Prof. Anchordoqui

Problems set # 2	**Physics 169**	February 11, 2015

1. Figure 1 shows the electric field lines for two point charges separated by a small distance. *(i)* Determine the ratio q_1/q_2. *(ii)* What are the signs of q_1 and q_2?

Solution *(i)* The magnitude of q_2 is three times the magnitude of q_1 because 3 times as many lines emerge from q_2 as enter q_1. Then $|q_2| = 3|q_1|$, yielding $q_1/q_2 = -1/3$. *(ii)* $q_2 > 0$ because lines emerge from it and $q_1 < 0$ because lines terminate on it.

2. An ion milling machine uses a beam of gallium ions ($m = 70u$) to carve microstructures from a target. A region of uniform electric field between parallel sheets of charge is used for precise control of the beam direction. Single ionized gallium atoms with initially horizontal velocity of 1.8×10^4 m/s enter a 2.0 cm-long region of uniform electric field which points vertically upward, as shown in Fig. 2. The ions are redirected by the field, and exit the region at the angle θ shown. If the field is set to a value of $E = 90$ N/C, what is the exit angle θ?

Solution A singly-ionized gallium atom has a charge of $q = +e$, and the mass of $m = 70u$, means 70 atomic mass units, where one atomic mass unit is $1u = 1.66 \times 10^{-27}$ kg. What we really have here is a particle under the influence of a constant force, just as if we were to throw a ball horizontally and watch its trajectory under the influence of gravity (the only difference is that since we have negative charges, things can "fall up"). To start with, we will place the origin at the ions initial position, let the positive x axis run to the right, and let the positive y axis run straight up. Thus, the particle starts with a velocity purely in the x direction: $\vec{v} = v_x\hat{\imath}$. While the particle is in the electric-field-containing region, it will experience a force pointing along the $+y$ direction, with a constant magnitude of qE. Since the force acts only in the y direction, there will be a net acceleration only in the y direction, and the velocity in the x direction will remain constant. Once outside the region, the particle will experience no net force, and it will therefore continue along in a straight line. It will have acquired a y component to its velocity due to the electric force, but the x component will still be v_x. Thus, the particle exits the region with velocity $\vec{v} = v_x\hat{\imath} + v_y\hat{\jmath}$. The angle at which the particle exits the plates, measured with respect to the x axis, must be $\tan\theta = v_y/v_x$. Thus, just like in any mechanics problem, finding the angle is reduced to a problem of finding the final velocity components, of which we already know one. So, how do we find the final velocity in the y direction? Initially, there is no velocity in the y direction, and while the particle is traveling between the plates, there is a net force of qE in the y direction. Thus, the particle experiences an acceleration $a_y = \frac{F_y}{m} = \frac{qE_y}{m}$. The electric field is purely in the y direction in this case, so $E_y = 90$ N/C. Now we know the acceleration in the y direction, so if we can find out the time the particle takes to transit the plates, we are done, since the the transit time Δt and acceleration a_y determine v_y, i.e., $v_y = a_y\Delta t$. Since the x component of the velocity is not changing, we can find the transit time by noting that the distance covered in the x direction must be the x component of the velocity times the transit time. The distance covered in the x direction is just the width of the plates, so $d_x = v_x\Delta t = 2.0$ cm $\Rightarrow \Delta t = d_x/v_x$. Putting the previous equations together, we can express v_y in terms of known quantities: $v_y = a_y \Delta t = a_y d_x / v_x = \frac{qE_y d_x}{mv_x}$. Finally, we can now find the angle θ as well: $\tan\theta = \frac{v_y}{v_x} = \frac{qE_y d_x}{mv_x^2}$. And that's that. Now we plug in the numbers we have, watching the units carefully: $\theta = \tan^{-1}\left[\frac{1.6\times10^{-19}\ \text{C}\cdot 90\ \text{N/C}\cdot 0.02\ \text{m}}{70\cdot 1.66\times10^{-27}\ \text{kg}\cdot(1.8\times10^4\ \text{m/s})^2}\right] = \tan^{-1} 7.6\times10^{-3} \approx 0.44°$.

3. Two 2.0-g spheres are suspended by 10.0-cm-long light strings, see Fig. 3. A uniform electric field is applied in the x direction. If the spheres have charges of -5.0×10^{-8} C and 5.0×10^{-8} C, determine the electric field intensity that enables the spheres to be in equilibrium at $\theta = 10°$.

Solution The sketch in Fig. 3 gives a free-body diagram of the positively charged sphere. Here, $F_1 = \frac{1}{4\pi\epsilon_0}\frac{|q|^2}{r^2}$ is the attractive force exerted by the negatively chaged sphere and $F_2 = qE$ is exerted by the electric field. This leads to $\sum F_y = 0 \Rightarrow T\cos 10° = mg$ or $T = \frac{mg}{\cos 10°}$ and $\sum F_x = 0 \Rightarrow F_2 = F_1 + T\sin 10°$ or $qE = \frac{1}{4\pi\epsilon_0}\frac{|q|^2}{r^2} + mg\tan 10°$. At equilibrium, the distance between the two spheres is $r = 2(L\sin 10°)$. Thus, the electric field strength required is $E = \frac{1}{4\pi\epsilon_0}\frac{|q|}{4(L\sin 10°)^2} + \frac{mg\tan 10°}{q} = \frac{8.99\times10^9\ \text{N}\cdot\text{m}^2/\text{C}^2 5.0\times10^{-8}\ \text{C}}{4[0.100\ \text{m}\ \sin 10°]^2} + \frac{2.0\times10^{-3}\ \text{kg}\, 9.80\ \text{m/s}^2\ \tan 10°}{5.0\times10^{-8}\ \text{C}} = 4.4\times10^5$ N/C.

4. Three charges of equal magnitude q are fixed in position at the vertices of an equilateral triangle (Fig. 4). A fourth charge Q is free to move along the positive x axis under the influence of the forces exerted by the three fixed charges. Find a value for s for which Q is in equilibrium. You will need to solve a transcendental equation.

Solution At an equilibrium position, the net force on the charge Q is zero. The equilibrium position can be located by determining the angle θ corresponding to equilibrium. In terms of lengths s, $\frac{\sqrt{3}}{2}a$, and r, shown in Fig. 4, the charge at the origin exerts an attractive force $\frac{1}{4\pi\epsilon_0}\frac{Qq}{(s+\frac{\sqrt{3}a}{2})^2}$. The other two charges exert equal repulsive forces of magnitude $\frac{1}{4pi\epsilon_0}\frac{Qq}{r^2}$. The horizontal components of the two repulsive forces add, balancing the attractive force, $F_{\text{net}} = \frac{1}{4\pi\epsilon_0}Qq\left(\frac{2\cos\theta}{r^2} - \frac{1}{[s+a(\sqrt{3}/2)]^2}\right) = 0$. From Fig. 4 it follows that $r = \frac{a}{2\sin\theta}$ and $s = \frac{a\cot\theta}{2}$. The equilibrium condition, in terms of θ, is $F_{\text{net}} = \frac{1}{4\pi\epsilon_0}\frac{4Qq}{a^2}\left[2\cos\theta\sin^2\theta - \frac{1}{(\sqrt{3}+\cot\theta)^2}\right] = 0$. Hence the equilibrium value of θ satisfies $2\cos\theta\sin^2\theta(\sqrt{3}+\cot\theta)^2 = 1$. One method for solving for θ is to tabulate the left side. To three significant figures a value of θ corresponding to equilibrium is 81.7°, see Table 4. The distance from the vertical side of the triangle to the equilibrium position is $s = \frac{a\cot 81.7°}{2} = 0.0729a$. A second zero-field point is on the negative side of the x-axis, where $\theta = -9.16°$ and $s = -3.10a$.

5. Eight solid plastic cubes, each 3.00 cm on each edge, are glued together to form each one of the objects *(i, ii, iii, iv)* shown in Fig. 5. *(a)* Assuming each object carries charge with uniform density 400 nC/m^3 throughout its volume, find the charge of each object. *(b)* Assuming each object carries charge with uniform density 15.0 nC/m^2 everywhere on its exposed surface, find the charge on each object. *(c)* Assuming charge is placed only on the edges where perpendicular surfaces meet, with uniform density 80.0 pC/m, find the charge of each object.

Solution *(a)* Every object has the same volume, $V = 8(0..030\ \text{m})^3 = 2.16\times10^{-4}\ \text{m}^3$. For each, $Q = \rho V = 4.00\times10^{-9}\ \text{C/m}^3\, 2.16\times10^{-4}\ \text{m}^3 = 8.64\times10^{-13}$ C. *(b)* We must count

Table 1: Problem 4.

θ	$2\cos\theta\ \sin^2\theta(\sqrt{3}+\cot\theta)^2$
$60°$	4
$70°$	2.654
$80°$	1.226
$90°$	0
$81°$	1.091
$81.5°$	1.024
$81.7°$	0.997

the 9.00 cm^2 squares painted with charge: *(i)* $6 \times 4 = 24$ squares, so $Q = \sigma A = (15.0 \times 10^{-9}\ \mathrm{C/m^2}\ 24.0\ 9.00 \times 10^{-4}\ \mathrm{m^2} = 3.24 \times 10^{-10}$ C; *(ii)* 34 squares exposed, so $Q = \sigma A = 15.0 \times 10^{-9}\ \mathrm{C/m^2}\ 34.0\ 9.00 \times 10^{-4}\ \mathrm{m^2} = 4.59 \times 10^{-10}$ C; *(iii)* 34 squares, so $Q = \sigma A = 15.0 \times 10^{-9}\ \mathrm{C/m^2}\ 34.0\ 9.00 \times 10^{-4}\ \mathrm{m^2} = 4.59 \times 10^{-10}$ C; *(iv)* 32 squares, so $Q = \sigma A = 15.0 \times 10^{-9}\ \mathrm{C/m^2}\ 32.0\ 9.00 \times 10^{-4}\ \mathrm{m^2} = 4.32 \times 10^{-10}$ C; *(c)* *(i)* Total edge length, $\ell = 24 \times 0.030$ m, so $Q = \lambda\ell = 80.0 \times 10^{-12}\ \mathrm{C/m} \times 24 \times 0.030 = 5.76 \times 10^{-11}$ C; *(ii)* total edge length, $\ell = 44 \times 0.030$ m, so $Q = \lambda\ell = 80.0 \times 10^{-12}\ \mathrm{C/m} \times 44 \times 0.030 = 1.06 \times 10^{-10}$ C; *(iii)* total edge length, $\ell = 64 \times 0.030$ m, so $Q = \lambda\ell = 80.0 \times 10^{-12}\ \mathrm{C/m} \times 64 \times 0.030 = 1.54 \times 10^{-10}$ C; *(iv)* total edge length, $\ell = 40 \times 0.030$ m, so $Q = \lambda\ell = 80.0 \times 10^{-12}\ \mathrm{C/m} \times 40 \times 0.030 = 0.960 \times 10^{-10}$ C.

6. *(i)* Consider a uniformly charged thin-walled right circular cylindrical shell having total charge Q, radius R, and height h. Determine the electric field at a point a distance d from the right side of the cylinder as shown in Fig. 6. [Hint: Use the result of Example 2 given in lecture 2 and treat the cylinder as a collection of ring charges.] *(ii)* Consider now a solid cylinder with the same dimensions and carrying the same charge, uniformly distributed through its volume. Use the result of Example 3 given in lecture 2 to find the field it creates at the same point.

Solution *(i)* We define $x = 0$ at the point where we are to find the field. One ring, with thickness dx (see Fig. 7), has charge Qdx/h and produces, at the chosen point, a field $dE = \frac{1}{4\pi\epsilon_0}\frac{x}{(x^2+R^2)^{3/2}}\frac{Qdx}{h}\hat{\imath}$. The total field is $E = \int_{\text{allcharge}} dE = \int_d^{d+h}\frac{1}{4\pi\epsilon_0}\frac{Qx}{h(x^2+R^2)^{3/2}dx}\hat{\imath} = \frac{1}{4\pi\epsilon_0}\frac{Q}{2h}\hat{\imath}\int_d^{d+h}(x^2+R^2)^{-3/2}2xdx$. Integration leads to $E = \frac{1}{4\pi\epsilon_0}\frac{Q}{h}\hat{\imath}\left\{\frac{1}{(d^2+R^2)^{1/2}} - \frac{1}{[(d+h)^2+R^2]^{1/2}}\right\}$. *(ii)* Think of the cylinder as a stack of disks, each with thickness dx, charge Qdx/h and charge per-area $\sigma = \frac{Qdx}{\pi R^2 h}$, see Fig. 8. One disk produces a field $d\vec{E} = \frac{1}{4\pi\epsilon_0}\frac{2\pi Q}{\pi R^2 h}dx\left(1 - \frac{x}{(x^2+R^2)^{1/2}}\right)\hat{\imath}$. Therefore, $\vec{E} = \int_{\text{allcharge}} d\vec{E} = \int_d^{d+h}\frac{1}{4\pi\epsilon_0}\frac{2Q}{R^2h}dx\left(1 - \frac{x}{(x^2+R^2)^{1/2}}\right)\hat{\imath}$. Integration leads to $\vec{E} = \frac{1}{4\pi\epsilon_0}\frac{2Q}{R^2h}\hat{\imath}\left[\ x\Big|_d^{d+h} - \frac{1}{2}\frac{(x^2+R^2)^{1/2}}{1/2}\Big|_d^{d+h}\right] = \vec{E} = \frac{1}{4\pi\epsilon_0}\frac{2Q}{R^2h}\hat{\imath}\left\{h + (d^2+R^2)^{1/2} - \left[(d+h)^2+R^2\right]^{1/2}\right\}$.

7. A uniformly charged rod of length 14.0 cm is bent into the shape of a semicircle as shown in Fig. 9. The rod has a total charge of $-7.50\ \mu$C. Find the magnitude and direction of the electric field at O, the center of the semicircle.

Solution Due to symmetry $E_y = \int dE_y = 0$ and $E_x = \int dE \sin\theta = \frac{1}{4\pi\epsilon_0}\int dq \frac{\sin\theta}{r^2}$ where $dq = \lambda ds = \lambda r d\theta$, so that $E_x = \frac{1}{4\pi\epsilon_0}\frac{\lambda}{r}(-\cos\theta)\Big|_0^\pi = \frac{1}{4\pi\epsilon_0}\frac{2\lambda}{r}$, where $\lambda q/L$ and $r = L/\pi$, see Fig. 9. Therefore, $E_x = \frac{1}{4\pi\epsilon_0}\frac{2q\pi}{L^2} = 2.16 \times 10^7$ N/C. Since the rod has a negative charge, $\vec{E} = -2.16 \times 10^7 \hat{\imath}$ N/C.

8. A line of charge with uniform density 35.0 nC/m lies along the line $y = -15.0$ cm, between the points with coordinates $x = 0$ and $x = 40.0$ cm. Find the electric field it creates at the origin.

Solution From the sketch in Fig. 10 it follows that $dE = \frac{1}{4\pi\epsilon_0}\frac{dq}{x^2+(0.150 \text{ m})^2}\frac{x\hat{\imath}+0.150 \text{ m}\hat{\jmath}}{\sqrt{x^2+(0.150 \text{ m})^2}} = \frac{1}{4\pi\epsilon_0}\frac{\lambda(-x\hat{\imath}+0.150 \text{ m}\hat{\jmath})dx}{[x^2+(0.150 \text{ m})^2]^{3/2}}$. Thus, $E = \int_{\text{allcharge}} dE = \frac{1}{4\pi\epsilon_0}\int_0^{0.400 \text{ m}} \frac{-x\hat{\imath}+0.150 \text{ m}\hat{\jmath}}{[x^2+(0.150 \text{ m})^2]^{3/2}}dx$. Integration yields $E = \frac{1}{4\pi\epsilon_0}\lambda\left[\frac{\hat{\imath}}{\sqrt{x^2+(0.150 \text{ m})^2}}\Big|_0^{0.400 \text{ m}} + \frac{0.150 \text{ m}x\hat{\jmath}}{(0.150 \text{ m})^2\sqrt{x^2+(0.150 \text{ m})^2}}\Big|_0^{0.400 \text{ m}}\right] = (-1.36\hat{\imath}+1.96\hat{\jmath})\times 10^{-15}$ N/C.[1]

9. *(i)* A rod of length ℓ has a uniform positive charge per unit length λ and a total charge Q. Calculate the electric field at a point P that is located along the long axis of the rod and a distance a from one end, see Fig. 11. *(ii)* Identical thin rods of length $2a$ carry equal charges $+Q$ uniformly distributed along their lengths. The rods lie along the x axis with their centers separated by a distance $b > 2a$ (Fig. 11). Show that the magnitude of the force exerted by the left rod on the right one is given by $F = \frac{1}{4\pi\epsilon_0}\frac{Q^2}{4a^2}\ln\left(\frac{b^2}{b^2-4a^2}\right)$.

Solution *(i)* Let us assume that the rod is lying along the x axis, that dx is the length of one small segment, and that dq is the charge on that segment, see Fig. 11. Because the rod has a charge per unit length λ, the charge dq on the small segment is $dq = \lambda dx$. The field $d\vec{E}$ at P due to this segment is in the negative x direction (because the source of the field carries a positive charge), and its magnitude is $dE = \frac{1}{4\pi\epsilon_0}\frac{dq}{x^2}$. Because every other element also produces a field in the negative x direction, the problem of summing their contributions is particularly simple in this case. The total field at P due to all segments of the rod, which are at different distances from P, is given by $E = \int_a^{\ell+a}\frac{1}{4\pi\epsilon_0}\lambda\frac{dx}{x^2} = -\frac{1}{4\pi\epsilon_0}\lambda\frac{1}{x}\Big|_a^{\ell+a} = \frac{\lambda}{4\pi\epsilon_0}\left(\frac{1}{a} - \frac{1}{\ell+a}\right) = \frac{1}{4\pi\epsilon_0}\frac{Q}{a(\ell+a)}$. *(ii)* According to the result of *(i)*, the left-hand rod creates a field at a distance d from its right-hand end equal to $E = \frac{1}{4\pi\epsilon_0}\frac{Q}{d(2a+d)}$. Now, since $dq = \lambda dx = \frac{Q}{2a}dx$, it follows that $dF = Edq = \frac{1}{4\pi\epsilon_0}\frac{QQ}{2a}\frac{dx}{x(x+2a)}$, yielding $F = \frac{1}{4\pi\epsilon_0}\frac{Q^2}{2a}\int_{b-2a}^b\frac{dx}{x(x+2a)} = -\frac{1}{4\pi\epsilon_0}\frac{Q^2}{2a}\;\frac{1}{2a}\ln\left(\frac{2a+x}{x}\right)\Big|_{b-2a}^b$. All in all $F = \frac{1}{4\pi\epsilon_0}\frac{Q^2}{4a^2}\left[-\ln\left(\frac{2a+b}{b}\right) + \ln\left(\frac{b}{b-2a}\right)\right] = \frac{1}{4\pi\epsilon_0}\frac{Q^2}{4a^2}\ln\left[\frac{b^2}{(b-2a)(b+2a)}\right] = \frac{1}{4\pi\epsilon_0}\frac{Q^2}{4a^2}\ln\left(\frac{b^2}{b^2-4a^2}\right)$.[2]

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[1] To calculate $\int(x^2+a)^{-3/2}dx$ consider the following change of variables: $x = \sqrt{a}\tan u$ such that $dx = \sqrt{a}\sec^2 u\, du$. Straightforward substitution leads to $\int\frac{\sqrt{a}\sec^2 u}{(a\tan^2 u+a)^{3/2}}du = \int\frac{\sqrt{a}\sec^2 u}{a^{3/2}(\sec^2 u)^{3/2}}du = \int\frac{\sqrt{a}\sec^2 u}{a^{3/2}\sec^3 u}du = \int\frac{\cos u}{a}du = \frac{\sin u}{a}+C$, where C is the integration constant. To get back to the original variables use $\tan u = x/\sqrt{a}$ which yields $\sin u = x/\sqrt{x^2+a}$ and hence the overall answer is $\int(x^2+a)^{-3/2}dx = \frac{x}{a\sqrt{x^2+a}} + C$.

[2] To calculate $\int\frac{1}{x^2+x\alpha}dx$ you must find a partial fractions decomposition for $\frac{1}{x^2+\alpha x}$. Begin by factoring the denominator, getting $\frac{1}{x^2+\alpha x} = \frac{1}{x(x+\alpha)}$. Now assume that there are constant A and B so that $\frac{1}{x(x+\alpha)} = \frac{A}{x} + \frac{B}{x+\alpha}$. It is easily seen that such constants always exist for the rational function $p(x)/q(x)$ if the following two conditions are met: *(i)* both $p(x)$ and $q(x)$ are polynomials and *(ii)* the degree of $p(x)$ is smaller than the degree of $q(x)$. Next get a

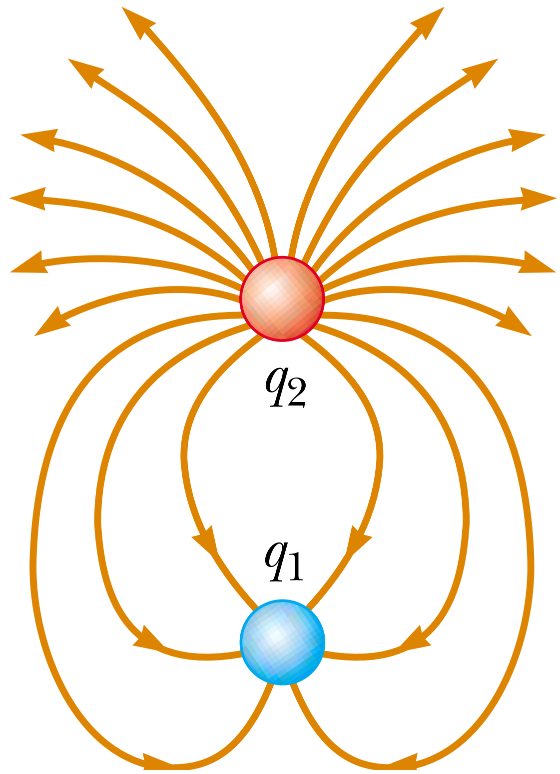

Figure 1: Problem 1.

10. An electric dipole in a uniform electric field is displaced slightly from its equilibrium position, as shown in Fig. 12, where $\theta$is small. The separation of the charges is $2a$, and the moment of inertia of the dipole is I. Assuming the dipole is released from this position, show that its angular orientation exhibits simple harmonic motion with a frequency $f = \frac{1}{2\pi}\sqrt{\frac{2qaE}{I}}$.

Solution The electrostatic forces exerted on the two charges result in a net torque $\tau = -2Fa\sin\theta = -2Eqa\sin\theta$. For small θ, $\sin\theta \approx \theta$ and using $p = 2qa$, we have $\tau = -Ep\theta$. The torque produces an angular acceleration given by $\tau = I\alpha = I\frac{d^2\theta}{dt^2}$. Combining these two expressions for torque, we have $\frac{d^2\theta}{dt^2} + \frac{Ep\theta}{I} = 0$. This equation can be written in the form $\frac{d^2\theta}{dt^2} = -\omega^2\theta$, where $\omega^2 = Ep/I$. The frequency of oscillation is $f = \frac{1}{2\pi}\sqrt{\frac{pE}{I}} = \frac{1}{2\pi}\sqrt{\frac{2qaE}{I}}$.

common denominator and add the fractions to obtain $\frac{1}{x(x+\alpha)} = \frac{A(x+\alpha)+Bx}{x(x+\alpha)}$. From this equality it follows that $A\alpha = 1$ and $A = -B$. The original integral can then be rewritten as $\frac{1}{\alpha}\int\left[\frac{1}{x} - \frac{1}{\alpha+x}\right]dx = \frac{1}{\alpha}\ln x - \frac{1}{\alpha}\ln(x+\alpha) = \frac{1}{\alpha}\frac{\ln x}{\ln(x+\alpha)}$.

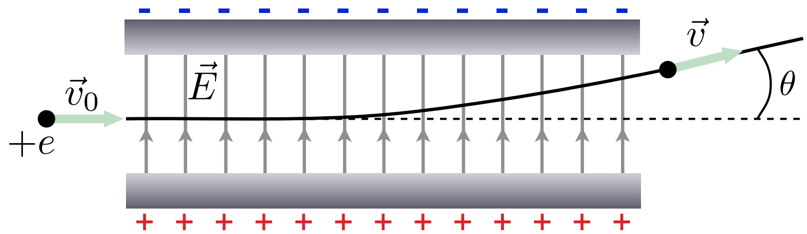

Figure 2: Problem 2.

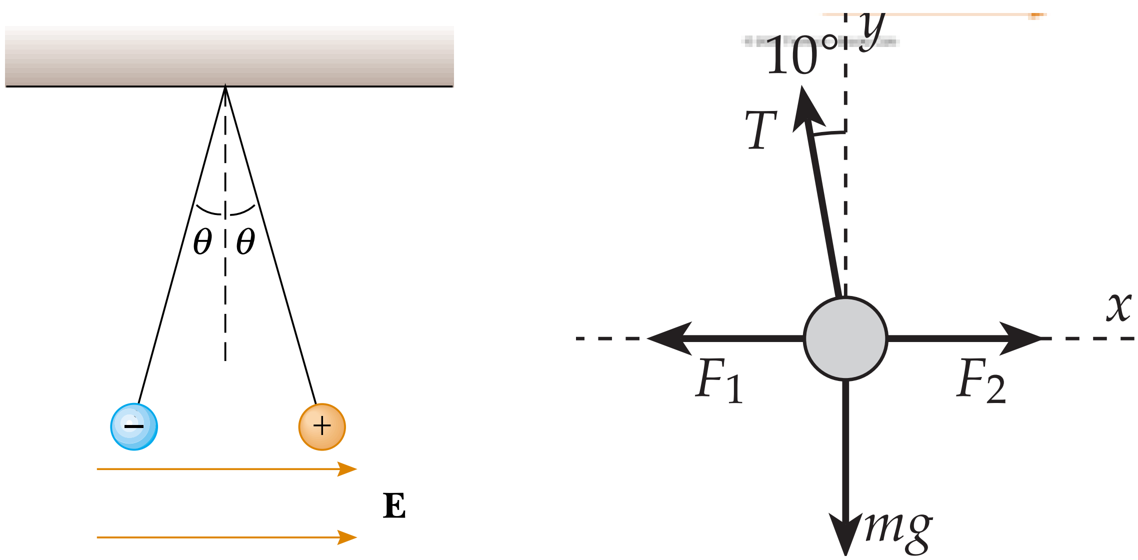

Figure 3: Problem 3.

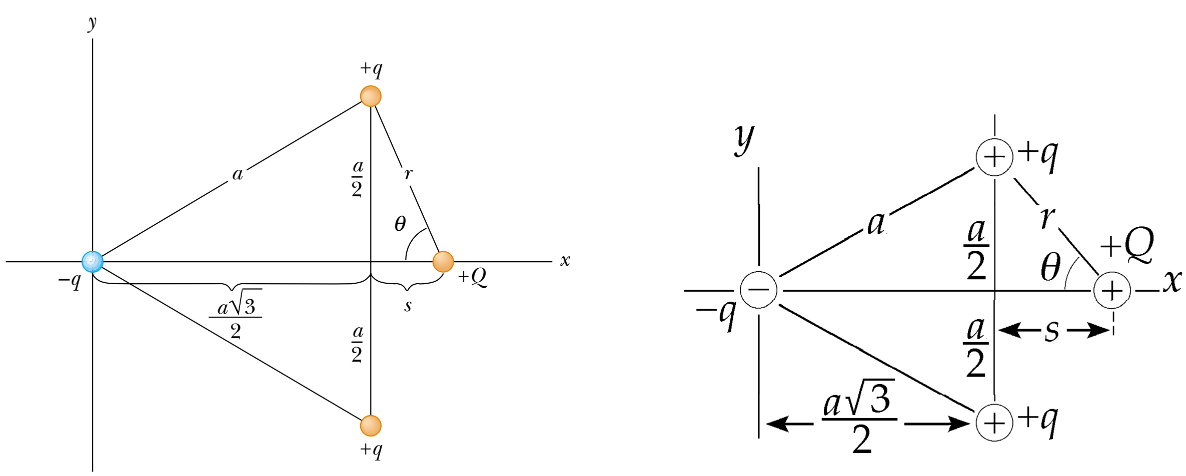

Figure 4: Problem 4.

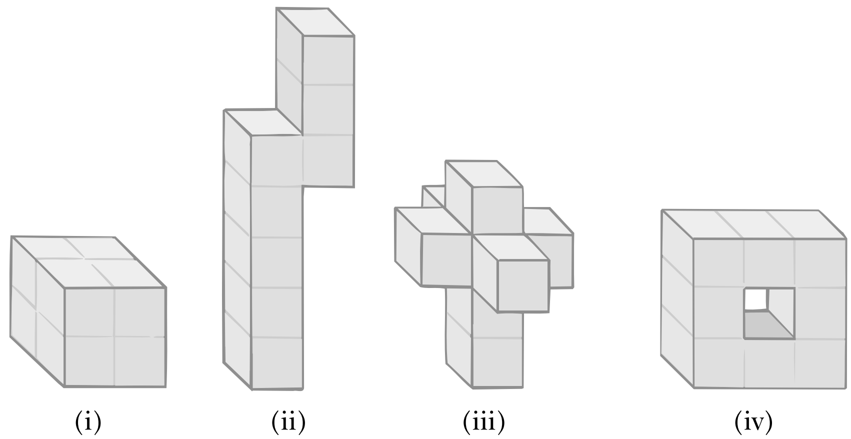

Figure 5: Problem 5.

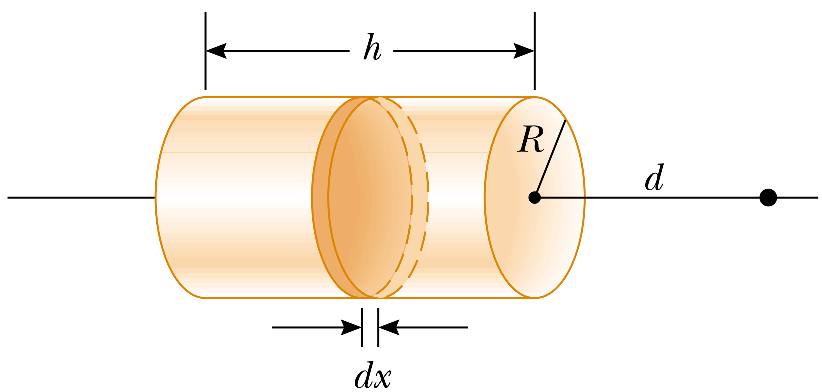

Figure 6: Problem 6.

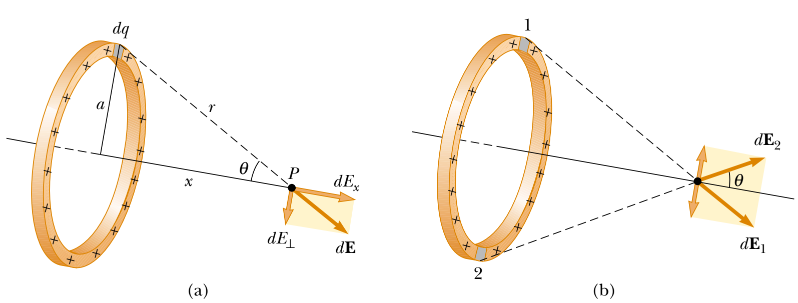

Figure 7: A uniformly charged ring of radius a. (a) The field at P on the x axis due to an element of charge dq. (b) The total electric field at P is along the x axis. The perpendicular component of the field at P due to segment 1 is canceled by the perpendicular component due to segment 2.

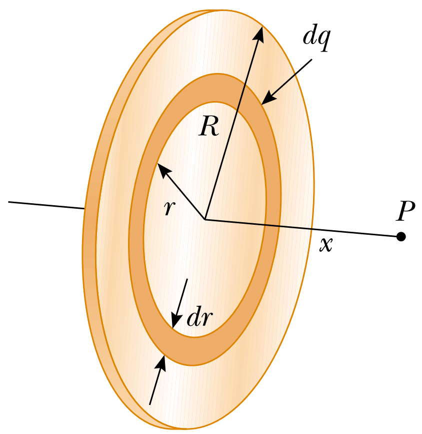

Figure 8: A uniformly charged disk of radius R. The electric field at an axial point P is directed along the central axis, perpendicular to the plane of the disk.

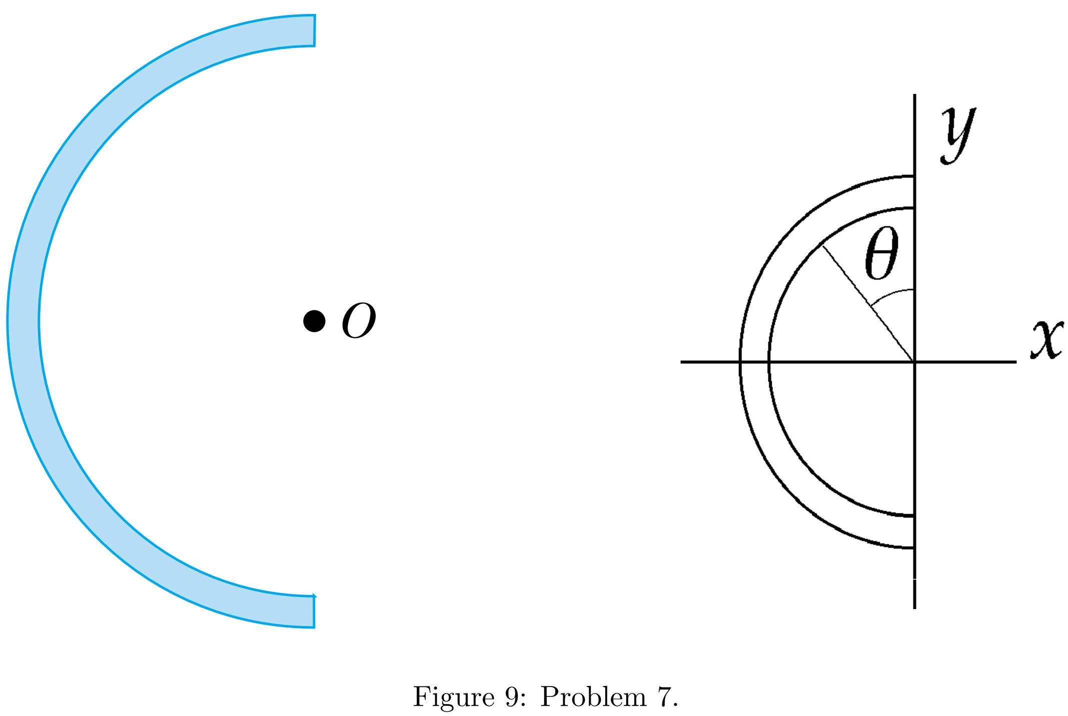

Figure 9: Problem 7.

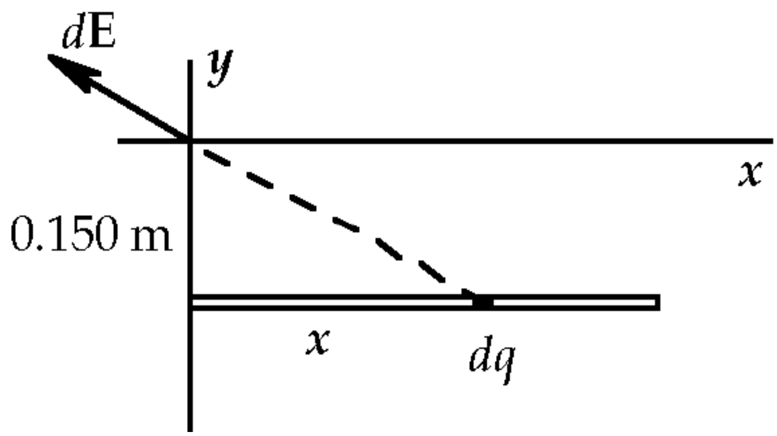

Figure 10: Problem 8.

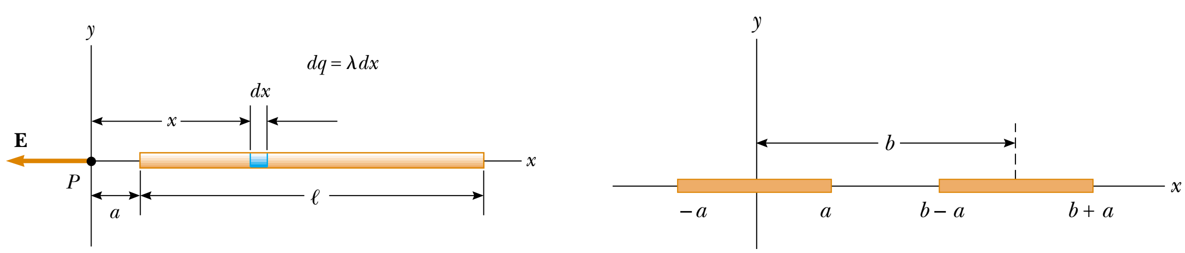

Figure 11: Problem 9; [*(i)* left and *(ii)* right].

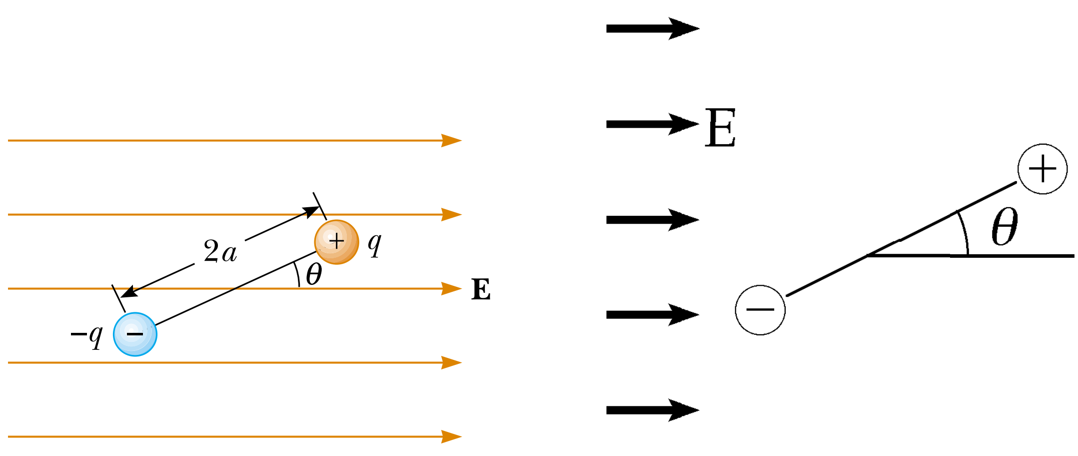

Figure 12: Problem 10.