\mu_0}{4\pi}\frac{Idtheta}{R}\hat{k}$. Therefore, $\vec{B} = -\int_0^{\pi/2}\frac{\mu_o I}{4\pi R}d\theta\hat{k} = -\frac{\mu_0 I}{8R}\hat{k}$ into the plane of Fig. 5

6. A nonconducting sphere has mass 80.0 g and radius 20.0 cm. A flat compact coil of wire with 5 turns is wrapped tightly around it, with each turn concentric with the sphere. As shown in Fig. 6, the sphere is placed on an inclined plane that slopes downward to the left, making an angle θ with the horizontal, so that the coil is parallel to the inclined plane. A uniform magnetic field of 0.350 T vertically upward exists in the region of the sphere. What current in the coil will enable the sphere to rest in equilibrium on the inclined plane? Show that the result does not depend on the value of θ.

Solution: The sphere is in translational equilibrium, thus $f_s - Mg\sin\theta = 0$, see Fig. 6. The sphere is in rotational equilibrium. If torques are taken about the center of the sphere, the magnetic field produces a clockwise torque of magnitude $\mu B\sin\theta$, and the frictional force a counterclockwise torque of magnitude f_sR, where R is the radius of the sphere. Hence, $f_sR - \mu B\sin\theta = 0$. Substituting the expression for f_s derived from the condition of translational equilibrium, $f_s = Mg\sin\theta$ into the condition for rotational equilibrium, and cancelling out $\sin\theta$, one obtains $\mu B = MgR$. Now, $\mu = NI\pi R^2$. Thus, $I = \frac{Mg}{\pi NBR} = \frac{0.08\text{ kg}\cdot 9.80\text{ m/s}^2}{5\pi\cdot 0.350\text{ T}\cdot 0.2\text{ m}} = 0.713$ A. The current must be counterclockwise as seen from above.

7. Two circular coils of radius a, each with N turns, are perpendicular to a common axis. Each coil carries a steady current I in the same direction, as shown in Fig. 7 *(i)* Find the total magnetic field on the axis $B_z^{\rm tot}$ as a function of z, using the midway point as the origin ($z = 0$). *(ii)* Show that $\frac{dB_z^{\rm tot}}{dz} = 0$ at the midpoint. This means that, functionally, at least at that one location, the field is not changing. *(iii)* If the separation b is picked correctly, then $\frac{d^2B_z^{\rm tot}}{dz^2} = 0$ at the midpoint. This configuration is known as Helmholtz coils. Determine the appropriate value of b. *(iv)* Show that the magnetic field on the axis near the origin, as an expansion in powers of z (up to z^4 inclusive) is given by

$$B_z^{\rm tot} = \frac{\mu_0 NIa^2}{d^3}\left[1 + \frac{3(b^2-a^2)z^2}{2d^4} + \frac{15(b^4 - 6b^2a^2 + 2a^4)z^4}{16d^8} + \cdots\right],$$

where $d^2 = a^2 + b^2/4$. *(v)* Show that the magnetic field on the z axis for large $|z|$ is given by the expansion in inverse odd powers of $|z|$ obtained from the small z expansion of part (iv) by the formal substitution $d \to |z|$. *(vi)* What is the maximum permitted value of $|z|/a$ if the field is to be uniform to one part in 10^4 (or one part in 10^2) in the Helmholtz configuration?

Solution *(i)* The magnetic field due to a circular hoop of radius a carrying a current I, which lies in the x-y plane with its center at the origin, is $B_z = \frac{\mu_0 I a^2}{2(z^2+a^2)^{3/2}}$. If the coils on the system have N turns, then the field from each coil is just N times larger. The magnetic field from both coils is then given by

$$\begin{aligned} B_z^{\text{tot}} &= B_{z_1} + B_{z_2} \\ &= \frac{\mu_0 N I a^2}{2} \left\{ \frac{1}{[a^2 + (b/2+z)^2]^{3/2}} + \frac{1}{[a^2 + (b/2-z)^2]^{3/2}} \right\} \end{aligned} \tag{1}$$

(ii) The derivative is

$$\frac{dB_z^{\text{tot}}}{dz} = \frac{3\mu_0 N I a^2}{2} \left\{ \frac{b/2 - z}{[a^2 + (b/2-z)^2]^{5/2}} - \frac{b/2 + z}{[a^2 + (b/2+z)^2]^{5/2}} \right\}.$$

It is easy to see that $\left.\frac{dB_z^{\text{tot}}}{dz}\right|_{z=0} = 0$. This is true regardless of the separation b. *(iii)* Differentiating again gives $\frac{d^2 B_z^{\text{tot}}}{dz^2} = \frac{3\mu_0 N I a^2}{2} \left\{ \frac{5(b/2+z)^2}{[a^2+(b/2+z)^2]^{7/2}} - \frac{1}{[a^2+(b/2+z)^2]^{5/2}} + \frac{5(b/2-z)^2}{[a^2+(b/2-z)^2]^{7/2}} - \frac{1}{[a^2+(b/2-z)^2]^{5/2}} \right\}$. At the midpoint, the second derivative becomes

$$\begin{aligned} \left.\frac{d^2 B_z^{\text{tot}}}{dz^2}\right|_{z=0} &= \frac{3\mu_0 N I a^2}{2} \left\{ \frac{10b^2/4}{[a^2 + b^2/4]^{7/2}} - \frac{2}{[a^2 + (b/2)^2]^{5/2}} \right\} \\ &= 3\mu_0 N I a^2 \left[\frac{a^2 + b^2/4 - 5b^2/4}{(a^2 + b^2/4)^{7/2}} \right] \\ &= 3\mu_0 N I a^2 \left[\frac{a^2 - b^2}{(a^2 + b^2/4)^{7/2}} \right]. \end{aligned}$$

Therefore, the condition $\left.\frac{d^2 B_z^{\text{tot}}}{dz^2}\right|_{z=0} = 0$ is met if $b = a$. *(iv)* All we must do now is to Taylor expand the terms to order z^4. Noting that we are seeking an expansion in powers of z/d^2, we may rewrite (1) as

$$\begin{aligned} B_z^{\text{tot}} &= \frac{\mu_0 N I a^2}{2} \left[(d^2 - bz + z^2)^{-3/2} + (d^2 + bz + z^2)^{-3/2} \right] \\ &= \frac{\mu_0 N I a^2}{2d^3} \left[(1 - b\zeta + d^2\zeta^2)^{-3/2} + (1 + b\zeta + d^2\zeta^2)^{-3/2} \right] \\ &= \frac{\mu_0 N I a^2}{2d^3} \left\{ \left[1 - b\zeta + (a^2 + b^2/4)\zeta^2\right]^{-3/2} + \left[1 + b\zeta + (a^2 + b^2/4)\zeta^2\right]^{-3/2} \right\}, \end{aligned} \tag{2}$$

where we have introduced $\zeta = z/d^2$. Expanding this in powers of ζ yields

$$B_z^{\text{tot}} = \frac{\mu_0 N I a^2}{2d^3} \left[1 + \frac{3}{2}(b^2 - a^2)\zeta^2 + \frac{15}{16}(b^4 - 6b^2a^2 + 2a^4)\zeta^4 + \cdots \right],$$

which is the desired result. The magnetic field along the axis changes very slowly when z is very small. *(v)* For large $|z|$ we Taylor expand B_z^{tot} in inverse powers of z, that is

$$B_z^{\text{tot}} = \frac{\mu_0 N I a^2}{2|z|^3} \left\{ \left[1 - bz^{-1} + (a^2 + b^2/4)z^{-2}\right]^{-3/2} + \left[1 + bz^{-1} + (a^2 + b^2/4)z^{-2}\right]^{-3/2} \right\}.$$

Comparing this with the last line of (2) shows that the Taylor series is formally equivalent under the substitution $\zeta \to z^{-1}$, which may be accomplished by taking $d \to |z|$. *(vi)* For $b = a$ the field is of the form

$$B_z^{\rm tot} = \frac{\mu_0 N I a^2}{2d^3}\left(1 - \frac{45}{16}\frac{a^4 z^4}{d^8} + \cdots\right) = \frac{4\mu_0 N I a^2}{5^{3/2}a^3}\left[1 - \frac{144}{125}\left(\frac{z}{a}\right)^4 + \cdots\right].$$

Taking the $(|z|/a)^4$ term as a small correction, the field non-uniformity is $\frac{\delta B_z^{\rm tot}}{B_z^{\rm tot}} \approx \frac{144}{125}\left(\frac{z}{a}\right)^4$. For uniformity to one part in 10^4, we find $|z|/a < 0.097$, while for uniformity to one part in 10^2, we instead obtain $|z|/a < 0.305$. These numbers are actually pretty good because of the fourth power. For example, the first value indicates we can move $\approx \pm 10\%$ of the distance between the coils while maintaining field uniformity at the level of 0.01%. Helmholtz coils are very useful in the lab for canceling out the Earth's magnetic field.

8. Let us treat the motion of an electron (charge $-e$, mass m) in a hydrogen atom classically. Suppose that an electron follows a circular orbit of radius r around a proton. What is the angular frequency ω_0 of the orbital motion? Suppose now that a small magnetic field $\vec{B}$ perpendicular to the plane of the orbit is switched on. Assuming that the radius of the orbit does not change, calculate the shift in the angular frequency $\Delta\omega = \omega_f - \omega_0$ of the orbital motion in terms of the magnitude B of the magnetic field, charge $-e$, mass m, and radius r. This is known as the "Zeeman effect."

Solution We first apply Newton's second law to find the angular frequency $\omega_)$ of the orbital motion. Coulombs law describes the force acting on the electron due to the electric interaction between the proton and the electron $\vec{F}_{\rm elec} = -\frac{1}{4\pi\epsilon_0}\frac{e^2}{r^2}\hat{r}$, where $\hat{r}$ is a unit vector in the plane of the circular orbit pointing radiallly outward. Because we are assuming the motion is circular the acceleration is $\vec{a} = -r\omega_0^2\hat{r}$. Newtons second law is then

$$-\frac{1}{4\pi\epsilon_0}\frac{e^2}{r^2}\hat{r} = -mr\omega_0^2\hat{r}. \tag{3}$$

We can now solve for the angular frequency

$$\omega_0 = \sqrt{\frac{e^2}{4\pi\epsilon_0 m r^3}}. \tag{4}$$

When a small magnetic field $\vec{B}$ perpendicular to the plane of the orbit is switched on there is magnetic force acting on the electron $\vec{F}_{\rm mag} = -e\vec{v} \times \vec{B}$. Let's first assume that the magnetic force points radially inward and that the radius of the orbit does not change. This will cause the electron to speed up hence increasing the angular frequency to $\omega_{\rm f}$. Choose coordinates as shown in Fig. 8. Assume that the magnetic field points in the positive z-direction. In order to have a radially inward force, we require that the velocity of the electron is $\vec{v} = r\omega_{\rm f}\hat{\theta}$. Then the magnetic force is given by $\vec{F}_{\rm mag} = -e\vec{v} \times \vec{B} = -er\omega_{\rm f}\hat{\theta} \times B\hat{k} = -er\omega_{\rm f}B\hat{r}$. Therefore Newton's second law is now $-\frac{e^2}{4\pi\epsilon_0 r^2}\hat{r} - er\omega_{\rm f}B\hat{r} = -mr\omega_{\rm f}^2\hat{r}$. Using (3) we can rewrite this as $-mr\omega_0^2\hat{r} - er\omega_{\rm f}B\hat{r} = -mr\omega_{\rm f}^2\hat{r}$ or $0 = \omega_{\rm f}^2 - \frac{e\omega_{\rm f}B}{m} - \omega_0^2$. We can solve this quadratic equation to obtain $\omega_{\rm f} = \frac{eB}{2m} \pm \sqrt{\frac{e^2B^2}{4m^2} + \omega_0^2}$. We choose the positive square root to keep $\omega_{\rm f} > 0$. We now

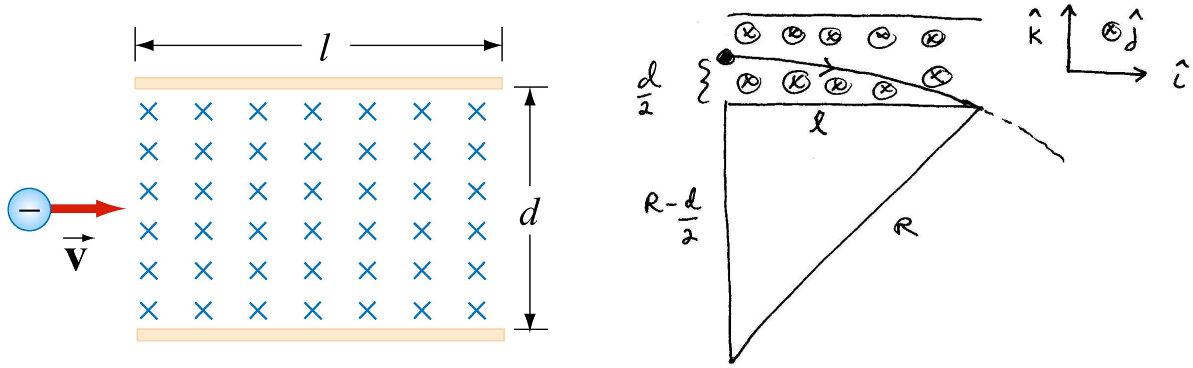

Figure 1: Problem 1.

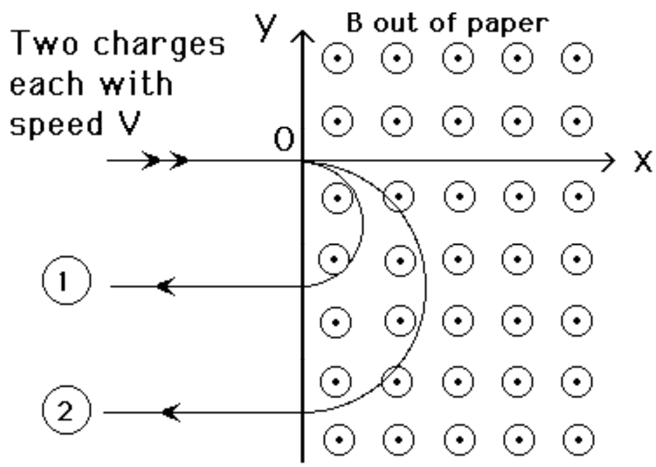

Figure 2: Problem 2.

substitute (4) into the above equation yielding $\omega_\mathrm{f} = \frac{eB}{2m} + \sqrt{\frac{e^2B^2}{4m^2} + \frac{e^2}{4\pi\epsilon_0 mr^3}}$. Then the change in angular frequency is $\Delta\Omega = \omega_\mathrm{f} - \omega_0 = \frac{eB}{2m} + \sqrt{\frac{e^2B^2}{4m^2} + \frac{e^2}{4\pi\epsilon_0 mr^3}} - \sqrt{\frac{e^2}{4\pi\epsilon_0 mr^3}}$. Suppose we reverse the direction of the magnetic field as shown in Fig. 8. Then the magnetic force points outward, $\vec{F}_\mathrm{mag} = -e\vec{v} \times \vec{B} = -er\omega_\mathrm{f}\hat{\theta} \times -B\hat{k} = +er\omega_\mathrm{f}B\hat{r}$, resulting in a smaller angular frequency ω_f . Repeating the analysis we just finished we have that $-mr\omega_0^2\hat{r} + er\omega_\mathrm{f}B\hat{r} = -mr\omega_\mathrm{f}^2\hat{r}$ or $0 = \omega_\mathrm{f}^2 + \frac{e\omega_\mathrm{f}B}{m} - \omega_0^2$. We can solve this quadratic equation for ω_f choosing the positive square root to keep $\omega_\mathrm{f} > 0$, that is $\omega_\mathrm{f} = -\frac{eB}{2m} \pm \sqrt{\frac{e^2B^2}{4m^2} + \omega_0^2}$. The change in angular frequency is now $\Delta\Omega = \omega_\mathrm{f} - \omega_0 = -\frac{eB}{2m} + \sqrt{\frac{e^2B^2}{4m^2} + \frac{e^2}{4\pi\epsilon_0 mr^3}} - \sqrt{\frac{e^2}{4\pi\epsilon_0 mr^3}}$.

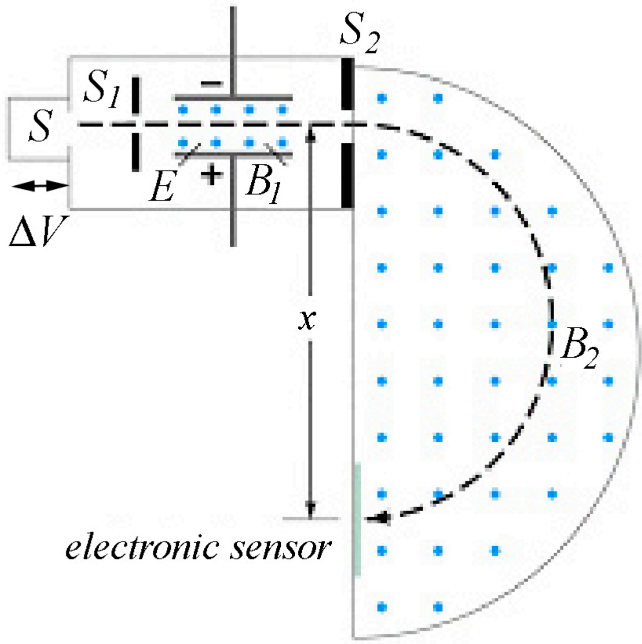

Figure 3: Problem 3.

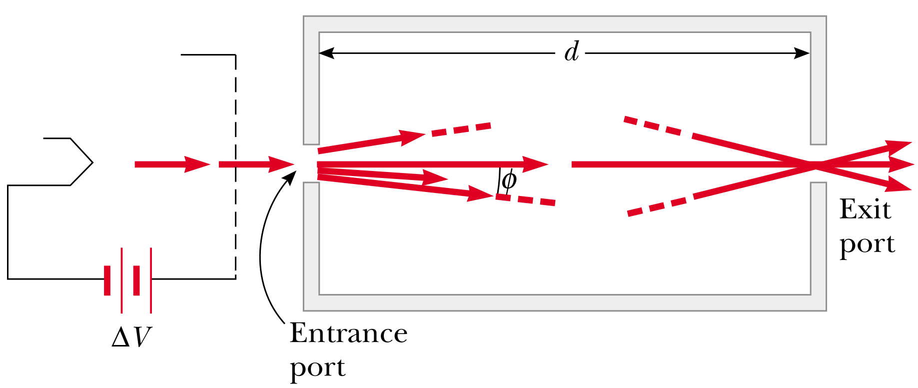

Figure 4: Problem 4.

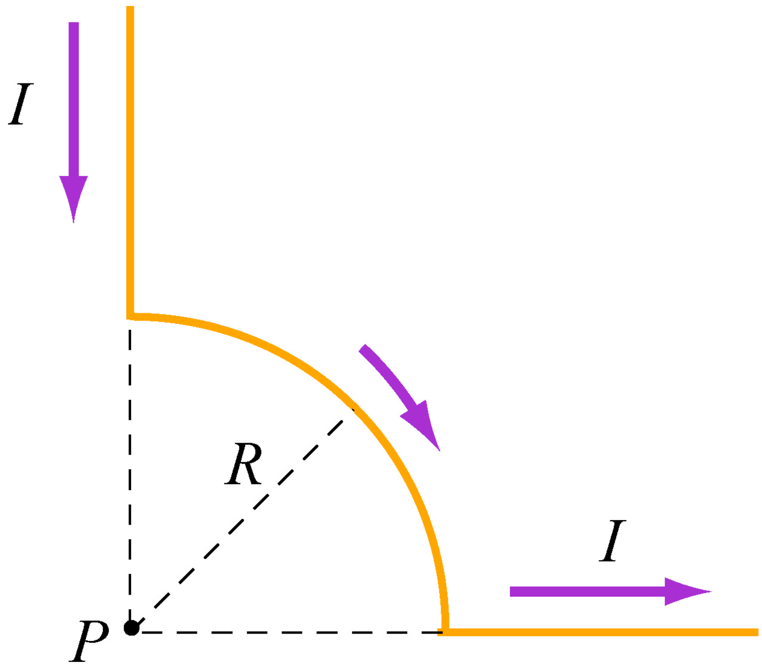

Figure 5: Problem 5.

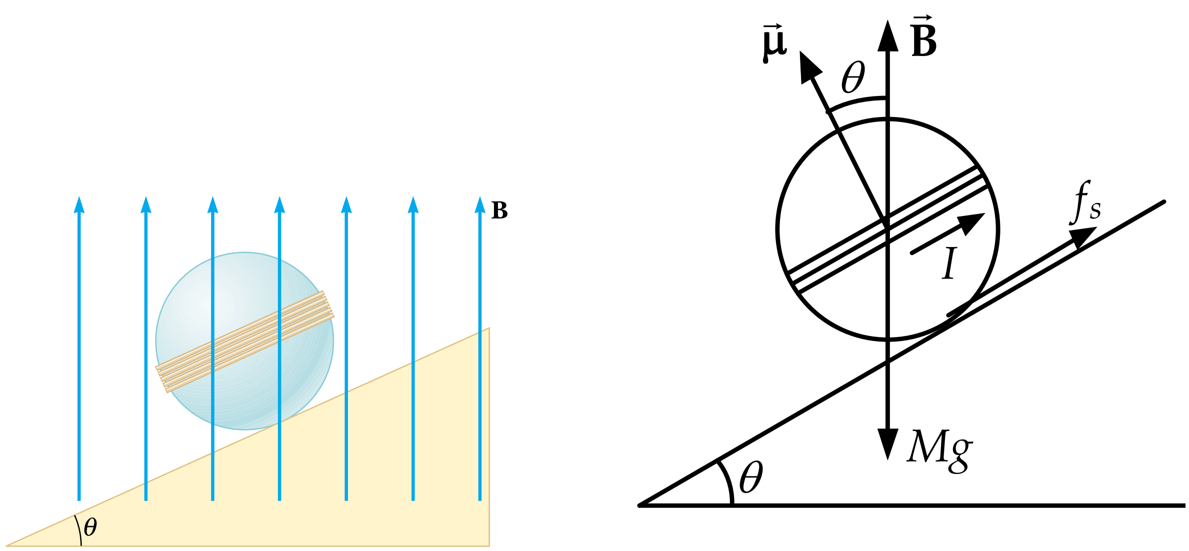

Figure 6: Problem 6.

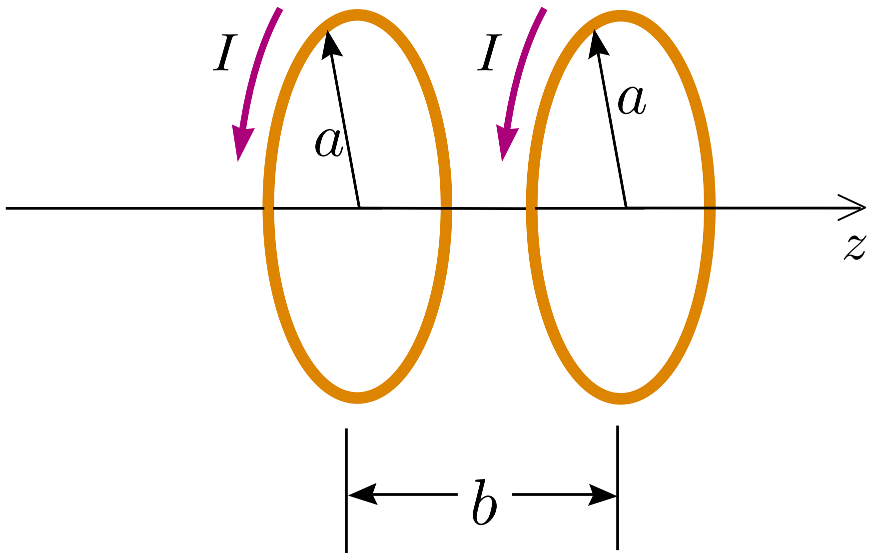

Figure 7: Problem 7.

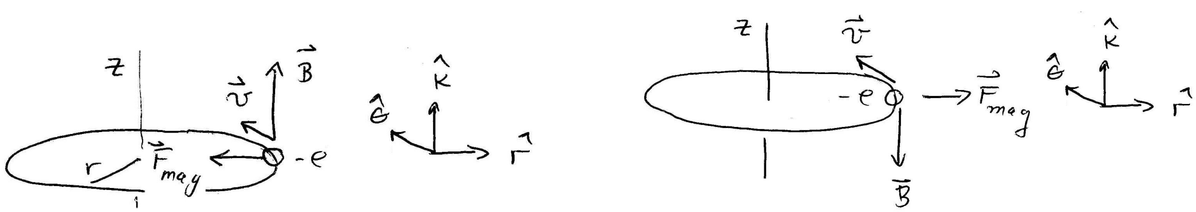

Figure 8: Problem 8.