Prof. Anchordoqui

Problems set # 10 **Physics 169** April 28, 2015

1. A 12 V battery is connected into a series circuit containing a 10 Ω resistor and a 2 H inductor. In what time interval will the current reach *(i)* 50% and *(ii)* 90% of its final value?

Solution This circuit has only one loop. From Kirchhoffs rule we get only one (differential) equation $\varepsilon - L\frac{dI}{dt} - IR = 0$. We can separate the variables, $\frac{dI}{\varepsilon - IR} = \frac{dt}{L}$, and integrate both sides of the equation (within appropriate limits). With the reference for time at the instant of closing the circuit $\int_0^{I(t)} \frac{dI}{\varepsilon - IR} = \int_0^t \frac{dt}{L}$. (Note that we used one symbol t for two different quantities!). From the fundamental theorem of calculus $\ln \frac{\varepsilon - I(t)R}{\varepsilon} = -\frac{R}{L}t$. The solution is a time dependent function $I(t) = \frac{\varepsilon}{R}\left(1 - e^{-\frac{Rt}{L}}\right)$. *(i)* The current reaches its final value after an infinite amount of time: $I(\infty) = \lim_{t\to infty} \frac{\varepsilon}{R}\left(1 - e^{-Rt/L}\right) = \frac{\varepsilon}{R}$. At instant t, the current is a fraction of its final value

$I(t_1) = I(\infty)\left(1 - e^{-Rt/L}\right)$. From which $I(t)/I(\infty) = \left(1 - e^{-Rt/L}\right)$. Solving this equation for t we obtain $t = -\frac{L}{R}\ln\left[1 - I(t)/I(\infty)\right]$. Hence *(i)* $t_{50\%} = -\frac{2\text{ H}}{10\ \Omega}\ln(1-0.5) = 0.14$ s; *(ii)* $t_{90\%} = -\frac{2\text{ H}}{10\ \Omega}\ln(1-0.9) = 0.46$ s.

2. In the circuit shown in Fig. 1, let $L = 7.00$ H, $R = 9.00\ \Omega$, and $\varepsilon = 120$ V. What is the self-induced emf 0.200 s after the switch is closed?

Solution Duplicating the procedure of problem 1 we obtain $I = \frac{\varepsilon}{R}(1 - e^{-t/\tau})$. Since $\tau = L/R = 0.78$ s, we have $I = \frac{120\text{ V}}{9.00\ \Omega}(1 - e^{-0.26}) = 3.05$ A. Now $\Delta V_R = IR = 27.4$ V and $\Delta V_L = \varepsilon - \Delta V_R = 92.6$ V.

3. The switch in Fig. 2 is open for $t < 0$ and then closed at time $t = 0$. Find the current in the inductor and the current in the switch as functions of time thereafter.

Solution Name the currents as shown in Fig. 2. Using Kirchhoff's laws we obtain $I_1 = I_2 + I_3$, $10.0\text{ V} - 4.00 I_1 - 4.00 I_2 = 0$, and $10.0\text{ V} - 4.00 I_1 - 8.00 I_3 - 1.00\frac{dI_3}{dt} = 0$. From the first two equations it follows that $10.0\text{ V} + 4.00 I_3 - 8.00 I_1 = 0$ and $I_1 = 0.50 I_+ 3 + 1.25$ A. Then the last equation can be rewritten as $10.0\text{ V} - 4.00(0.500 I_3 + 1.25\text{ A}) - 8.00 I_3 - 1.00\text{ H}\frac{dI_3}{dt} = 0$, yielding $1\text{ H}\frac{dI_3}{dt} + 10.0\ \Omega I_3 = 5.00$ V. We solve the differential equation to obtain $I_3(t) = \frac{5.00\text{V}}{10.0\ \Omega}\left[1 - e^{-10.0\ \Omega t/1.00\text{ H}}\right] = 0.50\text{ A}\left[1 - e^{-10t/s}\right]$. Then $I_1 = 1.25 + 0.50 I_3 = 1.50\text{ A} - 0.25\text{ A}e^{-10t/s}$.

4. Assume that the magnitude of the magnetic field outside a sphere of radius R is $B = B_0(R/r)^2$, where B_0 is a constant. Determine the total energy stored in the magnetic field outside the sphere and evaluate your result for $B_0 = 5.00\times10^{-5}$ T and $R = 6.00\times10^6$ m, values appropriate for the Earth's magnetic field.

Solution The total magnetic energy is the volume integral of the energy density, $u = \frac{B^2}{2\mu_0}$. Because B changes with position, u is not constant.For $B = B_0(R/r)^2$, we have $u = \frac{B_0^2}{2\mu_0}(R/r)^4$.

Next, we set up an expression for the magnetic energy in a spherical shell of radius r and thickness dr. Such a shell has a volume $4\pi r^2 dr$, so the energy stored in it is $d\mathcal{U} = u4\pi r^2 dr = \frac{2\pi B_0^2 R^4}{\mu_0}\frac{dr}{r^2}$. We integrate this expression for $r = R$ to $r = \infty$ to obtain the total magnetic energy outside the sphere. This gives $\mathcal{U} = \frac{2\pi B_0^2 R^3}{\mu_0} = 2.70 \times 10^{18}$ J.

5. A large coil of radius R_1 and having N_1 turns is coaxial with a small coil of radius R_2 and having N_2 turns. The centers of the coils are separated by a distance x that is much larger than R_1 and R_2. What is the mutual inductance of the coils? Suggestion: John von Neumann proved that the same answer must result from considering the flux through the first coil of the magnetic field produced by the second coil, or from considering the flux through the second coil of the magnetic field produced by the first coil. In this problem it is easy to calculate the flux through the small coil, but it is difficult to calculate the flux through the large coil, because to do so you would have to know the magnetic field away from the axis.

Solution The large coil produces this field at the center of the small coil: $\frac{N_1\mu_0 I_1 R_1^2}{2(x^2+R_1^2)^{3/2}}$. The field is normal to the area of the small coil and nearly uniform over this area, so it produces a flux $\Phi_{12} = \frac{N_1\mu_0 I_1 R_1^2}{2(x^2+R_1^2)^{3/2}}\pi R_2^2$ through the face area of the small coil. When the current I_1 varies, this is the emf induced in the small coil: $\varepsilon_2 = -N_2\frac{d}{dt}\frac{N_1\mu_0 R_1^2\pi R_2^2}{2(x^2+R_1^2)^{3/2}}I_1 = -\frac{N_1N_2\mu_0 R_1^2\pi R_2^2}{2(x^2+R_1^2)^{3/2}}\frac{dI_1}{dt} = -M\frac{dI_1}{dt}$, hence $M = \frac{N_1N_2\mu_0 R_1^2\pi R_2^2}{2(x^2+R_1^2)^{3/2}}$.

6. Two inductors having self-inductances L_1 and L_2 are connected in parallel as shown in Fig. 3(a). The mutual inductance between the two inductors is M. Determine the equivalent self-inductance $L_{\rm eq}$ for the system shown in Fig. 3(b).

Solution With $I = I_1 + I_2$, the voltage across the pair is: $\Delta V = -L_1\frac{dI_1}{dt} - M\frac{dI_2}{dt} = -L_2\frac{dI_2}{dt} - M\frac{dI_1}{dt} = -L_{\rm eq}\frac{dI}{dt}$. Hence, $-\frac{dI_1}{dt} = \frac{\Delta V}{L_1} + \frac{M}{L_1}\frac{dI_2}{dt}$ and $-L_2\frac{dI_2}{dt} + \frac{M}{L_1}\frac{\Delta V}{} + \frac{M^2}{L_1}\frac{dI_2}{dt} = \Delta V$, yiedling

$$(-L_1L_2 + M^2)\frac{dI_2}{dt} = \Delta V(L_1 - M). \qquad (1)$$

By substitution, $-\frac{dI_2}{dt} = \frac{\Delta V}{L_2} + \frac{M}{L_2}\frac{dI_1}{dt}$ leads to

$$(-L_1L_2 + M^2)\frac{dI_1}{dt} = \Delta V(L_2 - M). \qquad (2)$$

Adding (1) to (2), $(-L_1L_2+M^2)\frac{dI}{dt} = \Delta V(L_1+L_2-2M)$, and therefore $L_{\rm eq} = -\frac{\Delta V}{dI/dt} = \frac{L_1L_2-M^2}{L_1+L_2-2M}$.

7. A long solenoid, which has $n = 400$ turns per meter, carries a current given by $I = 30\ A(1 - e^{-1.6t})$. Inside the solenoid and coaxial with is a coil that has a radius of 6 cm and consists of a total of N = 250 turns of fine wire. *(i)* What emf is induced in the coil by the changing current? *(ii)* As indicated in Fig. 4, the current in the solenoid flows in the clockwise direction, what is the direction of the current induced in the coil, clockwise or anticlockwise? Explain how you arrived at your answer *(iii)* Consider now the situation in which an inductor is discharging, i.e., $I = 30\ Ae^{-1.6t}$, the direction of the current is still clockwise. What is the direction of the current

induced in the coil now (clockwise or anticlockwise)? Explain how you arrived at your answer.

Solution A long solenoid produces a uniform magnetic field. The strength B of the field depends on the number of B turns per unit length n and the current in the solenoid I, $B = \mu_0 n I$, where μ_0 is the permeability of free space. Therefore the flux through the surface spanned by the coil (in the direction of the field vector) is equal to the product of the magnetic field strength and the area of the coil $\Phi_{\rm B} = \int_{\rm surface} \vec{B} \cdot d\vec{A} = \int_{\rm surface} B dA \cos 0^\circ = BA = \mu_0 n I \pi R^2$. From the Faraday's law of induction, the electromotive force induced in the coil is $\varepsilon = -N \frac{d\Phi_{\rm B}}{dt} = -N \mu_0 n \pi R^2 \frac{dI}{dt} = -\mu_0 N n \pi r^2 I_0 \alpha e^{-\alpha t} = -4\pi 10^{-7} \frac{\rm N}{\rm A^2} \cdot 250 \cdot 400 \cdot \pi (0.06\ m)^2 \cdot 30\ {\rm A} 1.6 \frac{1}{s} e^{-1.6t} = -0.068\ {\rm V} \cdot e^{-1.6t}$. *(ii)* The magnetic field through the coil is to the left and the magnetic flux is increasing, we need to induce a current that oposes to the increase of the flux (Lenz law). The induced current is then counterclockwise producing a magnetic field to the right. *(iii)* The magnetic field on the coil is to the left as before, but the magnetic flux is now decreasing. The induced emf in the coil is now 0.068 V and the current is clockwise producing a magnetic field to the left.

8. In the circuit of Fig. 5, the battery emf is 50.0 V, the resistance is 250 Ω, and the capacitance is 0.500 μF. The switch S is closed for a long time and no voltage is measured across the capacitor. After the switch is opened, the potential difference across the capacitor reaches a maximum value of 150 V. What is the value of the inductance?

Solution When the switch has been closed for a long time, battery, resistor, and coil carry constant current $I_{\rm max} = \varepsilon / R$. When the switch is opened, the current in battery and resistor drops to zero, but the coil carries this same current for a moment as oscillations begin in the LC loop. We interpret the problem to mean that the voltage amplitude of these oscillations is ΔV, $\frac{1}{2} C (\Delta V)^2 = \frac{1}{2} L I_{\rm max}^2$. Thus, $L = \frac{C(\Delta V)^2}{I_{\rm max}^2} = \frac{C(\Delta V)^2 R^2}{\varepsilon^2} = \frac{0.5 \times 10^{-6}\ {\rm F}(150\ {\rm V}^2)(250\ \Omega)^2}{(50.0\ {\rm V})^2} = 0.281$ H.

9. An inductor consists of two very thin conducting cylindrical shells, one of radius a and one of radius b, both of length h. Assume that the inner shell carries current I out of the page, and that the outer shell carries current I into the page, distributed uniformly around the circumference in both cases. The z-axis is out of the page along the common axis of the cylinders and the r-axis is the radial cylindrical axis perpendicular to the z-axis. *(i)* Use Ampere's law to find the magnetic field between the cylindrical shells. Indicate the direction of the magnetic field on the sketch. What is the magnetic energy density as a function of r for $a < r < b$? *(ii)* Calculate the inductance of this long inductor recalling that $\mathcal{U} = \frac{1}{2} L I^2$ and using your results for the magnetic energy density in (i). *(ii)* Calculate the inductance of this long inductor by using the formula $\Phi = LI = \int_{\substack{\rm open \\ \rm surface}} \vec{B} \cdot d\vec{A}$ and your results for the magnetic field in (i). To do this you must choose an appropriate open surface over which to evaluate the magnetic flux. Does your result calculated in this way agree with your result in (ii)?

Solution *(i)* The enclosed current Ienc in the Ampere's loop with radius r is given by

$$I_{\text{enc}} = \begin{cases} 0 & r < a \\ I & a < r < b \\ 0 & r > b \end{cases} .$$

Applying Ampere's law, $\oint \vec{B} \cdot d\vec{s} = B(2\pi r) = \mu_0 I_{\text{enc}}$, we obtain

$$\vec{B} = \begin{cases} 0 & r < a \\ \frac{\mu_0 I}{2\pi r}\hat{\phi} & a < r < b \text{ (counterclockwise in the figure)} \\ 0 & r > b \end{cases} .$$

The magnetic energy density for $a < r < b$ is $u_B = \frac{B^2}{2\mu_0} = \frac{1}{2\mu_0}\left(\frac{\mu_0 I}{2\pi r}\right)^2 = \frac{\mu_0 I^2}{8\pi^2 r^2}$. It is zero elsewhere. *(ii)* The volume element in this case is $2\pi r h dr$. The magnetic energy is : $\mathcal{U}_B = \iiint_V u_B dV = \int_a^b \frac{\mu_0 I^2}{8\pi^2 r^2} 2\pi h r dr = \frac{\mu_0 I^2 h}{4\pi}\ln(b/a)$. Since $\mathcal{U}_B = \frac{1}{2}LI^2$, the inductance is $L = \frac{\mu_0 h}{2\pi}\ln(b/a)$. *(iii)* The magnetic field is perpendicular to a rectangular surface shown Fig. 6. The magnetic flux through a thin strip of area $dA = hdr$ is $d\Phi = BdA = \frac{\mu_0 I}{2\pi r}hdr = \frac{\mu_0 I h}{2\pi r}dr$. Thus, the total magnetic flux is $\Phi_B = \int d\Phi_B = \int_a^b \frac{\mu_0 I h}{2\pi r}dr = \frac{\mu_0 I h}{2\pi}\ln(b/a)$. Finally, the inductance is $L = \frac{\Phi_B}{I} = \frac{\mu_0 h}{2\pi}\ln(b/a)$, which agrees with that obtained in (ii).

10. The energy of an RLC circuit decreases by 1.00% during each oscillation when $R = 2.00\ \Omega$. If this resistance is removed, the resulting LC circuit oscillates at a frequency of 1.00 kHz. Find the values of the inductance and the capacitance.

Solution The period of damped oscillation is $T = \frac{2\pi}{\omega_{\text{d}}}$. After one oscillation the charge returning to the capacitor is $Q = Q_{\text{max}}e^{-\frac{RT}{2L}} = Q_{\text{max}}e^{-\frac{2\pi R}{2L\omega_{\text{d}}}}$. The energy is proportional to the charge squared,so after one oscillation it is $\mathcal{U} = \mathcal{U}_0 e^{-\frac{2\pi}{L\omega_{\text{d}}}}$. Then $e^{\frac{2\pi R}{L\omega_{\text{d}}}} = \frac{1}{0.99}$, which leads to $\frac{2\pi 2\Omega}{L\omega_{\text{d}}} = \ln(1.0101) = 0.001005$. It follows that $L\omega_{\text{d}} = \frac{2\pi 2\Omega}{0.001005} = 1250\Omega = L\left(\frac{1}{LC} - \frac{R^2}{4L^2}\right)^{1/2}$, yielding $1.563 \times 10^6\Omega^2 = \frac{L}{C} - \frac{(2\Omega)^2}{4}$ or equivalently $\frac{L}{C} = 1.563 \times 10^6\Omega^2$. We are also given $\omega = 2\pi \times 10^3/\text{s} = \frac{1}{\sqrt{LC}}$, which leads to $LC = \frac{1}{(2\pi\times 10^3/\text{s})^2} = 2.533 \times 10^{-8}\ \text{s}^2$. Solving simultaneously, $C = 2.533 \times 10^{-8}\ \text{s}^2/L$, yields $\frac{L^2}{2.533\times 10^{-8}\ \text{s}^2} = 1.563 \times 10^6\ \Omega^2$; therefore $L = 0.199$ H and $C = \frac{2.533\times 10^{-8}\ \text{s}^2}{0.199\ \text{H}} = 127$ nF.

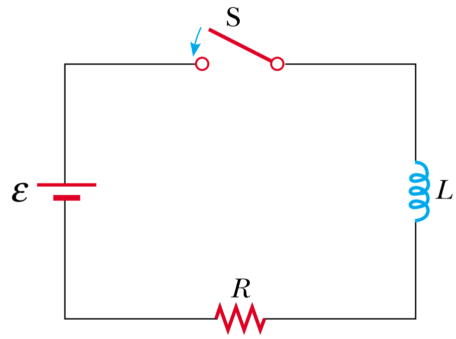

Figure 1: Problem 2.

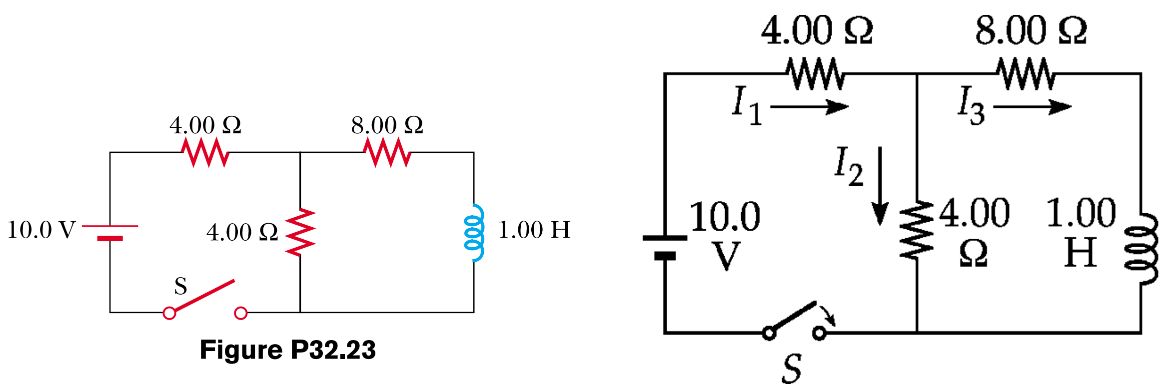

Figure 2: Problem 3.

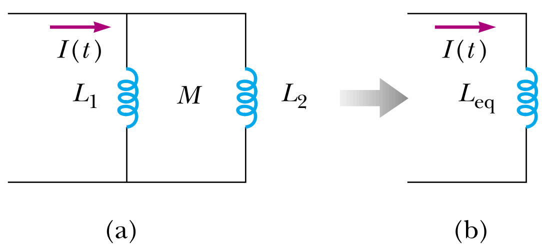

Figure 3: Problem 6.

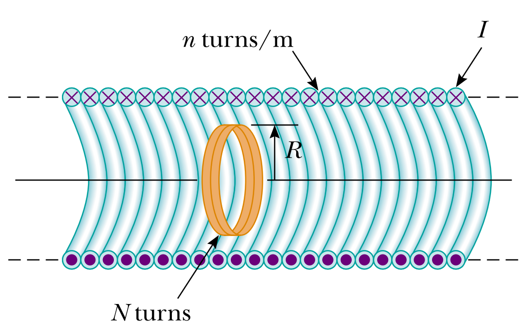

Figure 4: Problem 7.

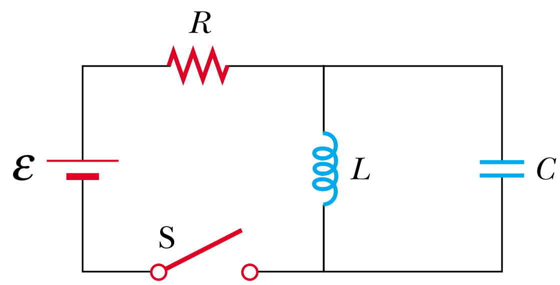

Figure 5: Problem 8.

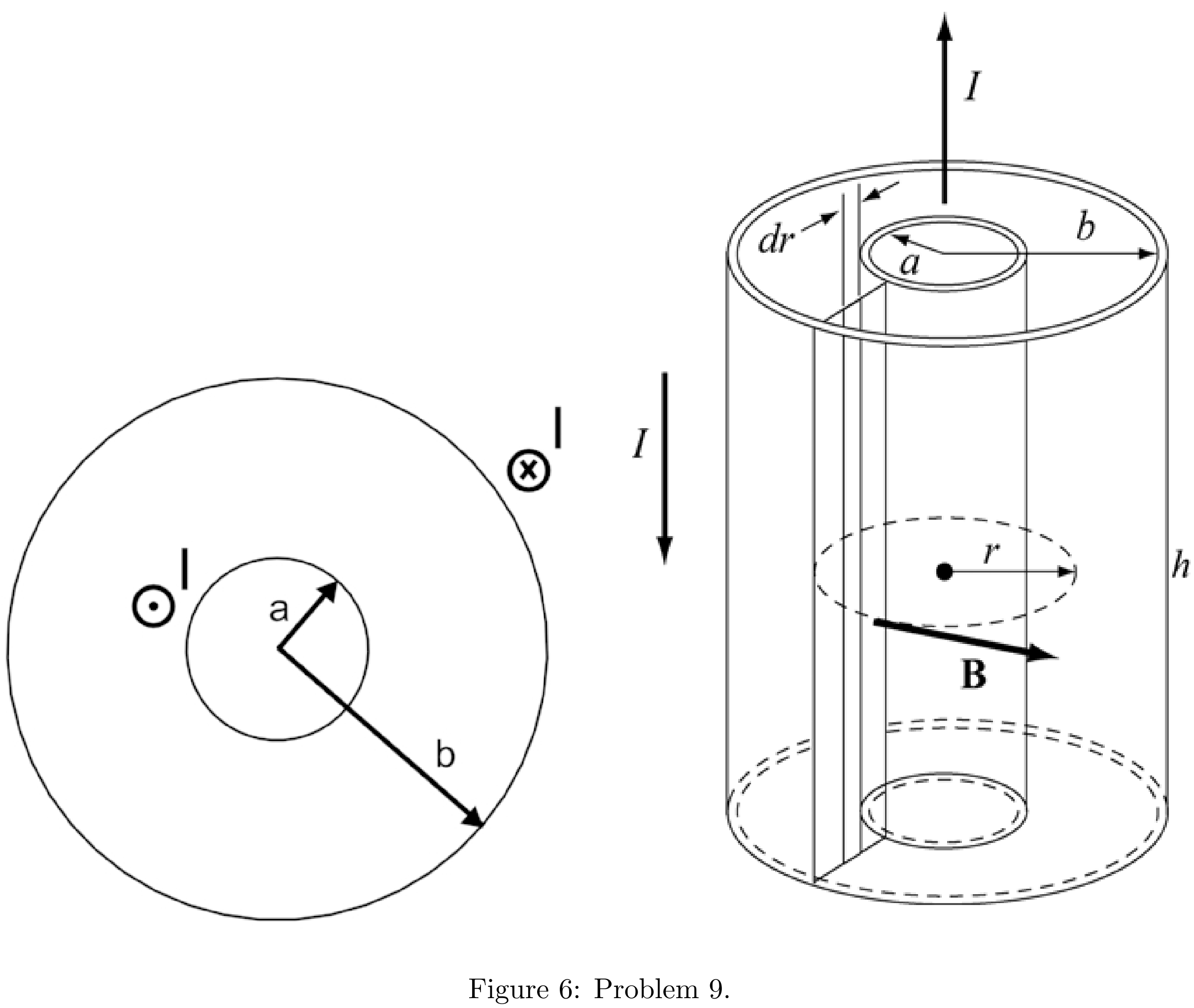

Figure 6: Problem 9.