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Problems set # 4 **Physics 169** March 3, 2015

1. *(i)* Eight equal charges q are located at the corners of a cube of side s, as shown in Fig. 1. Find the electric potential at one corner, taking zero potential to be infinitely far away. *(ii)* Four point charges are fixed at the corners of a square centered at the origin, as shown in Fig. 1. The length of each side of the square is $2a$. The charges are located as follows: $+q$ is at $(-a, +a)$, $+2q$ is at $(+a, +a)$, $-3q$ is at $(+a, -a)$, and $+6q$ is at $(-a, -a)$. A fifth particle that has a mass m and a charge $+q$ is placed at the origin and released from rest. Find its speed when it is a very far from the origin.

Solution: *(i)* To compute the potential all you need to know is that there are 3 charges a distance s away, 3 a distance $s\sqrt{2}$ away, and one charge a distance $s\sqrt{3}$ away. You find the potential due to each charge separately, and add the results via superposition: $V = \frac{q}{4\pi\epsilon_0 s}\left(3 + \frac{3}{\sqrt{2}} + \frac{1}{\sqrt{3}}\right) \approx 5.79\frac{q}{4\pi\epsilon_0 s}$. *(ii)* The diagram shows the four point charges fixed at the corners of the square and the fifth charged particle that is released from rest at the origin. We can use conservation of energy to relate the initial potential energy of the particle to its kinetic energy when it is at a great distance from the origin and the electrostatic potential at the origin to express $U_{\rm i}$. Use conservation of energy to relate the initial potential energy of the particle to its kinetic energy whenit is at a great distance from the origin: $\Delta K + \Delta U = 0$, or because $K_{\rm i} = U_{\rm f} = 0$, $K_{\rm f} - U_{\rm i} = 0$. Express the initial potential energy of the particle to its charge and the electrostatic potential at the origin: $U_{\rm i} = qV(0)$. Substitute for $K_{\rm f}$ and $U_{\rm i}$ to obtain: $\frac{1}{2}mv^2 - qV(0) = 0 \Rightarrow v = \sqrt{2qV(0)/m}$. Express the electrostatic potential at the origin: $V(0) = \frac{q}{4\pi\epsilon_0\sqrt{2}a}(1 + 2 - 3 + 6) = \frac{6q}{4\pi\epsilon_0\sqrt{2}a}$. Substitute for $V(0)$ and simplify to obtain: $v = q\sqrt{\frac{6\sqrt{2}}{4\pi\epsilon_0 ma}}$.

2. Five identical point charges $+q$ are arranged in two different manners as shown in Fig. 2: in once case as a face-centered square, in the other as a regular pentagon. Find the potential energy of each system of charges, taking the zero of potential energy to be infinitely far away. Express your answer in terms of a constant times the energy of two charges $+q$ separated by a distance a.

Solution: Using the principle of superposition, we know that the potential energy of a system of charges is just the sum of the potential energies for all the unique pairs of charges. The problem is then reduced to figuring out how many different possible pairings of charges there are, and what the energy of each pairing is. The potential energy for a single pair of charges, both of magnitude q, separated by a distance d is just: $PE_{\rm pair} = \frac{q^2}{4\pi\epsilon_0 d}$. Since all of the charges are the same in both configurations, all we need to do is figure out how many pairs there are in each situation, and for each pair, how far apart the charges are. How many unique pairs of charges are there? There are not so many that we couldn't just list them by brute force - which we will do as a check - but we can also calculate how many there are. In both configurations, we have 10 charges, and we want to choose all possible groups of 2 charges that are not repetitions. So far as potential energy is concerned, the pair $(2, 1)$ is the same as $(1, 2)$. Pairings like this are known as combinations, as opposed to permutations where $(1,2)$ and $(2,1)$ are not the same. It is straightforward to see that the ways of choosing pairs from five charges $= \binom{5}{2} = \frac{5!}{2!(5-2)!} = \frac{5\cdot4\cdot3\cdot2\cdot1}{2\cdot1\cdot3\cdot2\cdot1} = 10$. So there are 10 unique ways to choose 2 charges out of 5. First, lets consider the face-centered square lattice. In order to enumerate the possible pairings, we should label the charges to keep them straight. Label the corner charges $1-4$, and the center charge 5 (it doesnt matter which way you number the corners, just so long as 5 is the middle charge). Then our possible pairings are: $(1,2)$ $(1,3)$ $(1,4)$ $(1,5)$ $(2,3)$ $(2,4)$ $(2,5)$ $(3,4)$ $(3,5)$ $(4,5)$. There are ten, just as we expect. In this configuration, there are only three different distances that can separate a pair of charges: pairs on adjacent corners are a distance $a\sqrt{2}$ apart, a center-corner pairing is a distance a apart, and a far corner-far corner pair is $2a$ apart. We can take our list above, and sort it into pairs that have the same separation. We have four pairs of charges a distance a apart, four that are $a\sqrt{2}$ apart, and two that are $2a$ apart. Write down the energy for each type of pair listed in Table 1, multiply by the number of those pairs, and add the results together: $PE_{\text{square}} = 4(\text{center}-\text{corner pair}) + 2(\text{far corner pair}) + 4(\text{adjacent corner pair}) = \frac{q^2}{4\pi\epsilon_0 a}\left(4+1+4/\sqrt{2}\right) \approx 7.83\frac{q^2}{4\pi\epsilon_0 a}$. For the pentagon lattice, things are even easier. This time, just pick one charge as "1", and label the others from 2-5 in a clockwise or counter-clockwise fashion. Since we still have 5 charges, there are still 10 pairings, and they are the same as the list above. For the pentagon, however, there are only two distinct distances - either charges can be adjacent, and thus a distance a apart, or they can be next-nearest neighbors. What is the next-nearest neighbor distance? In a regular pentagon, each of the angles is $108°$, and in our case, each of the sides has length a, as shown in Fig. 2. We can use the law of cosines to find the distance d between next-nearest neighbors; $d^2 = a^2 + a^2 - 2a^2\cos 108° = 2a^2(1-\cos 10°) \Rightarrow d = a\sqrt{2-2\cos 108°} = a\phi \approx 1.618a$, where the number ϕ is known as the "Golden Ratio." The distances a and d automatically satisfy the golden ratio in a regular pentagon, $d/a = \phi$. Given the nearest neighbor distance in terms of a, we can then create a table of pairings for the pentagon; these are listed in Table 2. Now once again we write down the energy for each type of pair, and multiply by the number of pairs: $PE_{\text{pentagon}} = 5(\text{energy of adjacent pair}) + 5(\text{energy of next}-\text{nearest neighbor pair}) = \frac{5q^2}{4\pi\epsilon_0}\left(\frac{1}{a}+\frac{1}{d}\right) = \frac{5q^2}{4\pi\epsilon_0 a}\left[1+\frac{1}{\sqrt{2(1-\cos 180°)}}\right] \approx 8.09\frac{q^2}{4\pi\epsilon_0 a}$. So the energy of the pentagonal lattice is higher, meaning it is less favorable than the square lattice. Neither one is energetically favored though - since the energy is positive, it means that either configuration of charges is less stable than just having all five charges infinitely far from each other. This makes sense - if all five charges have the same sign, they dont want to arrange next to one another, and thus these arrangements cost energy to keep together. If we didnt force the charges together in these patterns, the positive energy tells us that they would fly apart given half a chance. For this reason, neither one is a valid sort of crystal lattice, real crystals have equal numbers of positive and negative charges, and are overall electrically neutral.

3. Consider a system of two charges shown in Fig. 3. Find the electric potential at an arbitrary point on the x axis and make a plot of the electric potential as a function of x/a.

Solution The electric potential can be found by the superposition principle. At a point on the x axis, we have $V(x) = \frac{1}{4\pi\epsilon_0}\frac{q}{x-a} + \frac{1}{4\pi\epsilon_0}\frac{(-q)}{|x+a|} = \frac{q}{4\pi\epsilon_0}\left[\frac{1}{|x-a|} - \frac{1}{|x+a|}\right]$. The above expression may be rewritten as $\frac{V(x)}{V_0} = \frac{1}{|x/a-1|} - \frac{1}{|x/a+1|}$, where $V_0 = \frac{q}{4\pi\epsilon_0 a}$. The plot of the dimensionless electric potential as a function of x/a is depicted in Fig. 3.

4. A point particle that has a charge of +11.1 nC is at the origin. *(i)* What is (are) the shapes of the equipotential surfaces in the region around this charge? *(ii)* Assuming the potential to be zero at $r = \infty$, calculate the radii of the five surfaces that have potentials equal to 20.0 V, 40.0 V, 60.0 V, 80.0 V and 100.0 V, and sketch them to scale centered on the charge. *(iii)* Are these surfaces equally spaced? Explain your answer. *(iv)* Estimate the electric field strength between the 40.0-V and 60.0-V equipotential surfaces by dividing the difference between the two potentials by the difference between the two radii. Compare this estimate to the exact value at the location midway between these two surfaces.

Solution: *(i)* The equipotential surfaces are spheres centered on the charge. *(ii)* From the relationship between the electric potential due to the point charge and the electric field of the point charge we have: $\int_a^b dV = -\int_{r_a}^{r_b} \vec{E} \cdot d\vec{r} = -\frac{Q}{4\pi\epsilon_0} \int_{r_a}^{r_b} r^{-2} dr$ or $V_b - V_a = \frac{1}{4\pi\epsilon_0}\left(\frac{1}{r_b} - \frac{1}{r_a}\right)$. Taking the potential to be zero at $r_a = \infty$ yields: $V_b - 0 = \frac{1}{4\pi\epsilon_0}\frac{1}{r_b} \Rightarrow V = \frac{Q}{4\pi\epsilon_0 r} \Rightarrow r = \frac{Q}{4\pi\epsilon_0 V}$. Because $Q = 1.1110^{-8}$ C, it follows that $r = 8.988 \times 10^9\ \text{N} \cdot \text{m}^2/\text{C}^2\ 1.11 \times 10^{-8}\ \text{C}\frac{1}{V}$. Now you can use the previous equation to determine the values of r:

V [V]	20.0	40.0	60.0	80.0	100.0
r [m]	4.99	2.49	1.66	1.25	1.00

The equipotential surfaces are shown in cross-section in Fig. 5. *(iii)* No. The equipotential surfaces are closest together where the electric field strength is greatest. *(iv)* The average value of the magnitude of the electric field between the 40.0-V and 60.0-V equipotential surfaces is given by: $E_{\text{est}} = -\frac{\Delta V}{\Delta r} = -\frac{40\ \text{V} - 60\ \text{V}}{2.49\ \text{m} - 1.66\ \text{m}} \simeq 29\frac{\text{V}}{\text{m}}$. The exact value of the electric field at the location midway between these two surfaces is given by $E = \frac{Q}{4\pi\epsilon_0 r^2}$, where r is the average of the radii of the 40.0-V and 60.0-V equipotential surfaces. Substitute numerical values and evaluate $E_{\text{exact}} = \frac{8.988\times10^9\ \text{N}\cdot\text{m}^2/\text{C}^2\ 1.11\times10^{-8}\ \text{C}}{(1.66\ \text{m}+2.49\ \text{m})^2/4} \simeq 23\frac{\text{V}}{\text{m}}$. The estimated value for E differs by about 21% from the exact value.

5. Two coaxial conducting cylindrical shells have equal and opposite charges. The inner shell has charge $+q$ and an outer radius a, and the outer shell has charge $-q$ and an inner radius b. The length of each cylindrical shell is L, and L is very long compared with b. Find the potential difference, $V_a - V_b$ between the shells.

Solution: The diagram shown in Fig. 5 is a cross-sectional view showing the charges on the inner and outer conducting shells. A portion of the Gaussian surface over which well integrate E in order to find V in the region $a < r < b$ is also shown. Once weve determined how E varies with r, from $V_b - V_a = -\int_a^b E_r dr$ we can find $V_a - V_b = \int_a^b E_r dr$ Apply Gauss' law to a cylindrical Gaussian surface of radius r and length L, $\oint \vec{E} \cdot \hat{n} dA = E_r 2\pi r L = \frac{q}{\epsilon_0}$. Solving for E_r yields: $E_r = \frac{q}{2\pi\epsilon_0 r L}$. Substitute for E_r and integrate from $r = a$ to b: $V_a - V_b = \frac{q}{2\pi\epsilon_0 L}\int_a^b \frac{dr}{r} = \frac{q}{2\pi\epsilon_0 L}\ln\left(\frac{b}{a}\right)$.

6. An electric potential $V(z)$ is described by the function

$$V(z) = \begin{cases} -2 \text{ V} \cdot \text{m}^{-1}\, z + 4 \text{ V}, & z > 2.0 \text{ m} \\ 0, & 1.0 \text{ m} < z < 2.0 \text{ m} \\ \frac{2}{3} \text{ V} - \frac{2}{3}\text{V} \cdot \text{m}^{-3}\, z^3, & 0 \text{ m} < z < 1.0 \text{ m} \\ \frac{2}{3} \text{ V} + \frac{2}{3}\text{V} \cdot \text{m}^{-3}\, z^3, & -1.0 \text{ m} < z < 0 \text{ m} \\ 0, & -2.0 \text{ m} < z < -1.0 \text{ m} \\ 2 \text{ V} \cdot \text{m}^{-1} z + 4 \text{ V}, & z < -2.0 \text{ m} \end{cases}$$

The graph in Fig. 6 shows the variation of an electric potential $V(z)$ as a function of z. *(i)* Give the electric field vector $\vec{E}$ for each of the six regions. *(ii)* Make a plot of the z-component of the electric field, E_z, as a function of z. Make sure you label the axes to indicate the numeric magnitude of the field.

<u>Solution</u> *(i)* Using $\vec{E} = -\vec{\nabla} V$, and noting that the electric potential only depends on the variable z, we have that the z-component of the electric field is given by $E_z = -\frac{dV}{dz}$. The electric field vector is then given by $\vec{E} = -\frac{dV}{dz}\hat{k}$. For $z > 2.0$ m, $\vec{E} = -\frac{dV}{dz}\hat{k} = -\frac{d}{dz}(-2 \text{ V} \cdot \text{m}^{-1}\, z + 4 \text{ V})\hat{k} = 2 \text{ V} \cdot \text{m}^{-1}\hat{k}$. For 1.0 m $< z <$ 2.0 m, $\vec{E} = -\frac{dV}{dx}\hat{k} = \vec{0}$. For 0 m $< z <$ 1.0 m, $\vec{E} = -\frac{dV}{dz}\hat{k} = -\frac{d}{dz}\left(\frac{2}{3} \text{ V} - \frac{2}{3} \text{ V} \cdot \text{m}^{-3} z^3\right)\hat{k} = 2 \text{ V} \cdot \text{m}^{-3} z^2 \hat{k}$. Note that the z^2 has units of $[m^2]$, so the value of the electric field at a point just inside, $z^- = 1.0 \text{ m} - \epsilon$ m (where $\epsilon > 0$ is a very small number), is given by $\vec{E}^- = 2 \text{ V} \cdot \text{m}^{-3}\ (1 \text{ m})^2 \hat{k} = 2\frac{\text{V}}{\text{m}}\hat{k}$. Note that the z-component of the electric field, $2 \text{ V} \cdot \text{m}^{-1}$, has the correct units. For -1.0 m $< z <$ 0 m, $\vec{E} = -\frac{dV}{dz}\hat{k} = -\frac{d}{dz}\left(\frac{2}{3} \text{ V} + \frac{2}{3} \text{ V} \cdot \text{m}^{-3} z^3\right)\hat{k} = -2 \text{ V} \cdot \text{m}^{-3} z^2 \hat{k}$. The value of the electric field at a point just inside, $z^+ = -1.0 \text{ m} + \epsilon$ m, is given by $\vec{E}^+ = -2 \text{ V} \cdot \text{m}^{-3}\ (1 \text{ m})^2 \hat{k} = -2\frac{\text{V}}{\text{m}}\hat{k}$. For -2.0 m $< z < -1.0$ m, $\vec{E} = -\frac{dV}{dz}\hat{k} = \vec{0}$. For $z < -2.0$ m, $\vec{E} = -\frac{dV}{dz}\hat{k} = -\frac{d}{dz}(2 \text{ V} \cdot \text{m}^{-1} z + 4 \text{ V})\hat{k} = -2 \text{ V} \cdot \text{m}^{-1}\hat{k}$. *(ii)* The z-component of the electric field as a function of z is shown in Fig. 6.

7. Two conducting, concentric spheres have radii a and b. The outer sphere is given a charge Q. What is the charge on inner sphere if it is earthed?.

<u>Solution:</u> The system of conducting concentric spheres is shown in Fig. 7. When the object is earthed, it means its potential is zero, but note that the charge on it may not be zero. To determine the charge, take the potential on the inner sphere as zero and assume that the charge on it is q. Since $V(r) - V(\infty) = -\int_\infty^r E(r')dr'$, the potential difference at $r = a$ is then $V(a) - V(\infty) = \frac{1}{4\pi\epsilon_0}\left(\frac{q}{a} - \frac{q}{b} + \frac{Q+q}{b}\right) = 0$, where we have taken the zero of potential at infinity. Therefore, $\frac{q}{a} - \frac{q}{b} + \frac{Q}{b} + \frac{q}{b} = 0$, yielding $q = -Q\frac{a}{b}$.

8. Consider two nested, spherical conducting shells. The first has inner radius a and outer radius b. The second has inner radius c and outer radius d. The system is shown in Fig. 8. In the following four situations, determine the total charge on each of the faces of the conducting spheres (inner and outer for each), as well as the electric field and potential everywhere in space (as a function of distance r from the center of the spherical shells). In all cases the shells begin uncharged, and a charge is then instantly introduced somewhere. *(i)* Both shells are not connected

to any other conductors (floating) – that is, their net charge will remain fixed. A positive charge $+Q$ is introduced into the center of the inner spherical shell. Take the zero of potential to be at infinity. *(ii)* The inner shell is not connected to ground (floating) but the outer shell is grounded – that is, it is fixed at $V = 0$ and has whatever charge is necessary on it to maintain this potential. A negative charge $-Q$ is introduced into the center of the inner spherical shell. *(iii)* The inner shell is grounded but the outer shell is floating. A positive charge $+Q$ is introduced into the center of the inner spherical shell. *(iv)* Finally, the outer shell is grounded and the inner shell is floating. This time the positive charge $+Q$ is introduced into the region in between the two shells. In this case the question "What are $\vec{E}(r)$ and $V(r)$?" cannot be answered analytically in some regions of space. In the regions where these questions can be answered analytically, give answers. In the regions where they cannot be answered analytically, explain why, but try to draw what you think the electric field should look like and give as much information about the potential as possible.

Solution: *(i)* There is no electric field inside a conductor. In addition, the net charge on an isolated conductor is zero (i.e. $Q_a + Q_b = Q_c + Q_d = 0$), yielding $Q_a = -Q$, $Q_b = +Q$, $Q_c = -Q$, $Q_d = +Q$. Using Gauss' law,

$$\vec{E}(r) = \begin{cases} \frac{Q}{4\pi\epsilon_0 r^2}\hat{r}, & r > d \\ \vec{0}, & c < r < d \\ \frac{Q}{4\pi\epsilon_0 r^2}\hat{r}, & b < r < c \\ \vec{0}, & a < r < b \\ \frac{Q}{4\pi\epsilon_0 r^2}\hat{r}, & r < a \end{cases}.$$

The field lines are shown in Fig. 9. Since $V(r) - V(\infty) = -\int_\infty^r E(r')dr'$, the potential difference is then:

$$V(r) - V(\infty) = \begin{cases} \frac{Q}{4\pi\epsilon_0 r}, & r > d \\ \frac{Q}{4\pi\epsilon_0 d}, & c < r < d \\ \frac{Q}{4\pi\epsilon_0}\left(\frac{1}{r} - \frac{1}{c} + \frac{1}{d}\right), & b < r < c \\ \frac{Q}{4\pi\epsilon_0}\left(\frac{1}{b} - \frac{1}{c} + \frac{1}{d}\right), & a < r < b \\ \frac{Q}{4\pi\epsilon_0}\left(\frac{1}{r} - \frac{1}{a} + \frac{1}{b} - \frac{1}{c} + \frac{1}{d}\right), & r < a \end{cases}.$$

(ii) Since the outer shell is now grounded, $Q_d = 0$ to maintain $\vec{E}(r) = \vec{0}$ outside the outer shell. We have, $Q_a = Q$, $Q_b = -Q$, $Q_c = +Q$, $Q_d = 0$. Again using Gauss law yields:

$$\vec{E}(r) = \begin{cases} \vec{0}, & r > c \\ -\frac{Q}{4\pi\epsilon_0 r^2}\hat{r}, & b < r < c \\ \vec{0}, & a < r < b \\ -\frac{Q}{4\pi\epsilon_0 r^2}\hat{r}, & r < a \end{cases}.$$

The field lines are shown in Fig. 9. The potential difference is then

$$V(r) - V(\infty) = \begin{cases} 0, & r > c \\ -\frac{Q}{4\pi\epsilon_0}\left(\frac{1}{r} - \frac{1}{c}\right), & b < r < c \\ -\frac{Q}{4\pi\epsilon_0}\left(\frac{1}{b} - \frac{1}{c}\right), & a < r < b \\ -\frac{Q}{4\pi\epsilon_0}\left(\frac{1}{r} - \frac{1}{a} + \frac{1}{b} - \frac{1}{c}\right), & r < a \end{cases}.$$

(iii) The inner shell is grounded and $Q_b = 0$ to maintain $\vec{E}(r) = \vec{0}$ outside the inner shell. Because there is no electric field on the outer shell, $Q_a = -Q$, $Q_b = Q_c = Q_d = 0$. Gauss law then yields

$$\vec{E}(r) = \begin{cases} \vec{0}, & r > a \\ \frac{Q}{4\pi\epsilon_0 r^2}\hat{r}, & r < a \end{cases}.$$

The field lines are shown in Fig. 9. The potential difference is then

$$V(r) - V(\infty) = \begin{cases} 0, & r > a \\ \frac{Q}{4\pi\epsilon_0}\left(\frac{1}{r} - \frac{1}{a}\right), & r < a \end{cases}.$$

(iv) The electric field within the cavity is zero. If there is any field line that began and ended on the inner wall, the integral $\oint \vec{E} \cdot d\vec{s}$ over the closed loop that includes the field line would not be zero. This is impossible since the electrostatic field is conservative, and therefore the electric field must be zero inside the cavity. The charge Q between the two conductors pulls minus charges to the near side on the inner conducting shell and repels plus charges to the far side of that shell. However, the net charge on the outer surface of the inner shell (Q_b) must be zero since it was initially uncharged (floating). Since the outer shell is grounded, $Q_d = 0$ to maintain $\vec{E}(r) = \vec{0}$ outside the outer shell. Thus, $Q_a = Q_b = Q_d = 0$, $Q_c = -Q$ and $\vec{E}(r) = \vec{0}$, for $r < b$ or $r > c$. For $b < r < c$, $\vec{E}(r)$ is in fact well defined but the functional form is very complicated. The field lines are shown in Fig. 9. What can we say about the electric potential? $V(r) = 0$ for $r > c$, and $V(r) =$ constant for $r < b$, but between the two shells the functional form of the potential is very complicated.

9. The hydrogen atom in its ground state can be modeled as a positive point charge of magnitude $+e$ (the proton) surrounded by a negative charge distribution that has a charge density (the electron) that varies with the distance from the center of the proton r as: $\rho(r) = -\rho_0 e^{-2r/a}$ (a result obtained from quantum mechanics), where $a = 0.523$ nm is the most probable distance of the electron from the proton. *(i)* Calculate the value of ρ_0 needed for the hydrogen atom to be neutral. *(ii)* Calculate the electrostatic potential (relative to infinity) of this system as a function of the distance r from the proton.

Solution: *(i)* Express the charge dq in a spherical shell of volume $dV = 4\pi r^2 dr$ at a distance r from the proton: $dq = \rho dV = -\rho_0 e^{-2r/a} 4\pi r^2 dr$. Express the condition for charge neutrality: $e = -4\pi\rho_0 \int_0^\infty r^2 e^{-2r/a} dr$. From the table of integrals we have $\int x^2 e^{bx} dx = \frac{e^{bx}}{b^3}(b^2x^2 - 2bx + 2)$. Using this result yields $\int_0^\infty r^2 e^{-2r/a} dr = a^3/4$. Substitute in the expression for e to obtain: $e = -\pi\rho_0 a^3 \Rightarrow \rho_0 = -\frac{e}{\pi a^3}$. Substitute numerical values to obtain $\rho_0 = -\frac{1.602\times10^{-19}\text{ C}}{\pi(0.523\text{ nm})^3} = -3.56 \times 10^8\text{ C/m}^3$. *(ii)* The electrostatic potential of this proton-electron system is the sum of the electrostatic potentials due to the proton and the electron's charge density: $V = V_1 + V_2$, where $V_1 = \frac{1}{4\pi\epsilon_0}\left(\frac{e}{r} + \frac{Q_1}{r}\right)$, $V_2 = \int_r^\infty \frac{1}{4\pi\epsilon_0}\frac{\rho(r')}{r'} 4\pi r'^2 dr'$, and $Q_1 = \int_0^r \rho(r') 4\pi r'^2 dr'$. Substituting for $\rho(r')$ in the expression for Q_1 yields: $Q_1 = 4\pi\rho_0 \int_0^r r'^2 e^{-2r'/a} dr'$. Using again $\int x^2 e^{bx} dx = \frac{e^{bx}}{b^3}(b^2x^2 - 2bx + 2)$ we evaluate $\int_0^r x^2 e^{-2x/a} dx = -\frac{a^3 e^{-2x/a}}{8}\left(\frac{4}{a^2}x^2 + 2\frac{2}{a}x + 2\right)\Big|_0^r = -\frac{a^3 e^{-2r/a}}{8}\left(\frac{4}{a^2}r^2 + \frac{4}{a}r + 2\right) + \frac{a^3}{4}$ and $Q_1 = 4\pi\rho_0\left[-\frac{a^3 e^{-2r/a}}{8}\left(\frac{4}{a^2}r^2 + \frac{4}{a}r + 2\right) + \frac{a^3}{4}\right]$. Substituting for Q_1 in the expression for V_1 yields: $V_1 = \frac{1}{4\pi\epsilon_0}\left\{\frac{e}{r} + \frac{4\pi\rho_0}{r}\left[-\frac{a^3 e^{-2r/a}}{8}\left(\frac{4}{a^2}r^2 + \frac{4}{a}r + 2\right) + \frac{a^3}{4}\right]\right\}$. Substitute for ρ_0 from *(i)* and simplify to obtain:

$V_1 = \frac{1}{4\pi\epsilon_0}\left\{\frac{e}{r} - \frac{4e}{ra^3}\left[-\frac{a^3e^{-2r/a}}{8}\left(\frac{4}{a^2}r^2 + \frac{4}{a}r + 2\right) + \frac{a^3}{4}\right]\right\} = \frac{1}{4\pi\epsilon_0}\frac{e}{r}e^{-2r/a}\left(\frac{2}{a^2}r^2 + \frac{2}{a}r + 1\right)$. Substituting for $\rho(r')$ and simplifying yields: $V_2 = \int_r^\infty \frac{1}{4\pi\epsilon_0 r'}\rho_0 e^{-2r'/a}4\pi r'^2 dr' = \frac{\rho_0}{\epsilon_0}\int_r^\infty e^{-2r'/a}r'dr'$. From a table of integrals we have: $\int xe^{bx}dx = \frac{e^{bx}}{b^2}(bx-1)$. Using this result we evaluate $\int_r^\infty e^{-2x/a}xdx = \frac{a^2}{4}e^{-2x/a}\left(\frac{2}{a}x+1\right)\Big|_r^\infty = -\frac{a^2}{4}e^{-2x/a}\left(\frac{2}{a}r+1\right)$. Substitute for $\int_r^\infty e^{-2r/a}r'dr'$ and ρ_0 in the expression for V_2 to obtain $V_2 = \frac{1}{\epsilon_0}\left(-\frac{e}{\pi a^3}\right)e^{-2r/a}\left[-\frac{a^2}{4}\left(\frac{2}{a}r+1\right)\right] = \frac{1}{4\pi\epsilon_0}e\frac{1}{a}e^{-2r/a}\left(\frac{2}{a}r+1\right)$. Substituting for V_1 and V_2 in $V = V_1 + V_2$ and simplifying yields: $V = \frac{1}{4\pi\epsilon_0}\frac{e}{r}e^{-2r/a}\left(\frac{2}{a^2}r^2 + \frac{2}{a}r + 1\right) + \frac{1}{4\pi\epsilon_0}\frac{e}{a}e^{-2r/a}\left(\frac{2}{a}r+1\right) = \frac{e}{4\pi\epsilon_0}\left(\frac{1}{a}+\frac{1}{r}\right)e^{-2r/a}$.

10. A particle that has a mass m and a positive charge q is constrained to move along the x-axis. At $x = -L$ and $x = L$ are two ring charges of radius L. Each ring is centered on the x-axis and lies in a plane perpendicular to it. Each ring has a total positive charge Q uniformly distributed on it. *(i)* Obtain an expression for the potential $V(x)$ on the x axis due to the charge on the rings. *(ii)* Show that $V(x)$ has a minimum at $x = 0$. *(iii)* Show that for $x << L$, the potential approaches the form $V(x) = V(0) + \alpha x^2$. *(iv)* Use the result of Part *(iii)* to derive an expression for the angular frequency of oscillation of the mass m if it is displaced slightly from the origin and released. (Assume the potential equals zero at points far from the rings.)

Solution: *(i)* Express the potential due to the ring charges as the sum of the potentials due to each of their charges: $V(x) = V_{\substack{\text{ringto}\\\text{theleft}}} + V_{\substack{\text{ringto}\\\text{theright}}}$. The potential for a ring of charge is $V(x) = \frac{Q}{4\pi\epsilon_0}\frac{1}{\sqrt{x^2+a^2}}$ where a is the radius of the ring and Q is its charge. For the ring to the left we have: $V_{\substack{\text{ringto}\\\text{theleft}}} = \frac{Q}{4\pi\epsilon_0}\frac{1}{\sqrt{(x+L)^2+L^2}}$. For the ring to the right we have: $V_{\substack{\text{ringto}\\\text{theright}}} = \frac{Q}{4\pi\epsilon_0}\frac{1}{\sqrt{(x-L)^2+L^2}}$. Substitute for $V_{\substack{\text{ringto}\\\text{theleft}}}$ and $V_{\substack{\text{ringto}\\\text{theright}}}$ to obtain $V(x) = \frac{Q}{4\pi\epsilon_0}\left(\frac{1}{\sqrt{(x+L)^2+L^2}} + \frac{1}{\sqrt{(x-L)^2+L^2}}\right)$. *(ii)* To show that $V(x)$ is a minimum at $x = 0$, we must show that the first derivative of $V(x) = 0$ at $x = 0$ and that the second derivative is positive. Evaluate the first derivative to obtain $\frac{dV}{dx} = \frac{Q}{4\pi\epsilon_0}\left\{\frac{L-x}{[(L-x)^2+L^2]^{3/2}} - \frac{L+x}{[(L+x)^2+L^2]^{3/2}}\right\} = 0$ for extrema. Solving for x yields $x = 0$. Evaluate $\frac{d^2V}{dx^2} = \frac{Q}{4\pi\epsilon_0}\left\{\frac{3(L-x)^2}{[(L-x)^2+L^2]^{5/2}} - \frac{1}{[(L-x)^2+L^2]^{3/2}} + \frac{3(L+x)^2}{[(L+x)^2+L^2]^{5/2}} - \frac{1}{[(L+x)^2+L^2]^{3/2}}\right\}$. Evaluating this expression for $x = 0$ yields: $\frac{d^2V(0)}{dx^2} = \frac{Q}{4\pi\epsilon_0}\frac{1}{2\sqrt{2}L^3} > 0$. Thus, $V(x)$ is a minimum at $x = 0$. *(iii)* Use a Taylor expansion to show that, for $x \ll L$, the potential approaches the form $V(x) = V(0) + \alpha x^2$. The Taylor expansion of $V(x)$ is: $V(x) = V(0) + V'(0)x + \frac{1}{2}V''(0)x^2 +$ higher order terms. For $x \ll L$, $V(x) \approx V(x) \approx V(0) + V'(0)x + \frac{1}{2}V''(0)x^2$. Substitute the results from *(i)* and *(ii)* to obtain: $V(x) = \frac{Q}{4\pi\epsilon_0}\left(\frac{\sqrt{2}}{L} + \frac{1}{4\sqrt{2}L^3}x^2\right)$, or $V(x) = V(0) + \alpha x^2$, where $V(0) = \frac{Q}{4\pi\epsilon_0}\frac{\sqrt{2}}{L}$ and $\alpha = \frac{Q}{4\pi\epsilon_0}\frac{1}{4\sqrt{2}L^3}$. *(iv)* we can obtain the potential energy function from the potential function and, noting that it is quadratic in x, find the spring constant and the angular frequency of oscillation of the particle provided its displacement from its equilibrium position is small. Express the angular frequency of oscillation of a simple harmonic oscillator: $\omega = \sqrt{\frac{k}{m}}$, where k is the restoring constant. From the result in part *(iii)* and the definition of electric potential $U(x) = qV(0) + \frac{1}{2}\left(\frac{qQ}{8\pi\epsilon_0\sqrt{2}L^3}\right)x^2 = qV(0) + \frac{1}{2}kx^2$, where $k = \frac{qQ}{8\pi\epsilon_0\sqrt{2}L^3}$. Substituting for k in the expression for ω yields: $\omega = \sqrt{\frac{qQ}{8\pi\epsilon_0 m\sqrt{2}L^3}}$.

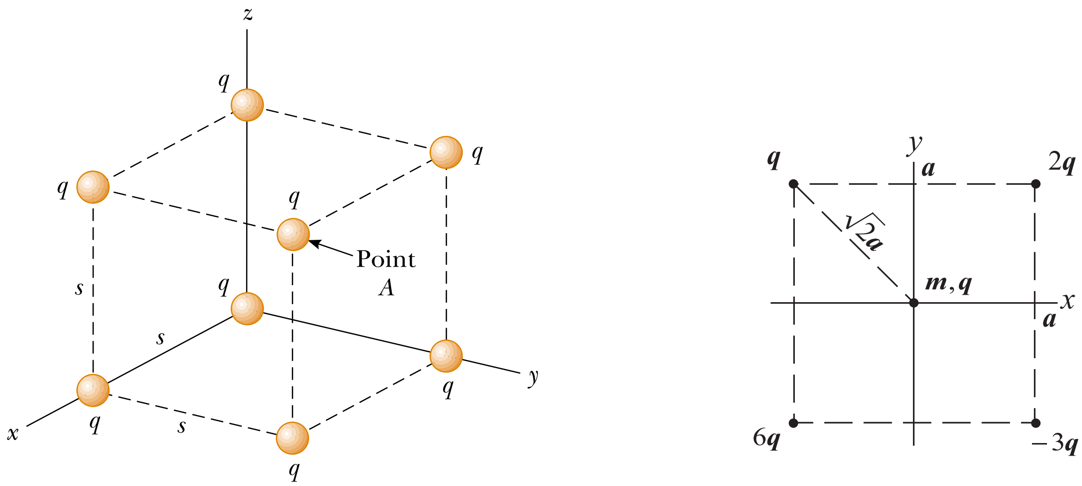

Figure 1: Problem 1.

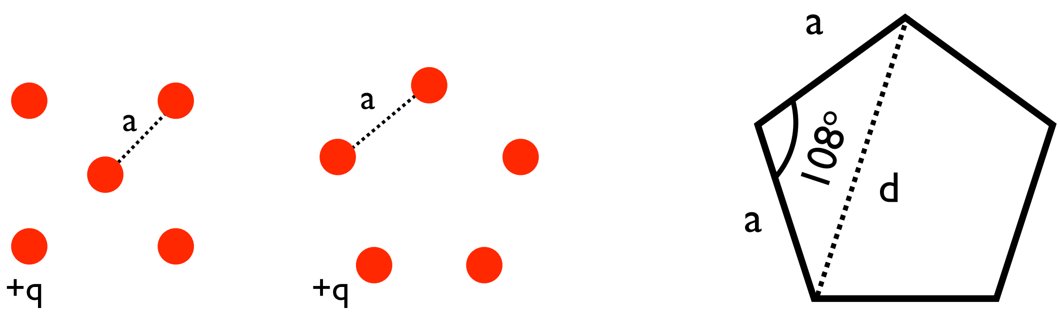

Figure 2: Problem 2.

Table 1: *Charge pairings in the square lattice*

#, pairing type	separation				pairs
4, center-corner	a	$(1,5)$	$(2,5)$	$(3,5)$	$(4,5)$
4, adjacent corners	$a\sqrt{2}$	$(1,4)$	$(3,4)$	$(2,3)$	$(1,2)$
2, far corner	$2a$			$(1,3)$	$(2,4)$

Table 2: *Charge pairings in the pentagonal lattice*

#, pairing type	separation					pairs
5, next-nearest neighbors	d	$(1,3)$	$(1,4)$	$(2,4)$	$(2,5)$	$(3,5)$
5, adjacent	a	$(1,2)$	$(2,3)$	$(3,4)$	$(4,5)$	$(5,1)$

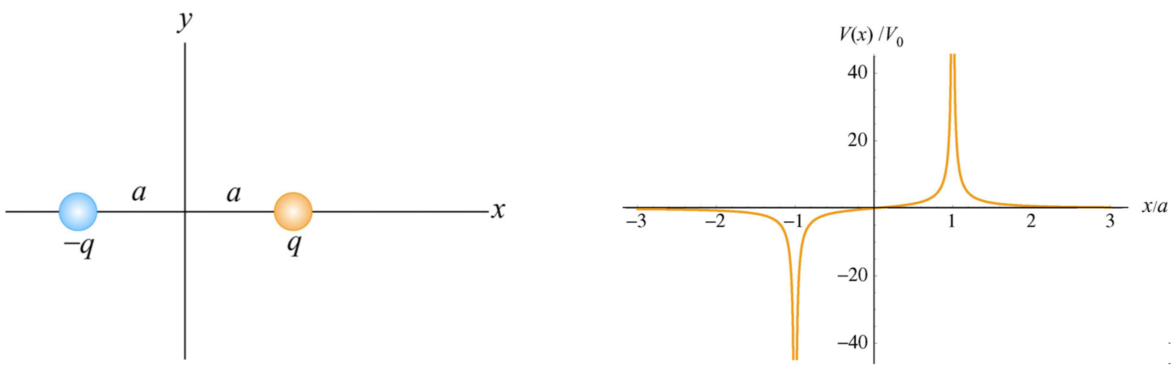

Figure 3: The lectric dipole of problem 3.

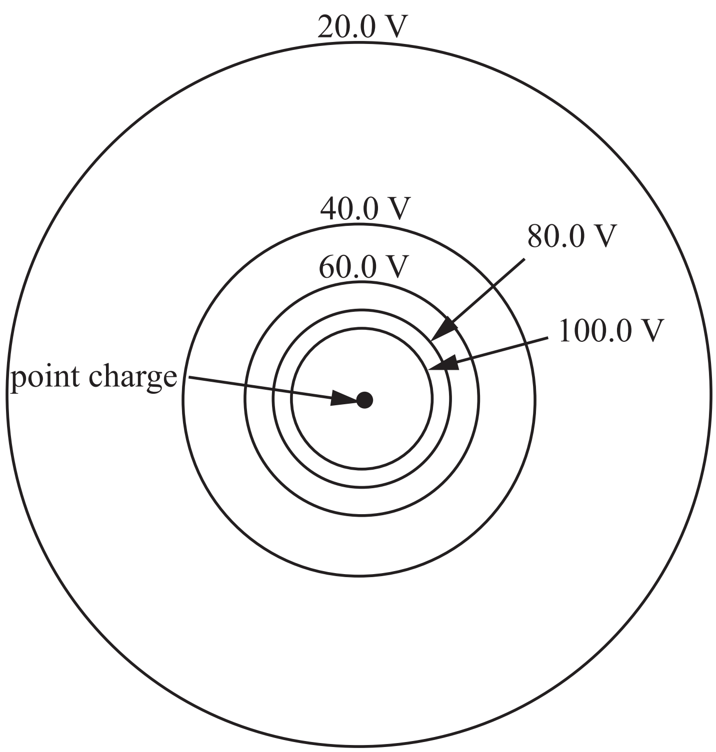

Figure 4: Problem 4.

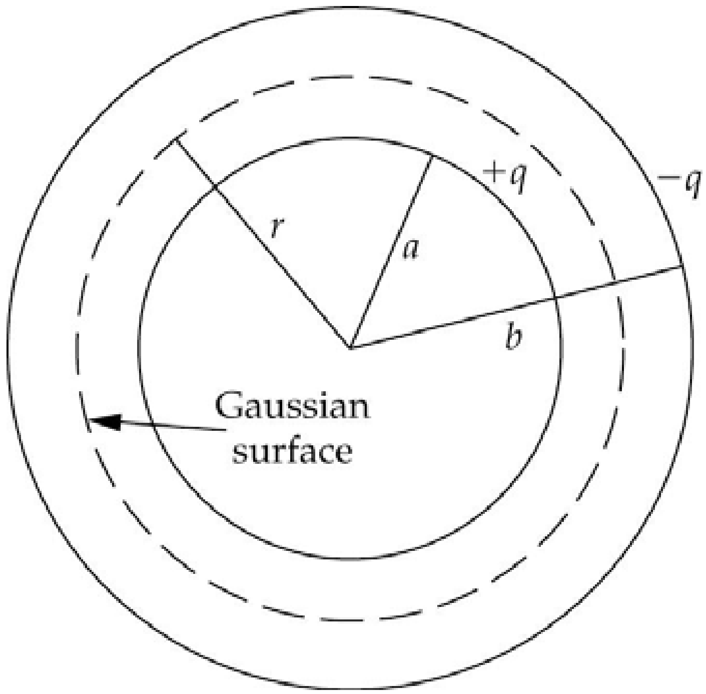

Figure 5: Problem 5.

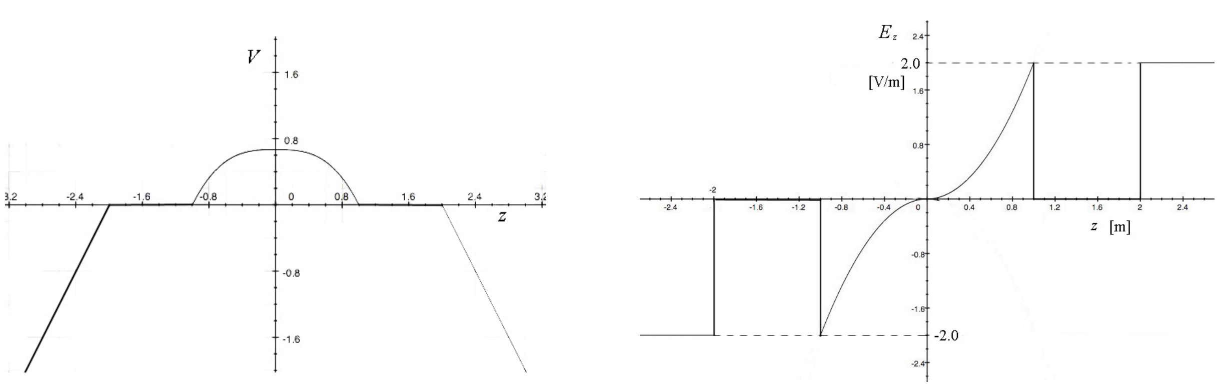

Figure 6: Problem 6.

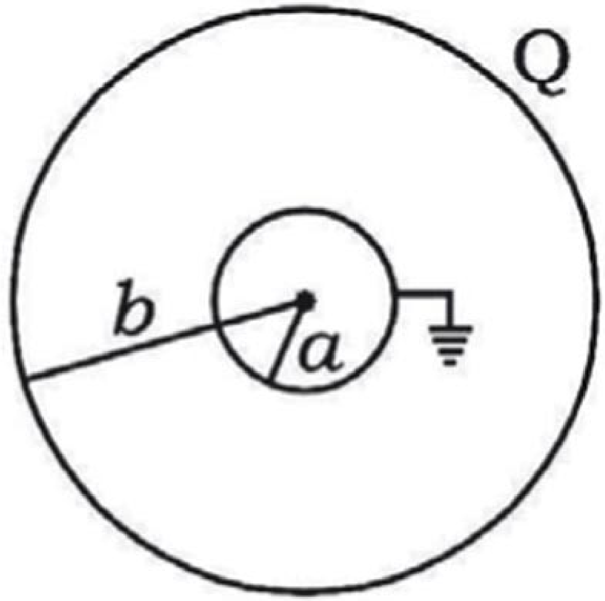

Figure 7: Problem 7.

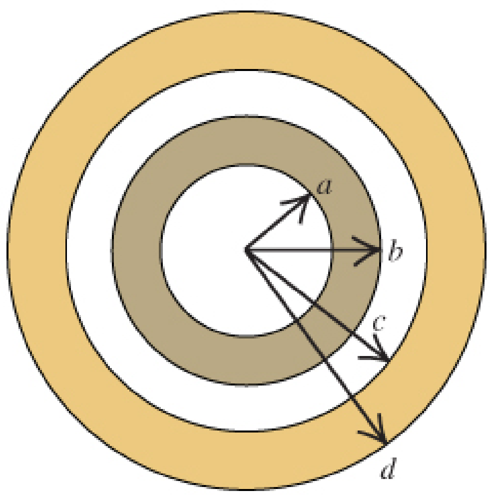

Figure 8: The Farady cage of problem 8.

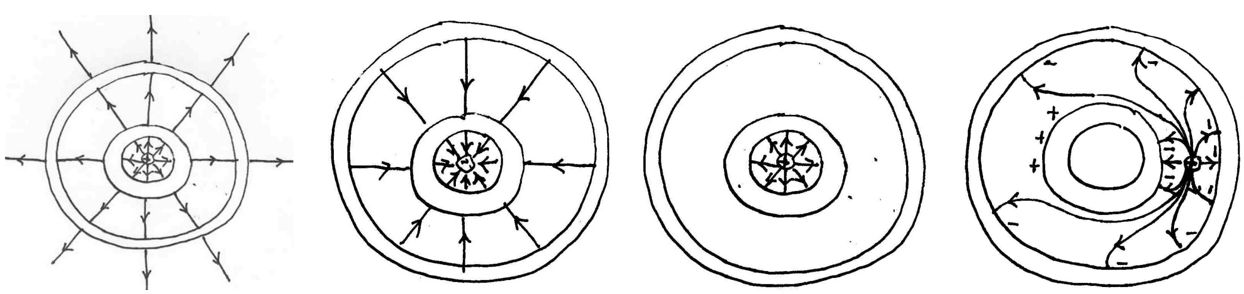

Figure 9: The electric field lines of problem 8; from left to right *(i), (ii), (iii), (iv)*.