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Problems set # 13	**Physics 169**	May 12, 2015

1. Consider a ray of light traveling in vacuum from point P_1 to P_2 by way of the point Q on a plane miror as shown in Fig. 1. Show that Fermat's principle implies that, on the actual path followed, Q lies in the same vertical plane as P_1 and P_2 and obeys the law of reflection, that is $\theta_1 = \theta_2$. [Hints: Let the miror lie in the x-z plane, and let P_1 lie on the y axis at $(0, y_1, 0)$ and P_2 in the x-y plane at $(x_2, y_2, 0)$. Finally, let $Q = (x, 0, z)$. Calculate the time for the light to traverse the path P_1QP_2 and show that it is minimum when Q has $z = 0$ and satisfies the law of reflection.]

Solution We already know that the actual path is a straight within one medium. Thus, the segments from P_1 to Q and from Q to P_2 are straight lines and the corresponding distances are $P_1Q = \sqrt{x^2 + y_1^2 + z^2}$ and $QP_2 = \sqrt{(x - x_1)^2 + y_2^2 + z^2}$. Then, the total time for the journay P_1QP_2 is $T = \left(\sqrt{x^2 + y_1^2 + z^2} + \sqrt{(x - x_1)^2 + y_2^2 + z^2}\right)/c$. To find the position of $Q = (x, 0, z)$ for which this is a minimum we must differentiate with respect to z and x and set the derivatives equal to zero: $\frac{\partial T}{\partial z} = \frac{z}{c\sqrt{\cdots}} + \frac{z}{c\sqrt{\cdots}} = 0 \Rightarrow z = 0$, which says that Q must lie in the same vertical plane as P_1 and P_2, and $\frac{\partial T}{\partial x} = \frac{x}{c\sqrt{\cdots}} + \frac{x-x_1}{c\sqrt{\cdots}} = 0 \Rightarrow \sin\theta_1 = \sin\theta_2$ or $\theta_1 = \theta_2$.

2. Nowadays dispersing prisms come in a great variety of sizes and shapes. Typically, a ray entering a dispersing prism will emerge having been deflected from its original direction by an angle δ, known as the angular deviation. Show that the minimum angle of deviation, $\delta_{\rm min}$, for a prism (with apex angle $\Phi$and index of refraction n) occurs when the angle of incidence θ_1 is such that the refracted ray inside the prism makes the same angle with the normal to the two prism faces, as shown in Fig. 2.

Solution At the first refraction the ray is deviated through an angle $\theta_1 - \theta_2$, and at the second refraction it is further deflected through $\theta_4 - \theta_3$. The total deviation is then $\delta = (\theta_1 - \theta_2) + (\theta_4 - \theta_3)$. We need to show that $\theta_3 = \theta_2$ and so $\theta_4 = \theta_1$. Since the polygon $ABCD$ contains two right angles, the angle BCD must be the supplement of the appex angle Φ, see Fig. 2. As the exterior angle to triangle BCD, Φ is also the sum of the alternate interior angles, that is $\Phi = \theta_2 + \theta_3$. Hence $\delta = \theta_1 + \theta_4 - \Phi$. What we would like to do now is write δ as a function of both the angle of incidence for the ray (i.e. θ_1) and the prism angle Φ. If the prism index is n and it is immersed in air ($n_{\rm air} \approx 1$) it follows from Snell's law that $\theta_4 = \sin^{-1}(n \sin\theta_3) = \sin^{-1}[\sin(\Phi - \theta_2)$. Upon expanding this expression, replacing $\cos\theta_2$ by $(1 - \sin\theta_2)^{1/2}$, and using Snell's law we have $\theta_4 = \sin^{-1}[(\sin\Phi)(n^2 - \sin^2\theta_1)^{1/2} - \sin\theta_1 \cos\Phi]$. The deviation is then $\delta = \theta_1 + \sin^{-1}[(\sin\Phi)(n^2 - \sin^2\theta_1)^{1/2} - \sin\theta_1\cos\Phi] - \Phi$. It is evident that the deviation suffered by a monochromatic beam on traversing a given prism (i.e. n and Φ are fixed) is a function only of the incident angle at the first face θ_1. The smallest deviation can be determined analytically by differentianting the expression for $\delta(\theta_1)$ and then setting $d\delta/d\theta_1 = 0$, but a more indirect route will certainly be simpler. Differentiating $\delta(\theta_1, \theta_2)$ and setting it equal to zero we get $\frac{d\delta}{d\theta_1} = 1 + \frac{d\delta}{d\theta_2} = 0$, or $d\theta_2/d\theta_1 = -1$. Taking the derivative of Snell's law at each interface, we have $\cos\theta_1 d\theta_1 = n\cos\theta_2 d\theta_2$ and $\cos\theta_4 d\theta_4 = n\cos\theta_3 d\theta_3$. Note as well that, since Φ is the sum of alternate interior angles, $d\theta_2 = -d\theta_3$, because $d\Phi = 0$. Dividing the expressions obtained for Snell's law and substituting for the derivatives it follows that $\cos\theta_1/\cos\theta_4 = \cos\theta_2/\cos\theta_3$. Making use of Snell's law once again, we can rewrite this as $\frac{1-\sin^2\theta_1}{1-\sin^2\theta_4} = \frac{n^2-\sin^2\theta_1}{n^2-\sin^2\theta 4}$. The value of θ_1 for which this is true is the one for which $d\delta/d\theta_1 = 0$. Provided $n \neq 1$, it follows that $\theta_1 = \theta_4$ and therefore $\theta_2 = \theta_3$. This means that the ray for which the deviation is a minimum traverses the prism symmetrically, i.e. parallel to the base.

3 An interesting effect called total internal reflection can occur when light is directed from a medium having a given index of refraction toward one having a lower index of refraction. Consider a light beam traveling in medium 1 and meeting the boundary between medium 1 and medium 2, where n_1 is greater than n_2. Various possible directions of the beam are indicated by rays 1 through 5 in Fig. 3. The refracted rays are bent away from the normal because n_1 is greater than n_2. At some particular angle of incidence θ_c, called the critical angle, the refracted light ray moves parallel to the boundary so that $\theta_2 = \pi/2$. For angles of incidence greater than θ_c the beam is entirely reflected at the boundary. Consider a triangular glass prism with apex angle Φ and index of refraction n. What is the smallest angle of incidence θ_1 for which a light ray can emerge from the other side?

Solution At the first refraction, $1.00\sin\theta_1 = n\sin\theta_2$. The critical angle at the second surface is given by $n\sin\theta_3 = 1.00$, or $\theta_3 = \sin^{-1}(1.00/n)$; but $(\pi/2 - \theta_2) + (\pi/2 - \theta_3) + \Phi = \pi$, which gives $\theta_2 = \Phi - \theta_3$. Thus, to have $\theta_3 < \sin^{-1}(1.00/n)$ and avoid total internal reflection at the second surface, it is necessary that $\theta_2 > \Phi - \sin^{-1}(1.00/n)$. Since $\sin\theta_1 = n\sin\theta_2$, this requirement becomes $\sin\theta_1 > n\sin[\Phi - \sin^{-1}(1.00/n)]$, or $\theta_1 > \sin^{-1}\{n\sin[\Phi - \sin^{-1}(1.00/n)]\}$. Through the application of trigonometric identities, $\theta_1 > \sin^{-1}(\sqrt{n^2-1}\sin\Phi - \cos\Phi)$.

4 The index of refraction for violet light in silica flint glass is 1.66, and that for red light is 1.62; see Fig. 4. What is the angular dispersion of visible light passing through a prism of apex angle 60.0° if the angle of incidence is 50.0°?

Solution For the incoming ray, $n\sin\theta_2 = \sin\theta_1$. Then $(\theta_2)_{\rm violet} = \sin^{-1}(\sin 50.0^\circ/1.66) = 27.48^\circ$ and $(\theta_2)_{\rm red} = \sin^{-1}(\sin 50.0^\circ/1.62) = 28.22^\circ$. For the outgoing ray, $\theta_3 = 60.0^\circ - \theta_2$ and $\sin\theta_4 = n\sin\theta_3$. This leads to $(\theta_4)_{\rm violet} = \sin^{-1}(1.66\sin 32.52^\circ) = 63.17^\circ$ and $(\theta_4)_{\rm red} = \sin^{-1}(1.62\sin 31.78^\circ) = 58.56^\circ$. The angular dispersion is the difference $\Delta\theta_4 = (\theta_4)_{\rm violet} - (\theta_4)_{\rm red} = 63.17^\circ - 58.56^\circ = 4.61^\circ$.

5. A spherical mirror has a focal length of 10.0 cm. *(i)* Locate and describe the image for an object distance of 25.0 cm. *(ii)* Locate and describe the image for an object distance of 10.0 cm. *(iii)* Locate and describe the image for an object distance of 5.00 cm.

Solution Because the focal length of the mirror is positive, it is a concave mirror. We expect the possibilities of both real and virtual images. *(i)* Because the object distance is larger than the focal length, we expect the image to be real. The image is at a distance q is found from the relation $\frac{1}{q} = \frac{1}{f} - \frac{1}{p}$, where p is the position. We have $q = 16.7$ cm. The magnification of the image is

$M = -\frac{q}{p} = -0.667$. The absolute value of M is less than unity, so the image is smaller than the object, and the negative sign for M tells us that the image is inverted. Because q is positive, the image is located on the front side of the mirror and is real. Look into the bowl of a shiny spoon or stand far away from a shaving mirror to see this image. *(ii)* Because the object is at the focal point, we expect the image to be infinitely far away. The distance q of the image is found by the relation $\frac{1}{q} = \frac{1}{f} - \frac{1}{p}$ and so as $p \to f$ the image distance $q \to \infty$. This result means that rays originating from an object positioned at the focal point of a mirror are reflected so that the image is formed at an infinite distance from the mirror; that is, the rays travel parallel to one another after reflection. Such is the situation in a flashlight or an automobile headlight, where the bulb filament is placed at the focal point of a reflector, producing a parallel beam of light. *(iii)* Because the object distance is smaller than the focal length, we expect the image to be virtual. We have $\frac{1}{q} = \frac{1}{f} - \frac{1}{p}$, yielding $q = -10.0$ cm. The magnification is $M = -\frac{q}{p} = 2.00$. The image is twice as large as the object, and the positive sign for M indicates that the image is upright. The negative value of the image distance tells us that the image is virtual, as expected. Put your face close to a shaving mirror to see this type of image.

6. An automobile rearview mirror shows an image of a truck located 10.0 m from the mirror. The focal length of the mirror is -0.60 m. *(i)* Find the position of the image of the truck. *(ii)* Find the magnification of the image.

<u>Solution</u> Because the mirror is convex, we expect it to form an upright, reduced, virtual image for any object position. *(i)* The image distance is $\frac{1}{q} = \frac{1}{f} - \frac{1}{p}$, so $q = -0.57$ m. *(ii)* The magnificastion is $M = -\frac{q}{p} = 0.057$. The negative value of q in part *(i)* indicates that the image is virtual, or behind the mirror. The magnification in part *(ii)* indicates that the image is much smaller than the truck and is upright because M is positive. The image is reduced in size, so the truck appears to be farther away than it actually is. Because of the image's small size, these mirrors carry the inscription, "Objects in this mirror are closer than they appear." Look into your rearview mirror or the back side of a shiny spoon to see an image of this type.

7. A small fish is swimming at a depth d below the surface of a pond. *(i)* What is the apparent depth of the fish as viewed from directly overhead? *(ii)* If your face is a distance d above the water surface, at what apparent distance above the surface does the fish see your face?

<u>Solution</u> Because $n_1 > n_2$, where $n_2 = 1.00$ is the index of refraction for air, the rays originating from the fish in Fig. 5 are refracted away from the normal at the surface and diverge outward. Extending the outgoing rays backward shows an image point under the water. Because the refracting surface is flat, R is infinite. Hence, we have $q = -\frac{n_2}{n_1}p$, with $p = d$, yielding $q = -\frac{1.00}{1.33}d = -0.752d$. Because q is negative, the image is virtual as indicated by the dashed lines in Fig. 5. The apparent depth is approximately three-fourths the actual depth. *(ii)* The light rays from your face are shown in Fig. 5. Because the rays refract toward the normal, your face appears higher above the surface than it actually is. The image distance is $q = -\frac{n_2}{n_1}p = -\frac{1.33}{1.00}d = -1.33d$. The negative sign for q indicates that the image is in the medium from which the light originated, which is the air above the water.

8. A converging lens has a focal length of 10.0 cm. *(i)* An object is placed 30.0 cm from the lens. Construct a ray diagram, find the image distance, and describe the image. *(ii)* An object is placed 10.0 cm from the lens. Find the image distance and describe the image. *(iii)* An object is placed 5.00 cm from the lens. Construct a ray diagram, find the image distance, and describe the image.

Solution Because the lens is converging, the focal length is positive. We expect the possibilities of both real and virtual images. *(i)* Because the object distance is larger than the focal length, we expect the image to be real. The ray diagram for this situation is shown in the left panel of Fig. 6. The image distance is found through the relation $\frac{1}{q} = \frac{1}{f} - \frac{1}{p}$, yielding $q = 15.0$ cm. The magnification is $M = -\frac{q}{p} = -0.5$. The positive sign for the image distance tells us that the image is indeed real and on the back side of the lens. The magnification of the image tells us that the image is reduced in height by one half, and the negative sign for M tells us that the image is inverted. *(ii)* Because the object is at the focal point, we expect the image to be infinitely far away. Indeed, we have $\frac{1}{q} = \frac{1}{f} - \frac{1}{p}$, yielding $q \to \infty$. This result means that rays originating from an object positioned at the focal point of a lens are refracted so that the image is formed at an infinite distance from the lens; that is, the rays travel parallel to one another after refraction. *(iii)* Because the object distance is smaller than the focal length, we expect the image to be virtual. The ray diagram for this situation is shown in the right panel of Fig. 6. The image distance follows from the relation $\frac{1}{q} = \frac{1}{f} - \frac{1}{p}$ and so $q = -10$ cm. The magnification is $M = \frac{-q}{p} = 2.00$. The negative image distance tells us that the image is virtual and formed on the side of the lens from which the light is incident, the front side. The image is enlarged, and the positive sign for M tells us that the image is upright.

9. A diverging lens has a focal length of 10.0 cm. *(i)* An object is placed 30.0 cm from the lens. Construct a ray diagram, find the image distance, and describe the image. *(ii) (ii)* An object is placed 10.0 cm from the lens. Construct a ray diagram, find the image distance, and describe the image. *(iii)* An object is placed 5.00 cm from the lens. Construct a ray diagram, find the image distance, and describe the image.

Solution *(i)* Because the lens is diverging, the focal length is negative. The ray diagram is shown in the left panel of Fig. 7. Because the lens is diverging, we expect it to form an upright, reduced, virtual image for any object position. The image distance is obtain from the relation $\frac{1}{q} = \frac{1}{f} - \frac{1}{p}$. We have, $q = -7.50$ cm. The magnification is $M = -\frac{q}{p} = 0.250$. This result confirms that the image is virtual, smaller than the object and upright. *(ii)* The ray diagram is shown in the middel panel of Fig. 7. The image is located at $q = -5.00$ cm and the magnification is $M = 0.5$. Note the difference between this situation and that for a converging lens. For a diverging lens, an object at the focal point does not produce an image infinitely far away. *(iii)* The ray diagram is shown in the right panel of Fig. 7. The image distance is $q = -3.33$ cm and the magnification is $M = 0.667$. For all three object positions, the image position is negative and the magnification is a positive number smaller than 1, which confirms that the image is virtual, smaller than the object, and upright.

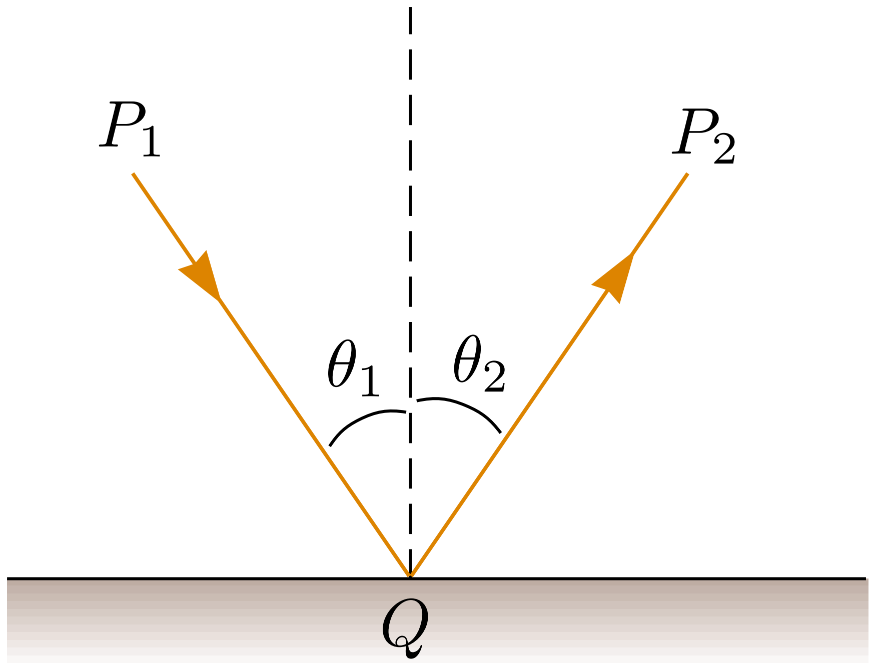

Figure 1: Reflection law. The incident ray, the reflected ray, and the normal all lie in the same plane; $\theta_1 = \theta_2$.

10. Two thin converging lenses of focal lengths $f_1 = 10.0$ cm and $f_2 = 20.0$ cm are separated by 20.0 cm. An object is placed 30.0 cm to the left of lens 1. Find the position and the magnification of the final image.

Solution Imagine light rays passing through the first lens and forming a real image (because $p > f$) in the absence of a second lens. Fig. 8 shows these light rays forming the inverted image I_1. Once the light rays converge to the image point, they do not stop. They continue through the image point and interact with the second lens. The rays leaving the image point behave in the same way as the rays leaving an object. Therefore, the image of the first lens serves as the object of the second lens. Using the thin lens equation, $\frac{1}{q_1} = \frac{1}{f} - \frac{1}{p_1}$, we find the location of the image formed by lens 1, $q_1 = 15.0$ cm. The magnification of the image is$M_1 = -\frac{q_1}{p_1} = -0.5$. The image formed by this lens acts as the object for the second lens. Therefore, the object distance for the second lens is $p_2 = 20.0 \text{ cm} - 15.0 \text{ cm} = 5.00$ cm. We find the location of the image formed by lens 2 from the thin lens equation, $q_2 = -6.67$ cm, and so the magnification is $M_2 = -\frac{q_2}{p_2} = 1.33$. The overall magnification of the system is $M = M_1 M_2 = -0.667$. The negative sign on the overall magnification indicates that the final image is inverted with respect to the initial object. Because the absolute value of the magnification is less than 1, the final image is smaller than the object. Because q_2 is negative, the final image is on the front, or left, side of lens 2. These conclusions are consistent with the ray diagram shown in Fig. 8.

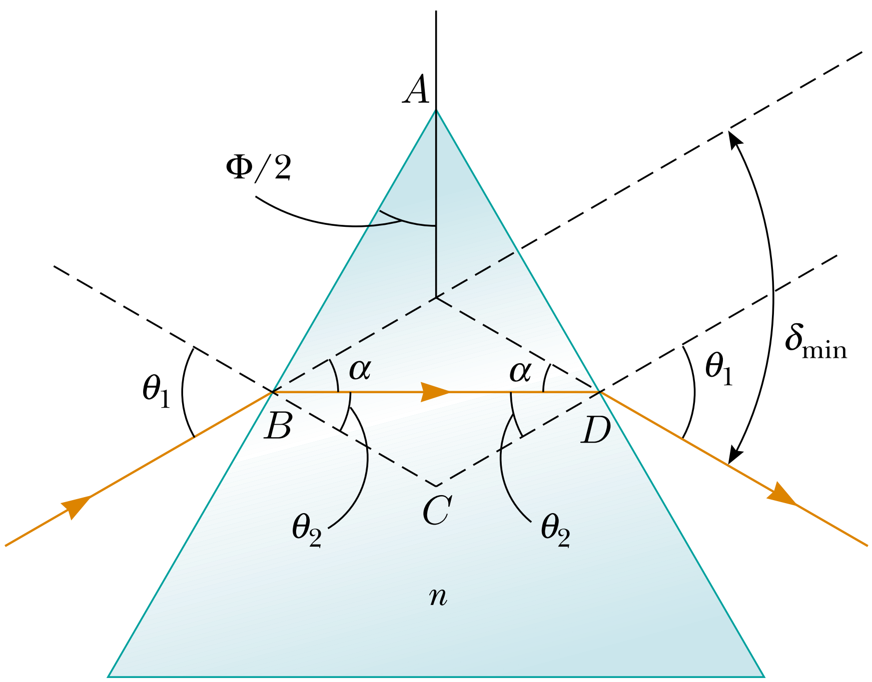

Figure 2: Geometry of a dispersing prism with a light ray passing through the prism at the minimum angle of deviation.

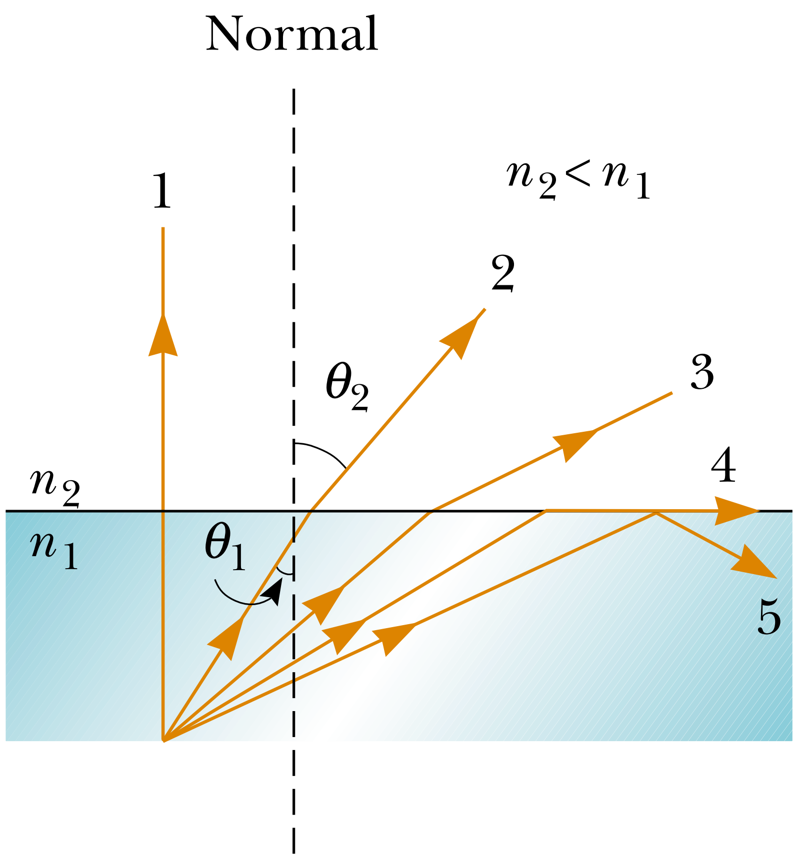

Figure 3: Rays travel from a medium of index of refraction n_1 into a medium of index of refraction n_2, where $n_2 < n_1$. As the angle of incidence θ_1 increases, the angle of refraction θ_2 increases until $\theta_2 = \pi/2$ (ray 4). For even larger angles of incidence, total internal reflection occurs (ray 5).

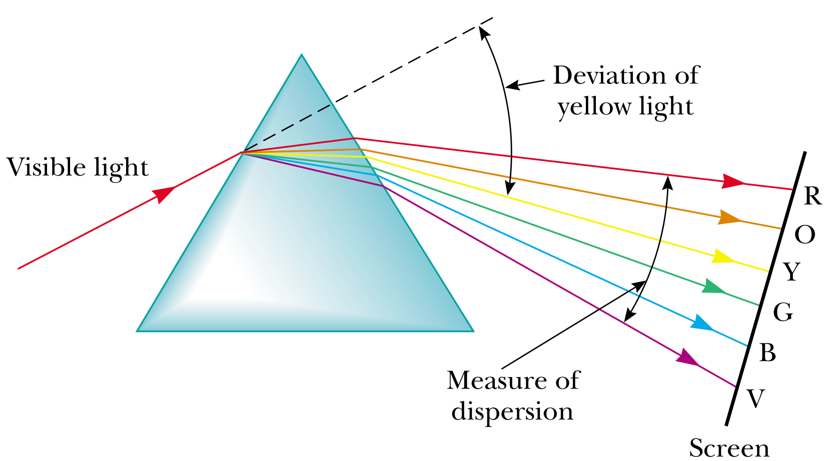

Figure 4: Newton's experiment showing that light is composed of colored components. A narrow beam of light is incident on a prism and produces a broadened and colored band which can be reconstituted back into a narrow white beam of light with a second prism.

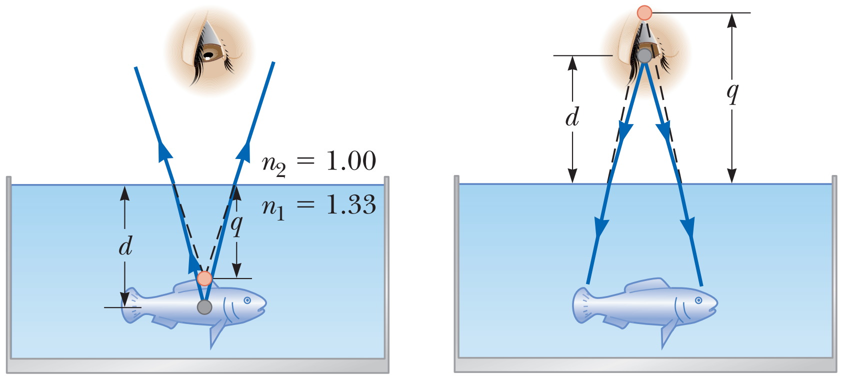

Figure 5: **Left panel.** The apparent depth q of the fish is less than the true depth d. All rays are assumed to be paraxial. **Right panel.** Your face appears to the fish to be higher above the surface than it is.

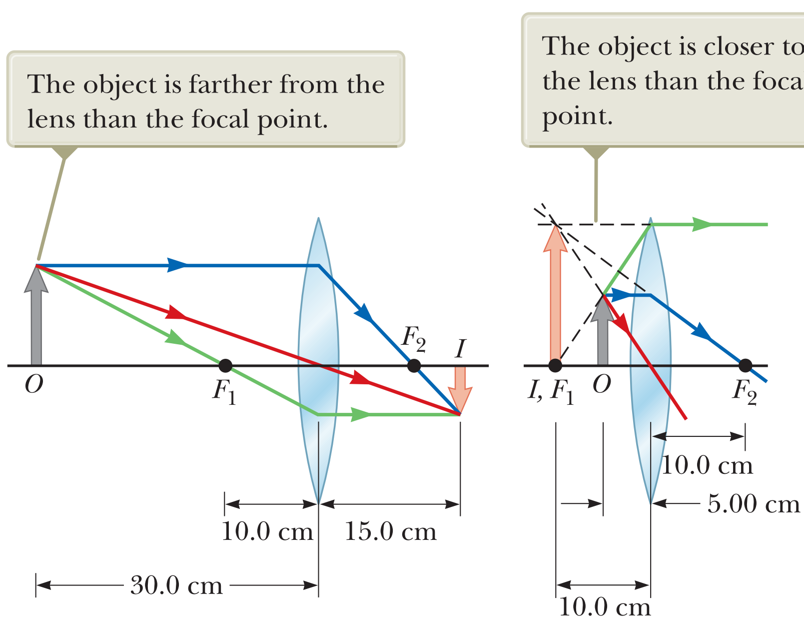

Figure 6: The situations in problem 8.

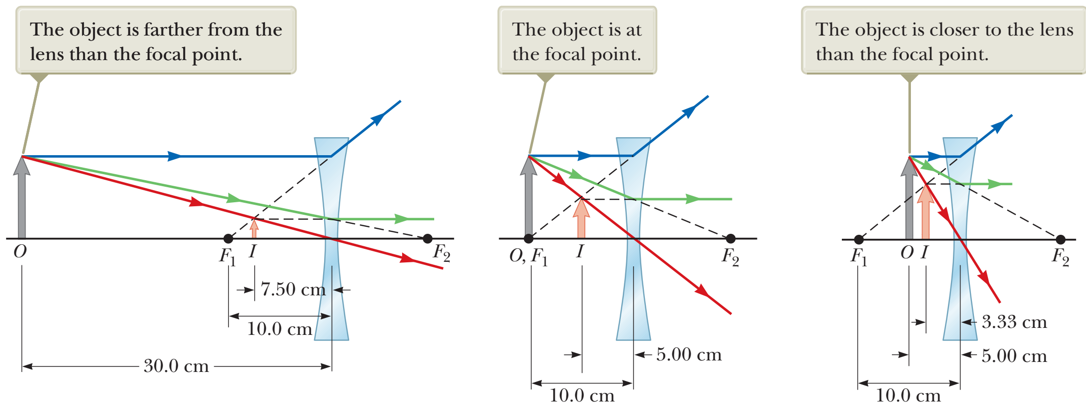

Figure 7: The situations in problem 9.

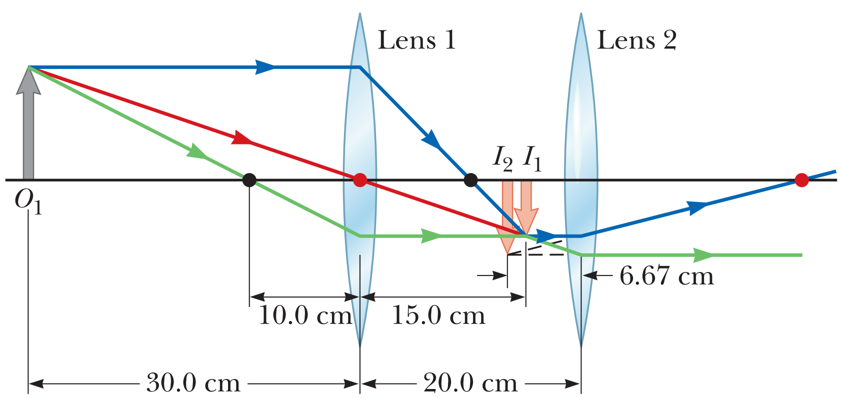

Figure 8: A combination of two converging lenses. The ray diagram shows the location of the final image (I_2) due to the combination of lenses. The black dots are the focal points of lens 1, and the red dots are the focal points of lens 2.