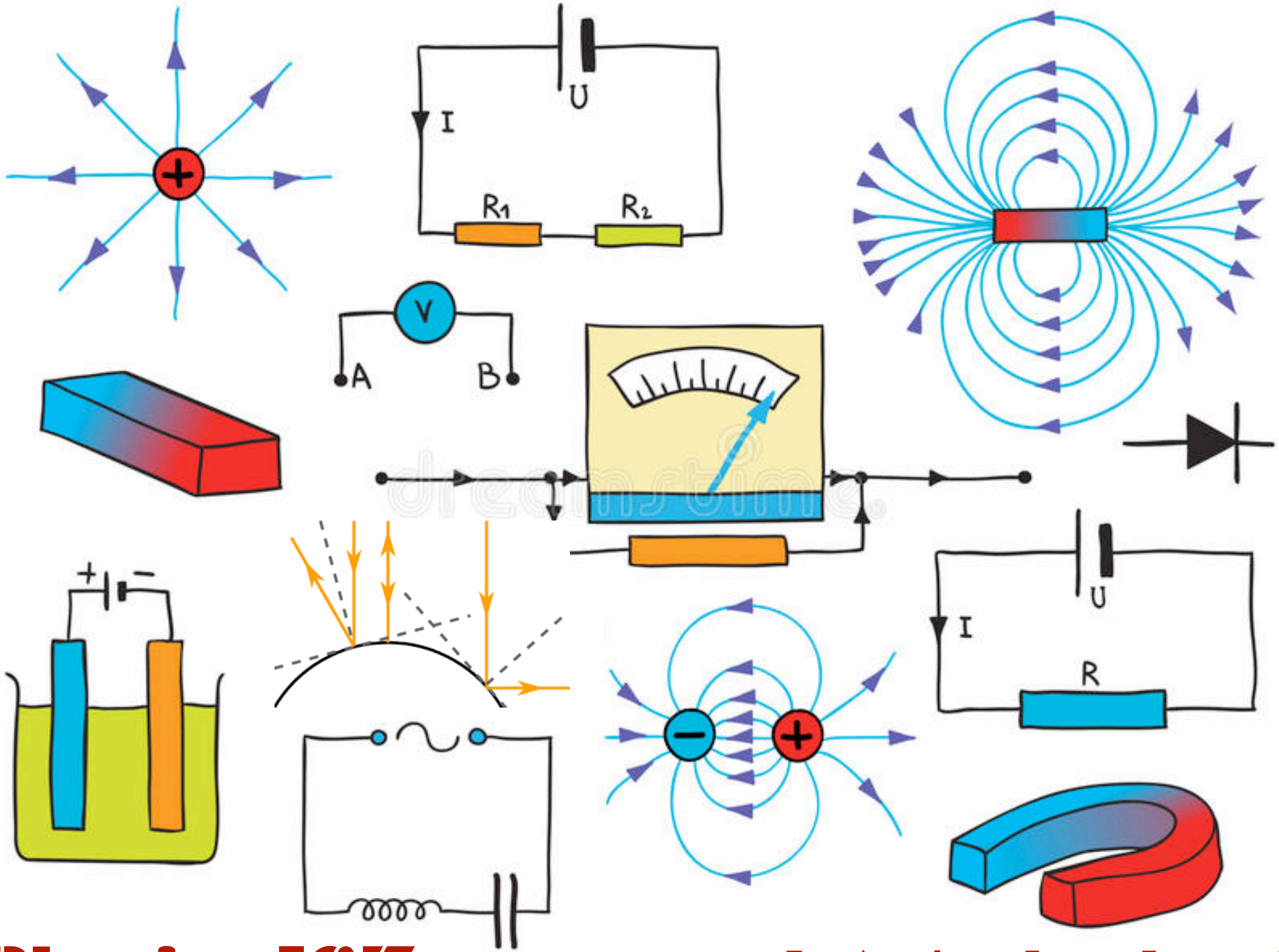


Physics 167

Luis Anchordoqui

Capacitors

➢ A capacitor is a system of **two conductors** that carries **equal and opposite charges**

➢ A capacitor **stores charge** and **energy** in the form of electro-static field

➢ We define capacitance as $C = \dfrac{Q}{V}$ Unit ☛ Farad(F)

$Q =$ Charge on one plate

$V =$ Potential difference between plates

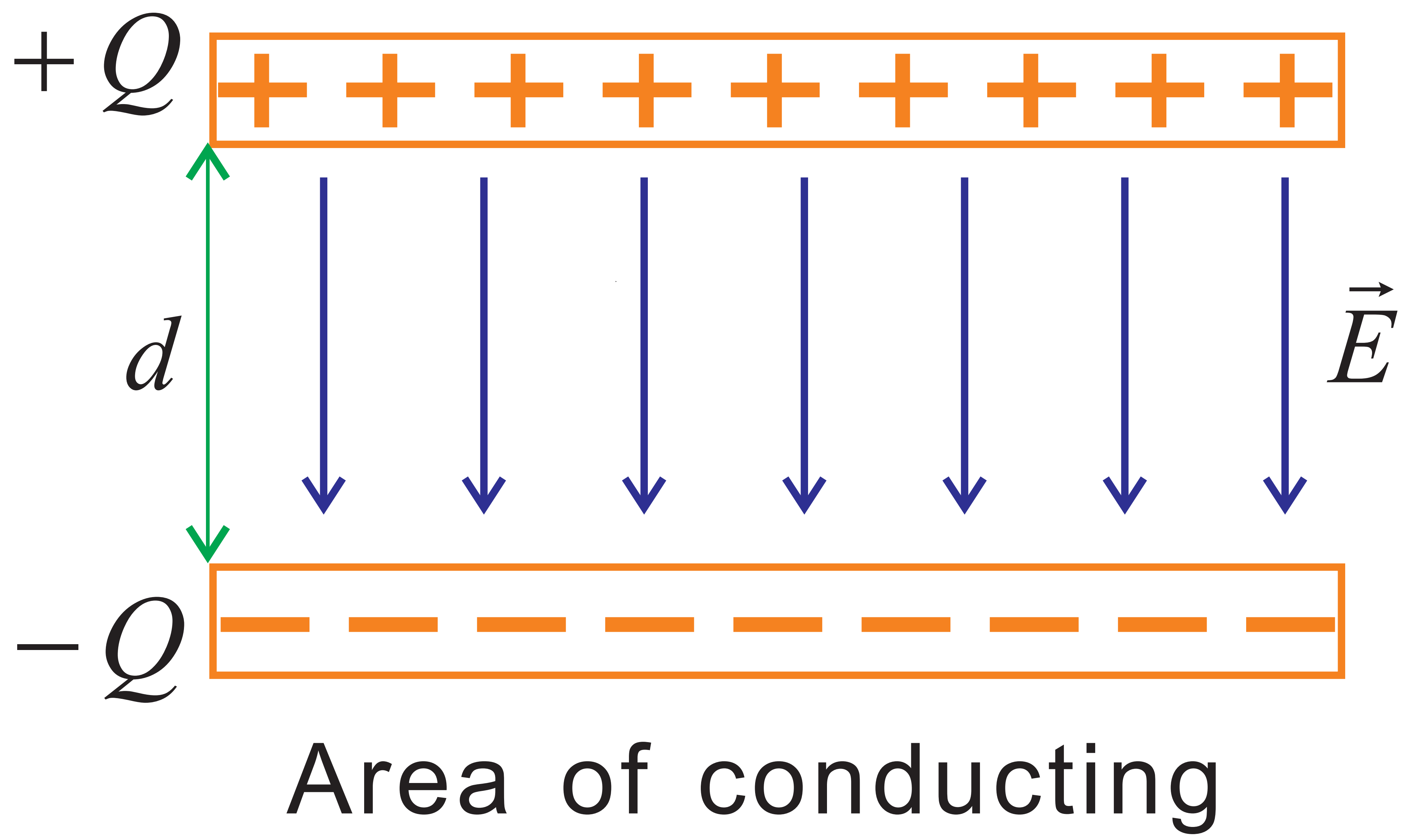
Note

capacitor's C is a constant that depends only on its shape and material

i.e. If we increase V for a capacitor we increase Q stored

Caculating Capacitance

Parallel - Plate Capacitors



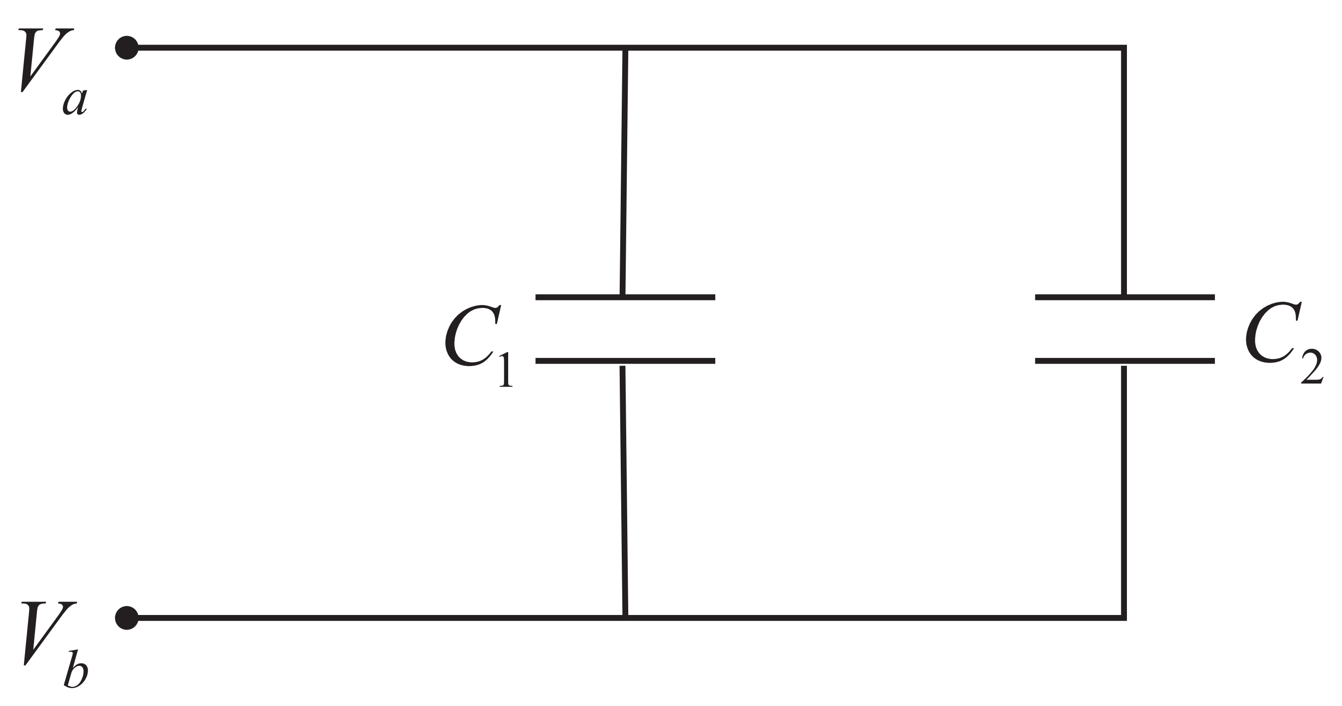
① Recall ☛ $|\vec{E}| = \dfrac{\sigma}{\epsilon_0} = \dfrac{Q}{\epsilon_0 A}$

② Recall ☛ $\Delta V = V_+ - V_- = Ed = \dfrac{Q}{\epsilon_0 A} d$

③ $\therefore \quad C = \dfrac{Q}{\Delta V} = \dfrac{\epsilon_0 A}{d}$

Capacitors in Parallel and Series

(a) Capacitors in Parallel ☛ potential difference $V = V_a - V_b$ is same across capacitors



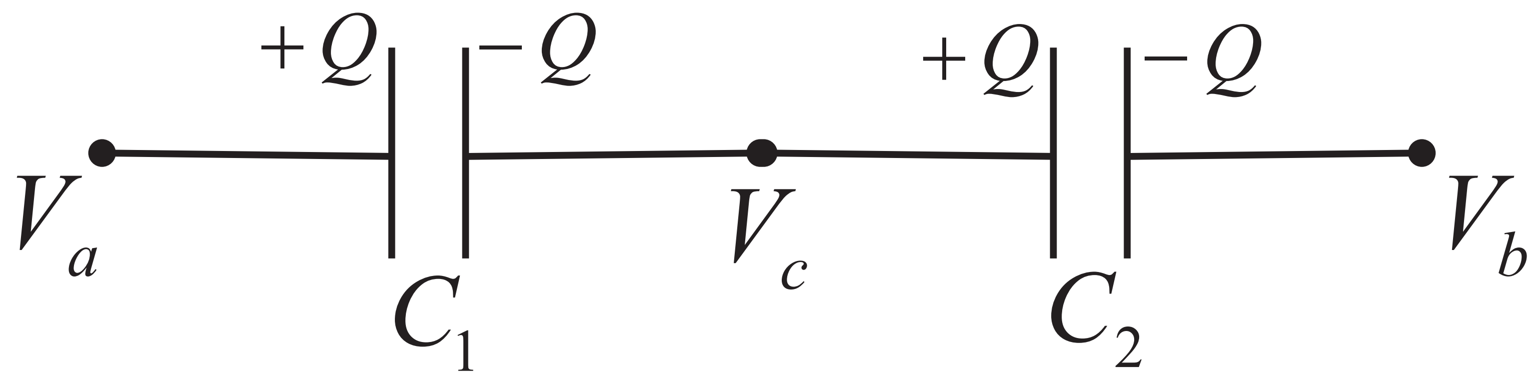
BUT ☛ **Charge** on each capacitor different

Total Charge

$$\begin{aligned} Q &= Q_1 + Q_2 \\ &= C_1 V + C_2 V \\ Q &= \underbrace{(C_1 + C_2)}_{\textbf{Equivalent capacitance}} V \end{aligned}$$

$\therefore$ For capacitors in parallel ☛ $C = C_1 + C_2$

(b) Capacitors in Series ☛ charge across capacitors are same



BUT ☛ **potential difference** (P.D.) across capacitors different

$$\Delta V_1 = V_a - V_c = \frac{Q}{C_1} \qquad \text{P.D. across } C_1$$

$$\Delta V_2 = V_c - V_b = \frac{Q}{C_2} \qquad \text{P.D. across } C_2$$

$\therefore$ Potential difference

$$\Delta V = V_a - V_b$$

$$= \Delta V_1 + \Delta V_2$$

$$\Delta V = Q\left(\frac{1}{C_1} + \frac{1}{C_2}\right) = \frac{Q}{C}$$

C ☛ **Equivalent Capacitance**

$$\therefore \quad \frac{1}{C} = \frac{1}{C_1} + \frac{1}{C_2}$$

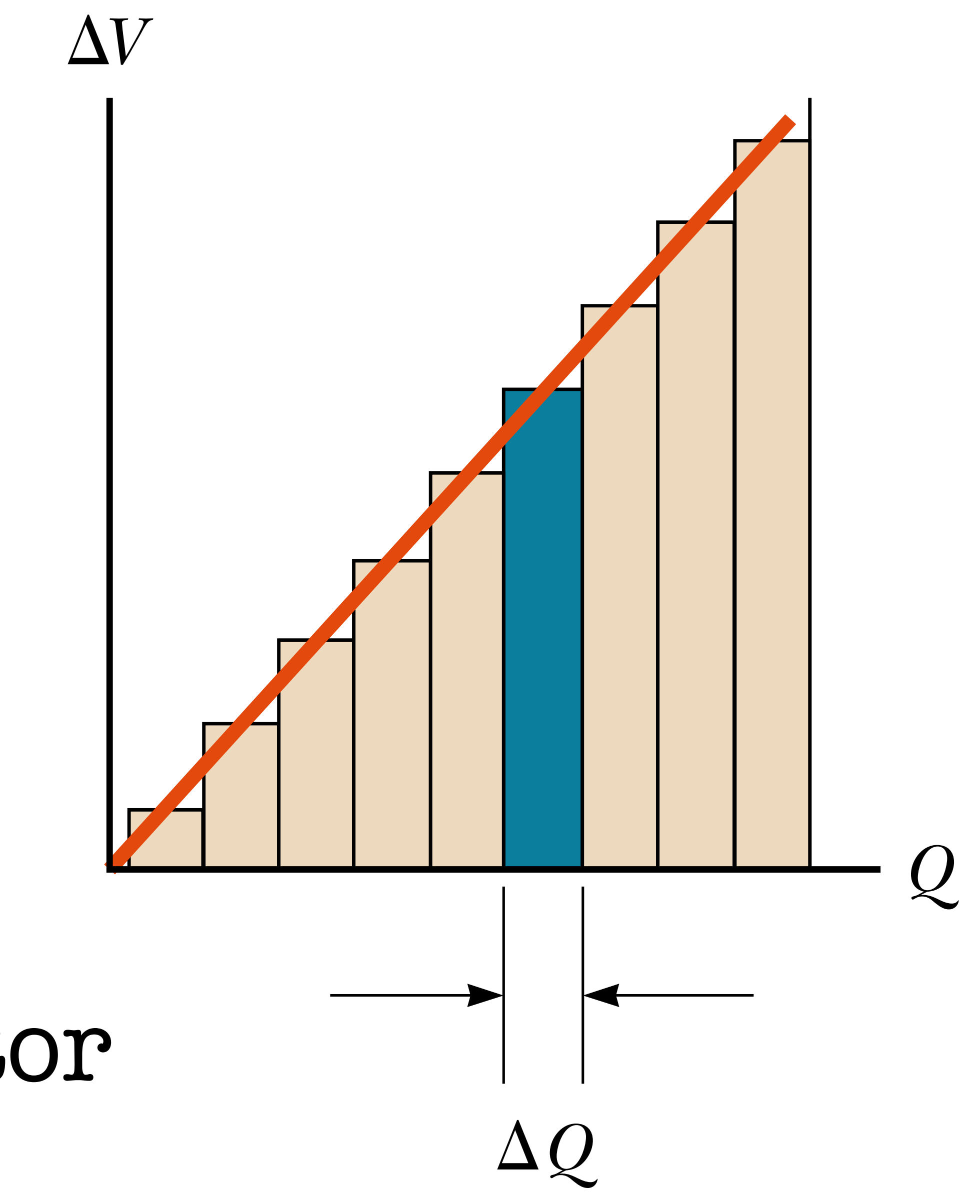
Energy Storage in Capacitors

➢ When capacitor is uncharged ☛ no work is required to move first bit of charge over

➢ As more charge is transferred ☛ work is needed to move charge against increasing voltage V

➢ Work needed to add a small amount of charge ΔQ when potential difference across plates is ΔV

$$\Delta W = \Delta V \; \Delta Q$$



➢ For capacitor with charge Q ☛ $\Delta V = Q/C$

➢ Plot of voltage versus total charge gives straight line with slope of $1/C$

➢ Work ΔW for particular ΔV ☛ area of blue rectangle

➢ Adding up all rectangles gives approximation of total work needed to fill capacitor

➢ For $\Delta Q/Q \ll 1$ ☛ total work needed to charge capacitor to final Q and ΔV is area under line

➢ Area of triangle ☛ $\Delta W = \frac{1}{2} Q \; \Delta V$

$$\text{Energy stored} = \frac{1}{2} Q \; \Delta V = \frac{1}{2} C (\Delta V)^2 = \frac{Q^2}{2C}$$

➢ Energy stored in capacitor is stored in electric field between plates

Note

➢ In parallel-plate capacitor ☛ $\vec{E}$ -field is constant between plates

$$\text{density } u = \frac{\text{total energy stored}}{\text{total volume with } \vec{E}\text{−field}}$$

$$\therefore \; u = \frac{U}{\underbrace{Ad}_{\text{Rectangular Volume}}}$$

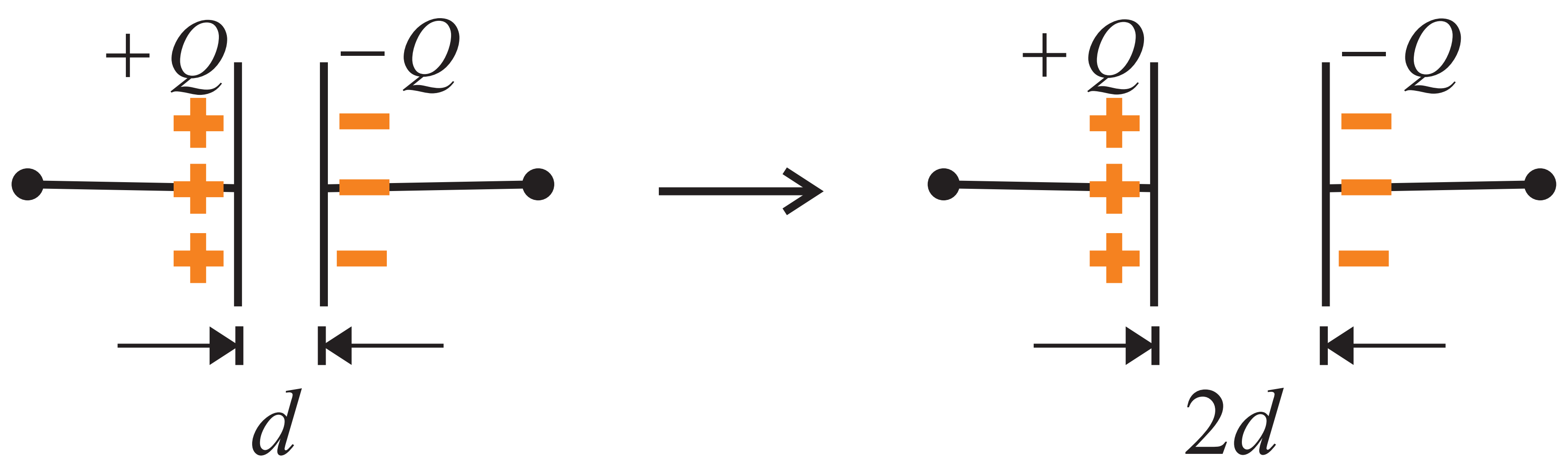
Recall ☛ $\begin{cases} C = \dfrac{\epsilon_0 A}{d} \\ E = \dfrac{\Delta V}{d} \quad \Rightarrow \quad \Delta V = Ed \end{cases}$

$$\therefore \; u = \frac{1}{2}\,(\overbrace{\frac{\epsilon_0 A}{d}}^{C}) \cdot (\overbrace{Ed}^{(\Delta V)^2})^2 \cdot \overbrace{\frac{1}{Ad}}^{\frac{1}{\text{Volume}}}$$

Energy per unit volume of electrostatic field ☛ $u = \dfrac{1}{2}\,\epsilon_0 E^2$

Example

Changing capacitance by pulling plates apart



① Isolated Capacitor

Charge on capacitor plates remains constant

BUT ☛

$$C_{new} = \frac{\epsilon_0 A}{2d} = \frac{1}{2} C_{old}$$

$$\therefore \quad U_{new} = \frac{Q^2}{2C_{new}} = \frac{Q^2}{2C_{old}/2} = 2U_{old}$$

$\therefore$ In pulling plates apart work done $W > 0$

Summary ☛

$$Q \rightarrow Q \qquad C \rightarrow C/2$$

$$(V = \frac{Q}{C}) \Rightarrow V \rightarrow 2V \qquad E \rightarrow E \qquad (E = \frac{V}{d})$$

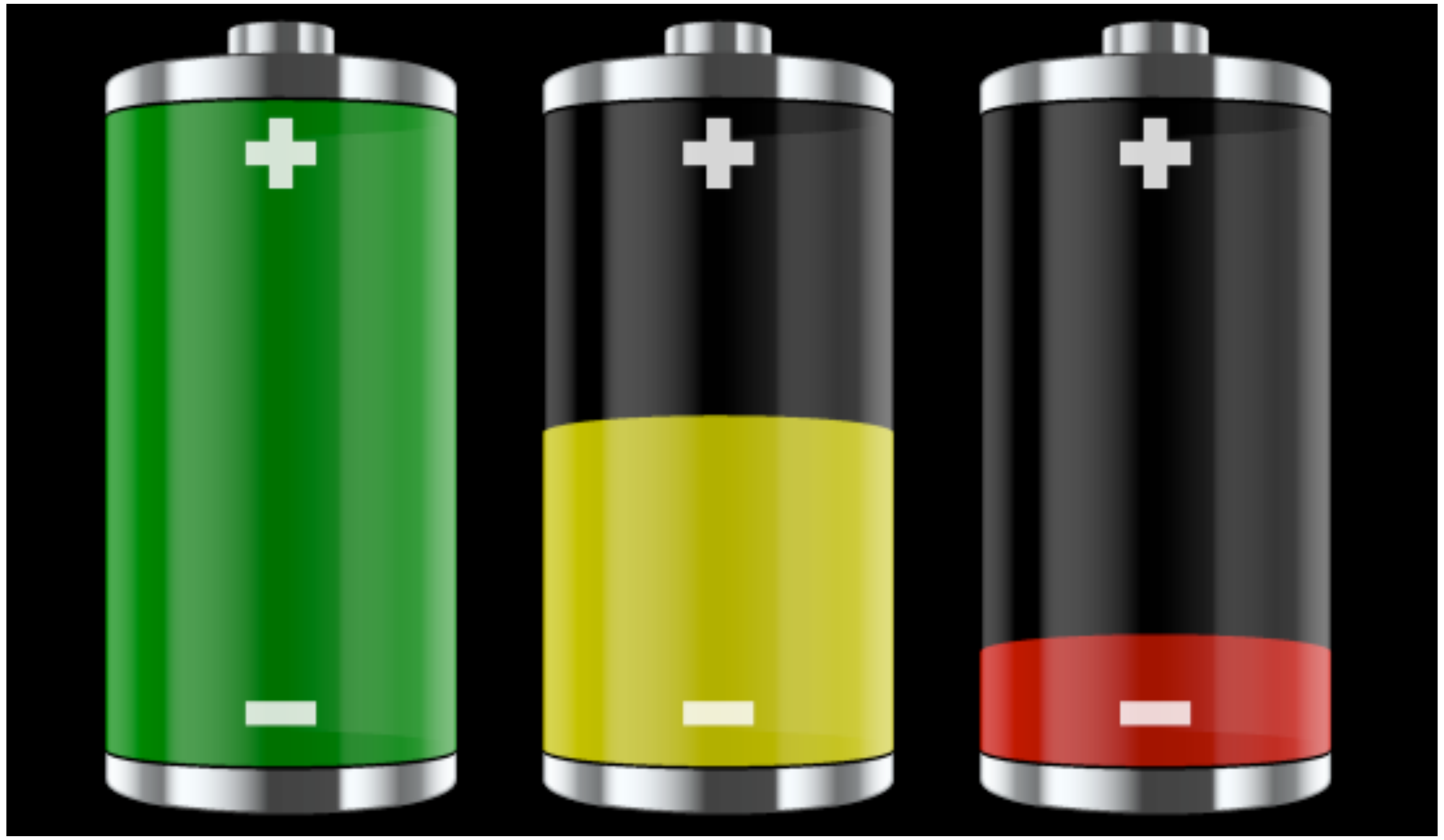
$$\frac{1}{2}\epsilon_0 E^2 = u \rightarrow u \qquad U \rightarrow 2U \qquad (U = u \cdot \text{volume})$$

Electric battery

Device consisting of 2 or more electrochemical cells that convert stored chemical energy into electrical energy

Each cell has

positive terminal (or **cathode**)



negative terminal (or **anode**)

② Capacitor connected to a battery

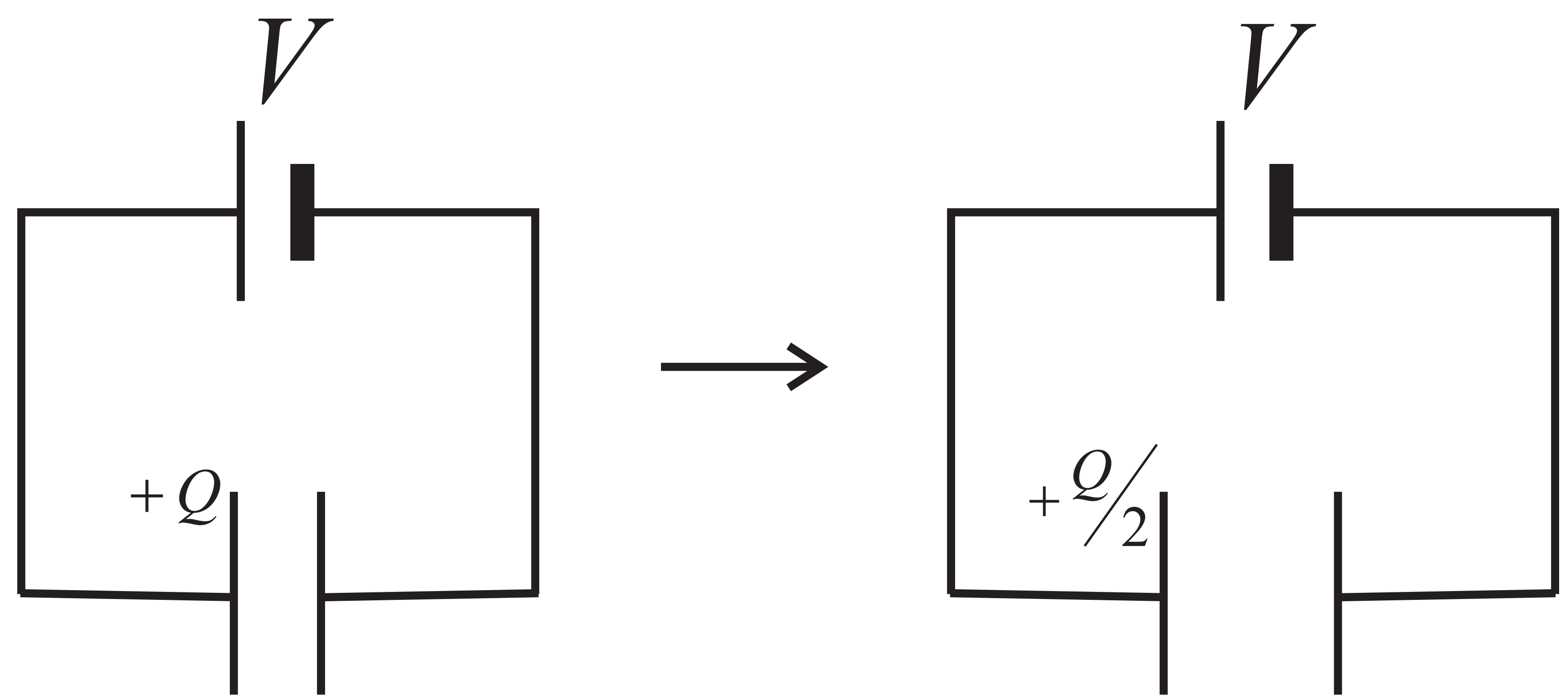
Potential difference between capacitor plates remains **constant**

$$U_{new} = \frac{1}{2} C_{new} \Delta V^2 = \frac{1}{2} \cdot \frac{1}{2} C_{old} \Delta V^2 = \frac{1}{2} U_{old}$$

$\therefore$ In pulling plates apart work done by battery < 0

Summary ☛

$$\begin{array}{ccc} Q & \to & Q/2 \\ V & \to & V \\ u & \to & u/4 \end{array} \qquad \begin{array}{ccc} C & \to & C/2 \\ E & \to & E/2 \\ U & \to & U/2 \end{array}$$

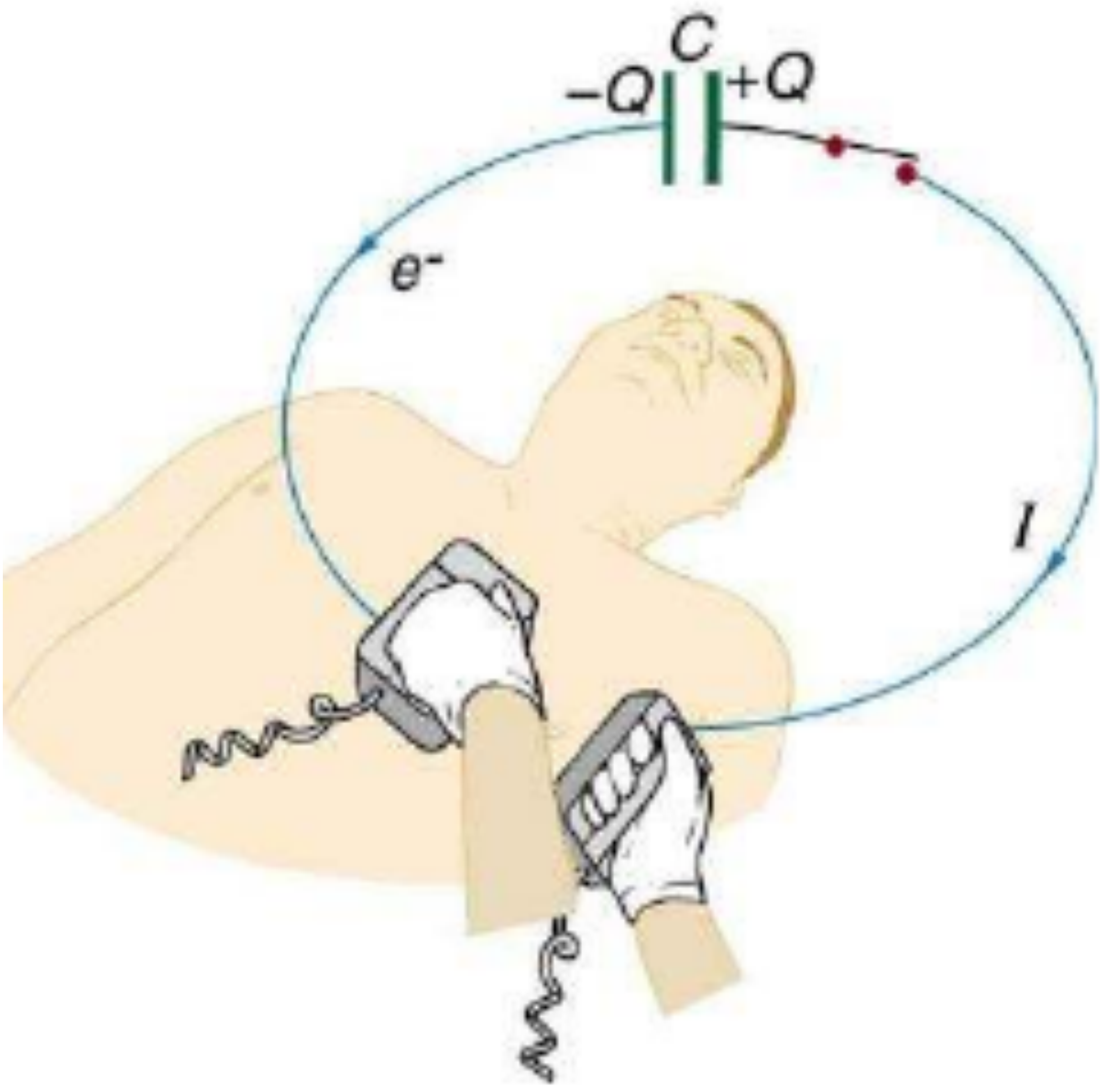


Defibrillator

A defibrillator uses a capacitor that is charged to a high voltage to create charge flow that gets heart going again

If the capacitor has a capacitance of 30 μF and is charged to 5,000 V,

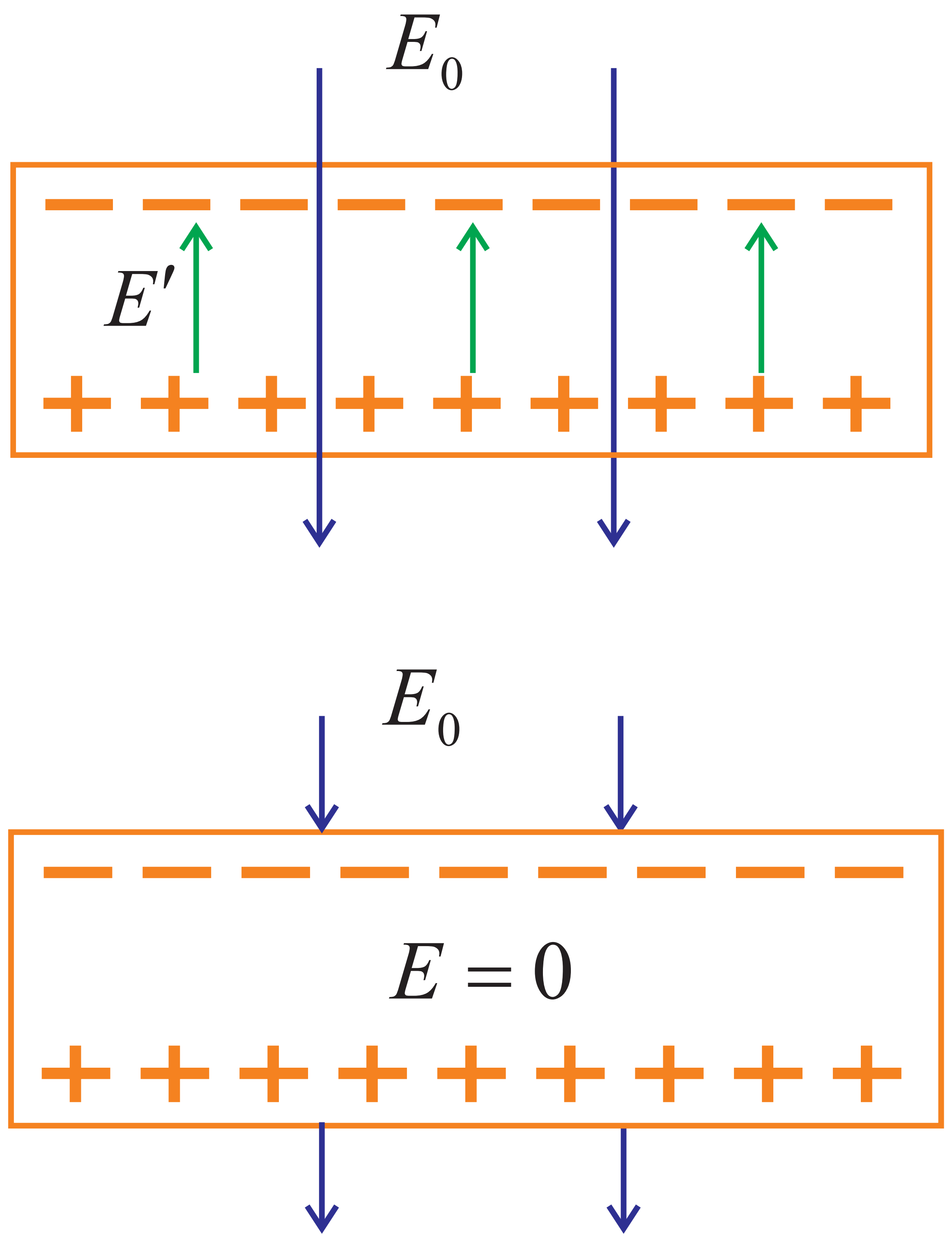
how much energy is stored in the capacitor?



Energy = $\frac{1}{2}CV^2 = \frac{1}{2}(30 \times 10^{-6}\ F)(5000\ V)^2 = 375\ J$

Dielectrics

➢ Consider **conductor** being placed in an **external** E_0-field



➢ In a conductor charges are free to move inside internal E'-field set up by these charges satisfies

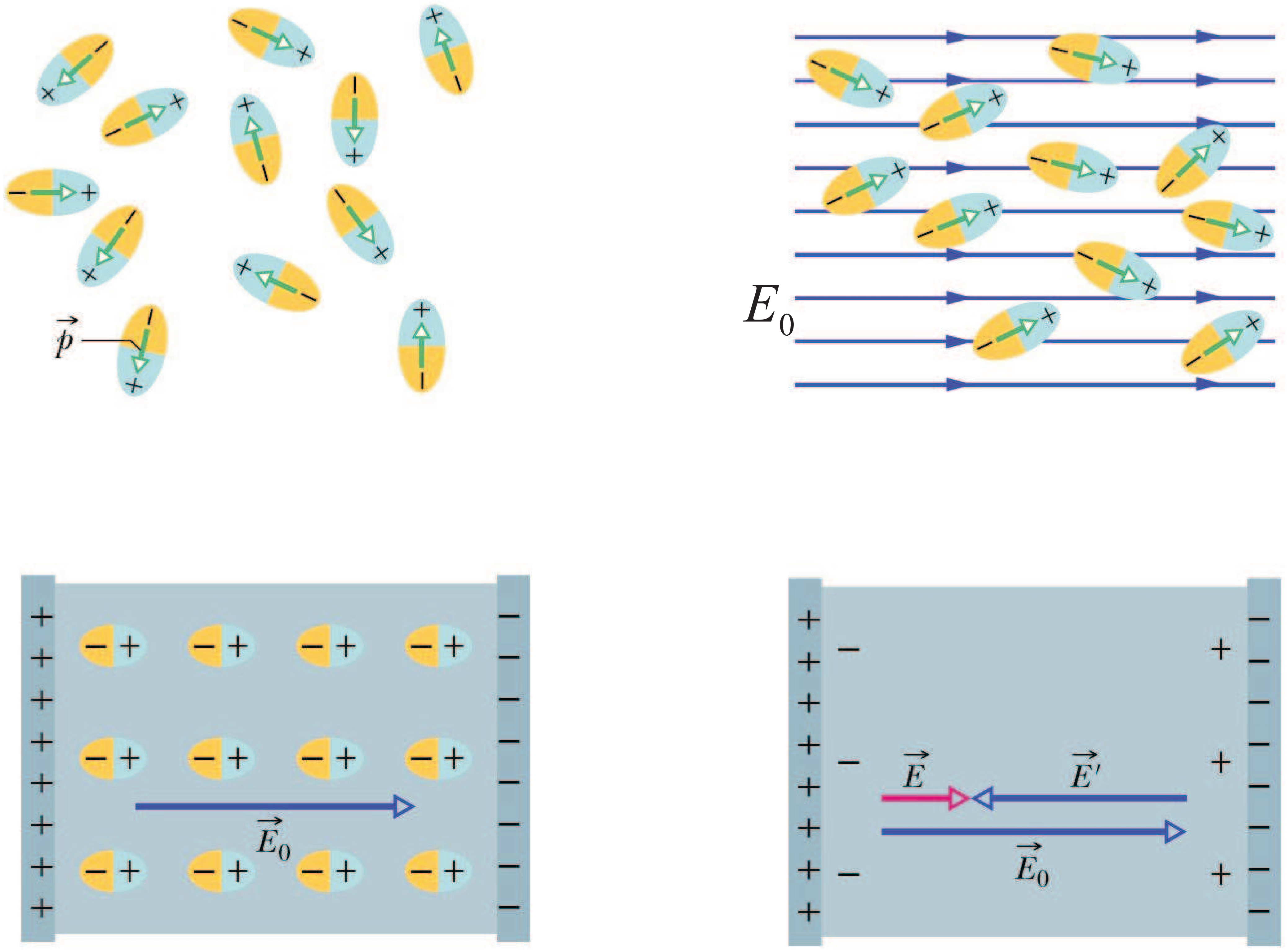
$$E' = -E_0$$

➢ so that E-field inside conductor = 0

➢ For **dielectric** ☛ atoms and molecules behave like **dipole** in $\vec{E}$ -field

- We can envision this so that in absence of $\vec{E}$ -field

direction of dipole in dielectric are randomly distributed



- Aligned dipoles will generate an induced E' -field satisfying $|E'| < |E_0|$
- We can observe aligned dipoles in form of **induced surface charge**

Dielectrics Constant

➢ When a dielectric is placed in an external E_0-field ☛ $\vec{E}$- field inside a dielectric is **induced**

$$\vec{E} = \frac{1}{\kappa}\vec{E}_0$$

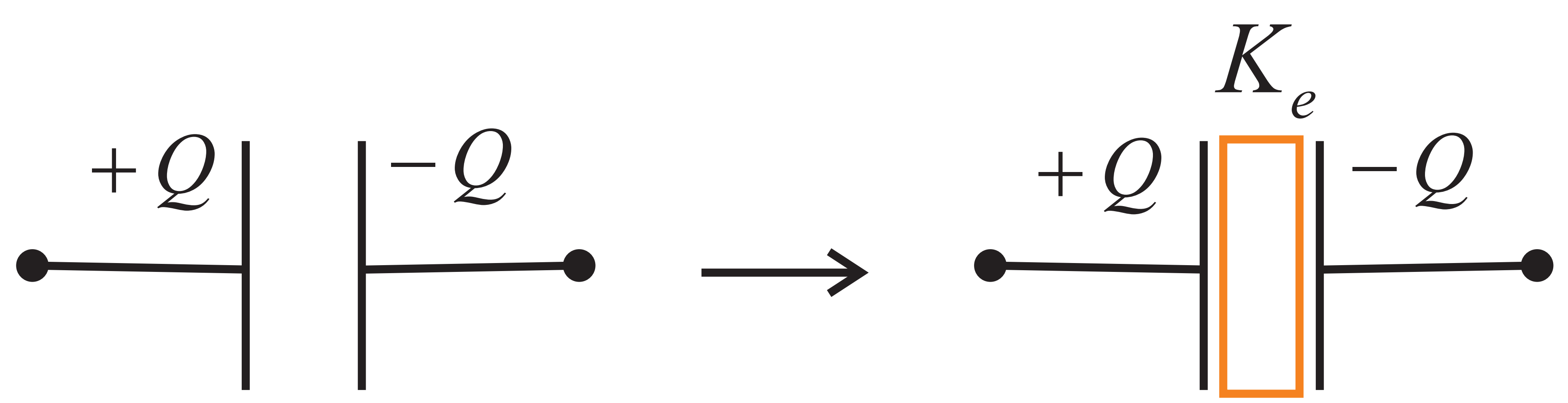
$\kappa \geq 1$ **☛ dielectric constant**

Example

Vacuum	$\kappa = 1$
Porcelain	$\kappa = 6.5$
Water	$\kappa \sim 80$
Perfect conductor	$\kappa \to \infty$
Air	$\kappa = 1.00059$

Capacitors with Dielectrics

Case I



➢ **Charge remains constant** after dielectric is inserted

BUT ☛

$$E_{\text{new}} = \frac{1}{\kappa} E_{\text{old}}$$

$$\therefore \ \Delta V = Ed \Rightarrow \Delta V_{\text{new}} = \frac{1}{\kappa} \Delta V_{\text{old}}$$

$$\therefore \ C = \frac{Q}{\Delta V} \Rightarrow C_{\text{new}} = \kappa \, C_{\text{old}}$$

➢ For a parallel-plate capacitor with dielectric

$$C = \frac{\kappa \ \epsilon_0 A}{d}$$

➢ We can also write $C = \dfrac{\epsilon A}{d}$ in general with

$\epsilon = \kappa\, \epsilon_0$ ☛ **permittivity of dielectric**

Recall ϵ_0 ☛ **permittivity of vacuum**

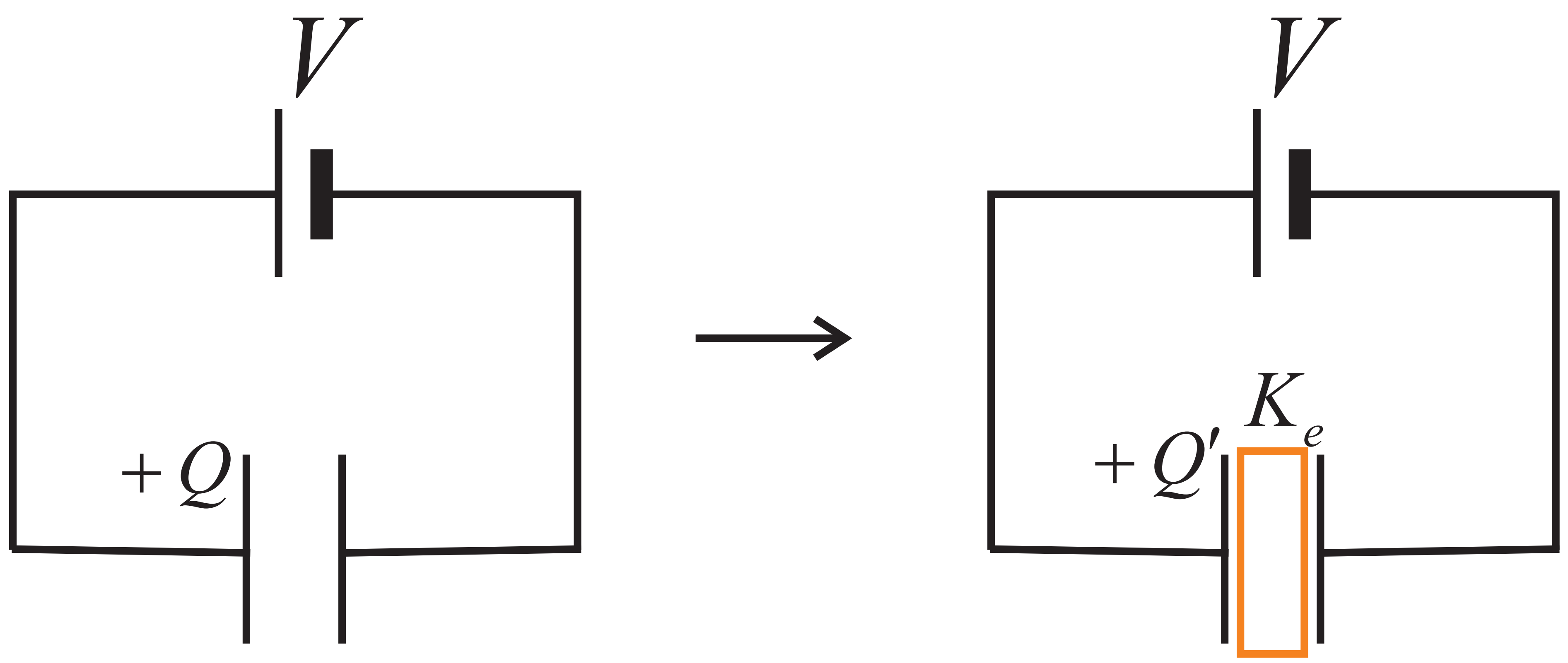
➢ Energy stored

$$U = \frac{Q^2}{2C}$$

$$\therefore U_{\text{new}} = \frac{1}{\kappa} U_{\text{old}} < U_{\text{old}}$$

∴ Work done in inserting dielectric < 0

Case II Capacitor connected to battery



Voltage across capacitor plates **remains constant** after insertion of dielectric

$\vec{E}$ -field inside capacitor remains constant

$$(\because E = V/d)$$

BUT ☛ How can E-field remain constant?

ANSWER ☛ By having extra charge on capacitor plates

Recall ☛

- ➢ For conductors

$$E = \frac{\sigma}{\epsilon_0}$$

$$\Rightarrow \quad E = \frac{Q}{\epsilon_0 A}$$

(σ charge per unit area = Q/A)

- ➢ After insertion of dielectric

$$E' = \frac{Q'}{\kappa \epsilon_0 A}$$

- ➢ But $\vec{E}$-field remains constant ☛ $E' = E \Rightarrow \dfrac{Q'}{\kappa\epsilon_0 A} = \dfrac{Q}{\epsilon_0 A}$

$$\Rightarrow Q' = \kappa Q > Q$$

- ➢ Capacitor $C = Q/V \quad \Rightarrow \quad C' \to \kappa C$

- ➢ Energy stored $U = \frac{1}{2} C V^2 \quad \Rightarrow \quad U' \to \kappa U$

$$U_{\rm new} > U_{\rm old}$$

$\therefore$ Work done to insert dielectric > 0

Energy Stored with Dielectrics

- Total energy stored $$U = \frac{1}{2} C V^2$$

- With dielectric recall $$C = \frac{\kappa \epsilon_0 A}{d}$$

$$V = Ed$$

- ∴ Energy stored per unit volume ☛ $$u = \frac{U}{Ad} = \frac{1}{2} \kappa \epsilon_0 E^2$$

and so ☛ $$u_{\text{dielectric}} = \kappa \, u_{\text{vacuum}}$$

∴ More energy is stored per unit volume in dielectric than in vacuum

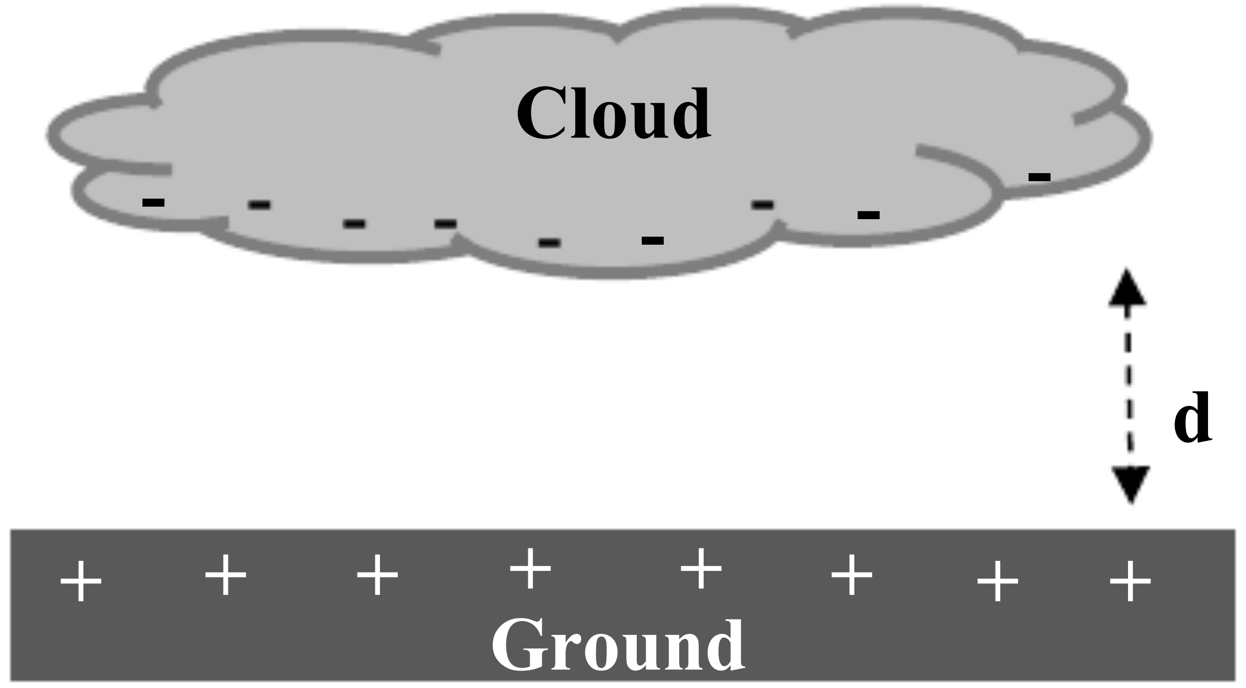
Thunder and Lighting

🌩️ **Thunder is created when lightning passes through the air**

⚡ Lightning discharge heats the air rapidly and causes it to expand

🔥 Temperature of the air in the lightning channel may reach as high as 50,000 degrees Fahrenheit, 5 times hotter than the surface of the sun

- When bottom of a cloud becomes negatively charged it attracts positive change from ground, this can be thought of as a type of capacitor, with cloud and ground being parallel plates with air in between
- If it takes an electric field strength of 3×10^6 V/m to breakdown (conduct charge) air between cloud and ground, how much energy is stored in cloud due to charge right before lightening is formed (before air breaks down)?
- Assume distance between cloud and ground is about d = 1,000 m and area of cloud is approximately 8×10^7 m^2 (about 5 km radius)



$V = Ed = (3 \times 10^6 \text{ V/m})(1{,}000 \text{ m}) = 3 \times 10^9 \text{ V}$

$C = \varepsilon_o A/d = (8.85 \times 10^{-12} \text{ C}^2/\text{Nm}^2)(8 \times 10^7 \text{ m}^2)/(1{,}000 \text{ m})$

$C = 7.1 \times 10^{-7} \text{ F}$

Energy = $1/2\, CV^2 = 1/2\, (7.1 \times 10^{-7} \text{ F})\, (3 \times 10^9 \text{ V})^2$

Energy = 3.2×10^{12} J

➢ If the average house uses about 2,000 W of electrical power, if it were possible to collect and store this energy (unfortunately it is not), how long would it run house?

Power = Energy/time therefore

time = Energy/Power = $(3.2 \times 10^{12}$ J$)/2{,}000$ W $= 1.6 \times 10^{9}$ s ≈ 50 years

Dielectric Strength and Breakdown Voltage

- When a dielectric is in an electric field the outer electrons in that dielectric material experience a force due to the electric field, the atoms/molecules become polarized

- If the electric field becomes large enough these electrons will be stripped off the molecules and free to move along the electric field, at this point avalanche of electron's become dislocated and a current is established between the charge separation

- Atoms/molecules are ionized

- This is known as the breakdown voltage and the properties of the material are destroyed

➢ Dielectric strength (DS) - of a material is the maximum electric field (V/m) that a material can experience before breakdown

➢ Breakdown voltage of a dielectric is given by ☛ $V_{Bd} = (DS)d$

* *DS* is the dialectic strength of the material (in V/m)
* d – is the thickness of the material along the electric field lines (in meters)

Material	Dielectric Constant κ	Dielectric Strength[a] (V/m)
Air (dry)	1.000 59	3×10^6
Bakelite	4.9	24×10^6
Fused quartz	3.78	8×10^6
Neoprene rubber	6.7	12×10^6
Nylon	3.4	14×10^6
Paper	3.7	16×10^6
Polystyrene	2.56	24×10^6
Polyvinyl chloride	3.4	40×10^6
Porcelain	6	12×10^6
Pyrex glass	5.6	14×10^6
Silicone oil	2.5	15×10^6
Strontium titanate	233	8×10^6
Teflon	2.1	60×10^6
Vacuum	1.000 00	—
Water	80	—

Dielectric breakdown in air

✧ Space between capacitor's plates is filled with air, spacing of plates is 0.5 mm

What is maximum voltage capacitor can have before breakdown?

$$V = (DS)d = (1 \times 10^6\ \mathrm{V/m})(0.5 \times 10^{-3}\mathrm{m}) = 500\ \mathrm{V}$$

What if the space was filled with nylon?

$$V = (DS)d = (14 \times 10^6\ \mathrm{V/m})(0.5 \times 10^{-3}\mathrm{m}) = 7{,}000\ \mathrm{V}$$

Capacitance of a Spherical Capacitor

- Spherical capacitors consist of two concentric conducting spherical shells of radii R_1 and R_2
- Shells are given equal and opposite charges $+Q$ and $-Q$ respectively
- Electric field between shells is directed radially outward
- Magnitude of field can be obtained by applying Gauss law over a spherical Gaussian surface of radius r concentric with the shells

$$\Phi_E = E \cdot 4\pi r^2 = \frac{Q}{\epsilon_0}$$

- electric field between the conductor is given as

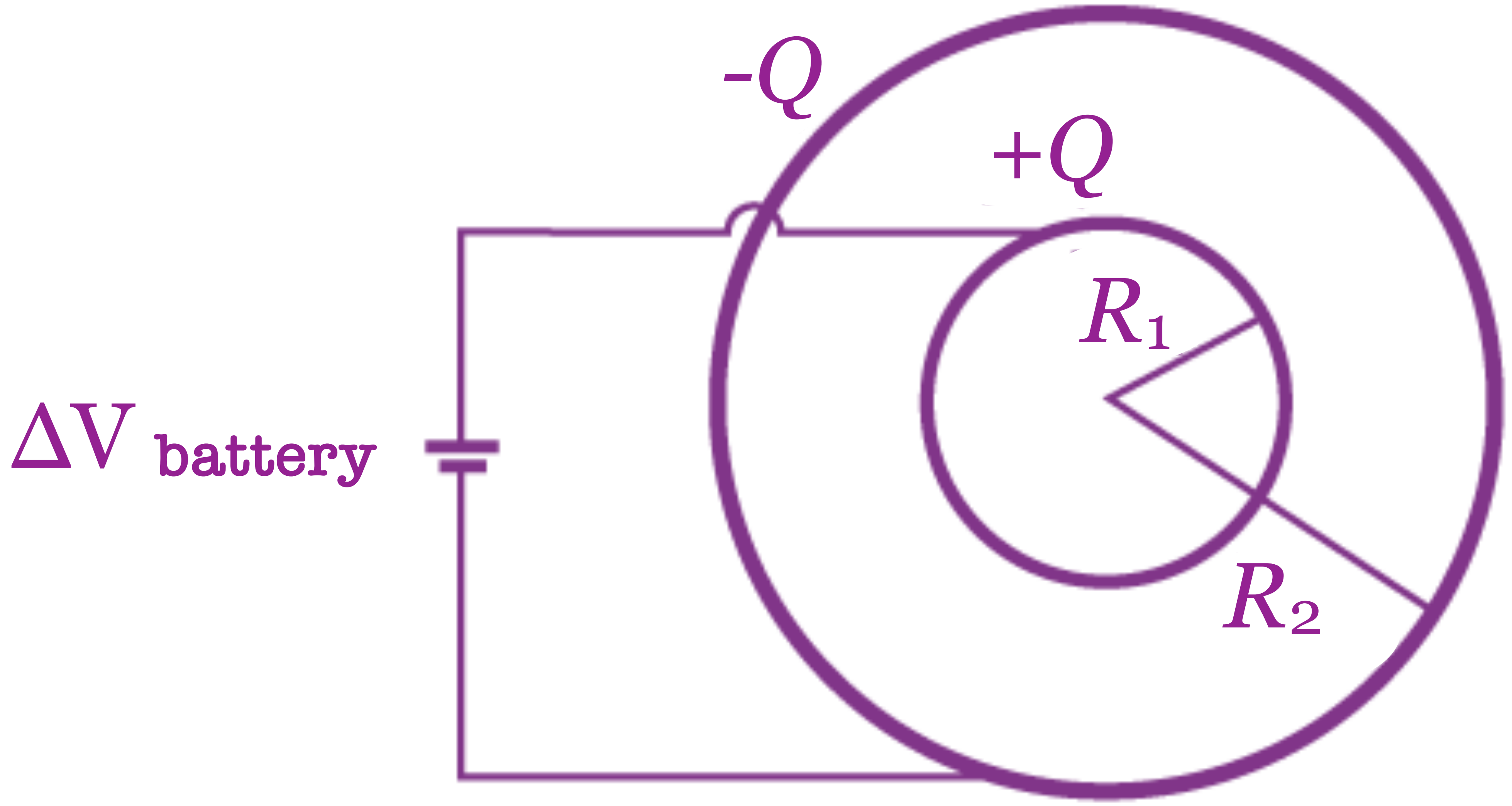
$$\vec{E} = \frac{1}{4\pi\epsilon_0}\frac{Q}{r^2}\hat{r}$$

- potential difference between the plates is

$$V = -(V_2 - V_1) = V_1 - V_2$$

- Substituting the value of V in the capacitance formula, we get

$$C = \frac{Q}{V} = 4\pi\epsilon_0 \frac{R_1 R_2}{R_2 - R_1}$$



Spherical Capacitor

✱ A capacitor consists of two concentric spherical shells

✱ Outer radius of inner shell is a = 0.1 m and inner radius of outer shell is b = 0.2 m

(i) What is capacitance C of this capacitor?

ANSWER ☛

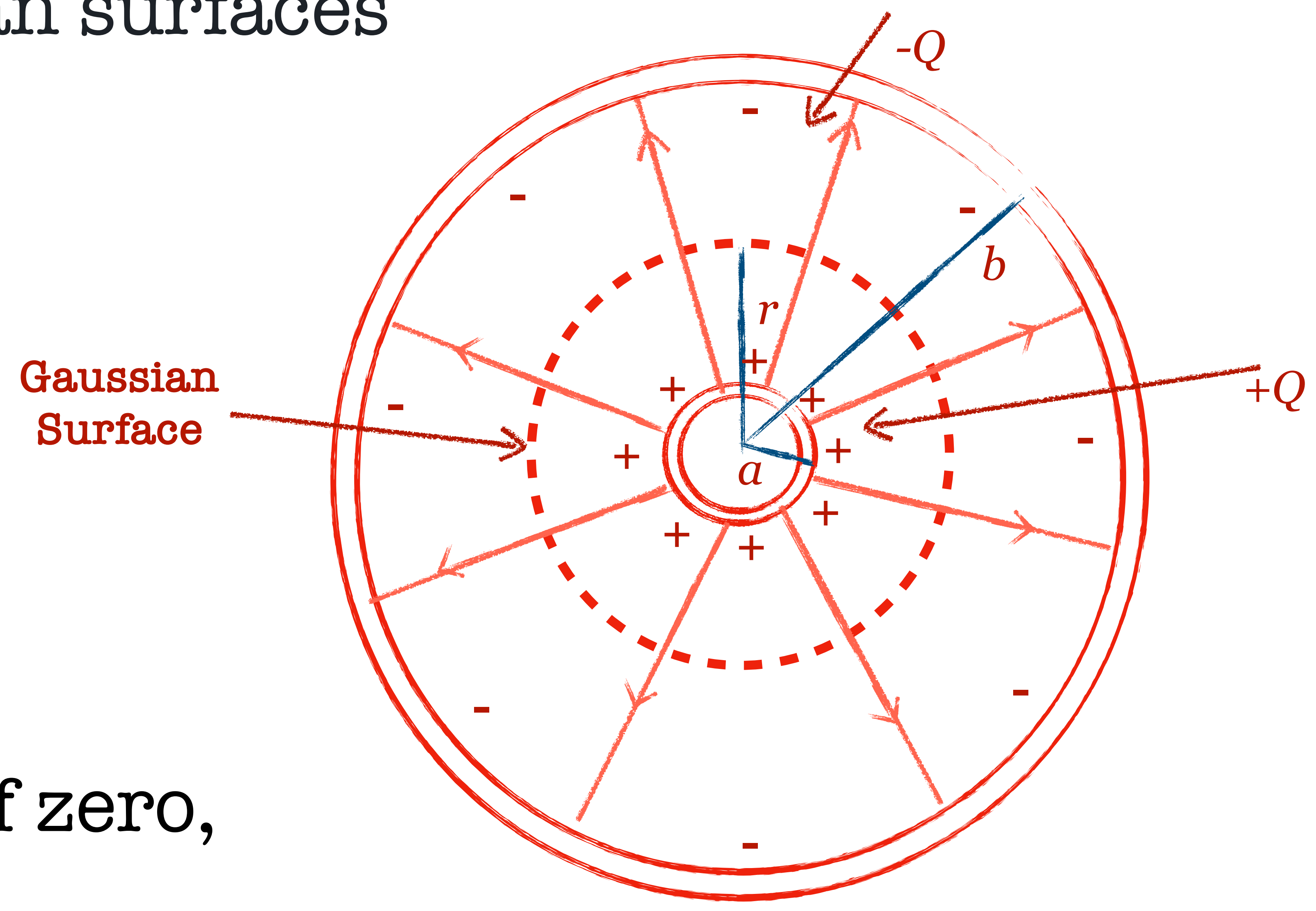
Shells have spherical symmetry so we need to use spherical Gaussian surfaces

Space is divided into three regions

I- oustide $r \geq b$

II- in between $a < r < b$

III- inside $r \leq a$



In each region electric field is purely radial (that is $\vec{E} = E\hat{r}$)

In regions I and III these Gaussian surfaces contain a total charge of zero, so the electric fields in these regions must be zero as well

In regions II, Gaussian sphere of radius r

Electric flux on surface is $\Phi_E = EA = E \cdot 4\pi r^2$

Enclosed charge is $Q_{\text{enc}} = +Q$, and electric field is everywhere perpendicular to surface

Thus Gauss law becomes $$E \cdot 4\pi r^2 = \frac{Q}{\epsilon_0} \Rightarrow E = \frac{Q}{4\pi\epsilon_0 r^2}$$

That is, the electric field is exactly the same as that for a point charge

$$\vec{E} = \begin{cases} \frac{Q}{4\pi\epsilon_0 r^2}\hat{r} & \text{for } a < r < b \\ \vec{0} & \text{elseware} \end{cases}$$

Positively charged inner sheet is at a higher potential so we shall calculate

$$\Delta V = V(a) - V(b) = \frac{Q}{4\pi\epsilon_0}\left(\frac{1}{a} - \frac{1}{b}\right) > 0 \quad \text{wich is positive}$$

We can now calculate capacitance using the definition

$$C = \frac{Q}{|\Delta V|} = \frac{Q}{\frac{Q}{4\pi\epsilon_0}\left(\frac{1}{a} - \frac{1}{b}\right)} = \frac{4\pi\epsilon_0}{\left(\frac{1}{a} - \frac{1}{b}\right)} = \frac{4\pi\epsilon_0 ab}{b-a} = \frac{0.1 \text{ m } 0.2 \text{ m}}{8.99 \times 10^9 \text{ Nm}^2/\text{C}^2 \text{ } 0.1 \text{ m}} = 2.2 \times 10^{-11} \text{ F}$$

Note that units of capacitance are ε_0 times an area ab divided by a length b - a, exactly same units as formula for a parallel-plate capacitor $C = \varepsilon_0 A/d$

Also note that if radii b and a are very close together, spherical capacitor begins to look very much like two parallel plates separated by a distance $d = b - a$ and area $A \approx 4\pi\left(\frac{a+b}{2}\right)^2 \approx 4\pi\left(\frac{a+a}{2}\right)^2 = 4\pi a^2 \approx 4\pi ab$

☛ when b approaches a, spherical formula is same at plate one $C = \frac{4\pi\epsilon_0 ab}{b-a} \sim \frac{\epsilon_0 4\pi a^2}{d} = \frac{\epsilon_0 A}{d}$

(ii) Suppose maximum possible electric field at outer surface of inner shell before air starts to ionize is

$$E_{\max}(a) = 3.0 \times 10^6 \, \mathrm{V} \cdot \mathrm{m}^{-1}$$

What is maximum possible charge on inner capacitor?

ANSWER ☛

Electric field is $E(a) = \dfrac{Q}{4\epsilon_0 a^2}$

Therefore maximum charge is $Q_{\max} = 4\pi\epsilon_0 E_{\max}(a) a^2 = \dfrac{3.0 \times 10^6 \ \mathrm{V} \cdot \mathrm{m}^{-1} \ (0.1 \ \mathrm{m})^2}{8.99 \times 10^9 \ \mathrm{N} \cdot \mathrm{m}^2/\mathrm{C}^2}$

(iii) What is the maximum amount of energy stored in this capacitor?

ANSWER ☛

Energy stored is $U_{\max} = \dfrac{Q_{\max}^2}{2C} = \dfrac{(3.3 \times 10^{-6} \ \mathrm{C})^2}{2 \cdot 2.2 \times 10^{-11} \ \mathrm{F}} = 2.5 \times 10^{-1} \ \mathrm{J}$

(iv) What is potential difference between shells when $E(a) = 3.0 \times 10^6\ \mathrm{V \cdot m^{-1}}$?

ANSWER ☛

Two different ways to find potential difference

Using definition of capacitance we have that

$$|\Delta V| = \frac{Q}{C} = \frac{4\pi\epsilon_0 E(a) a^2 (b-a)}{4\pi\epsilon_0 ab} = \frac{E(a)a(b-a)}{b} = \frac{3.0 \times 10^6\ \mathrm{V \cdot m^{-1}} (0.1\ \mathrm{m})^2}{0.2\ \mathrm{m}} = 1.5 \times 10^5\ \mathrm{V}$$

We already calculated potential difference in part (i)

$$\Delta V = \frac{Q}{4\pi\epsilon_0}\left(\frac{1}{a} - \frac{1}{b}\right)$$

Recall ☛

$$E(a) = \frac{Q}{4\pi\epsilon_0 a^2} \quad \text{or} \quad \frac{Q}{4\pi\epsilon_0} = E(a)a^2$$

Substitute this into our expression for potential difference yielding

$$\Delta V = E(a)a^2\left(\frac{1}{a} - \frac{1}{b}\right) = E(a)a^2\frac{b-a}{ab} = E(a)a\frac{b-a}{b}$$


aNy