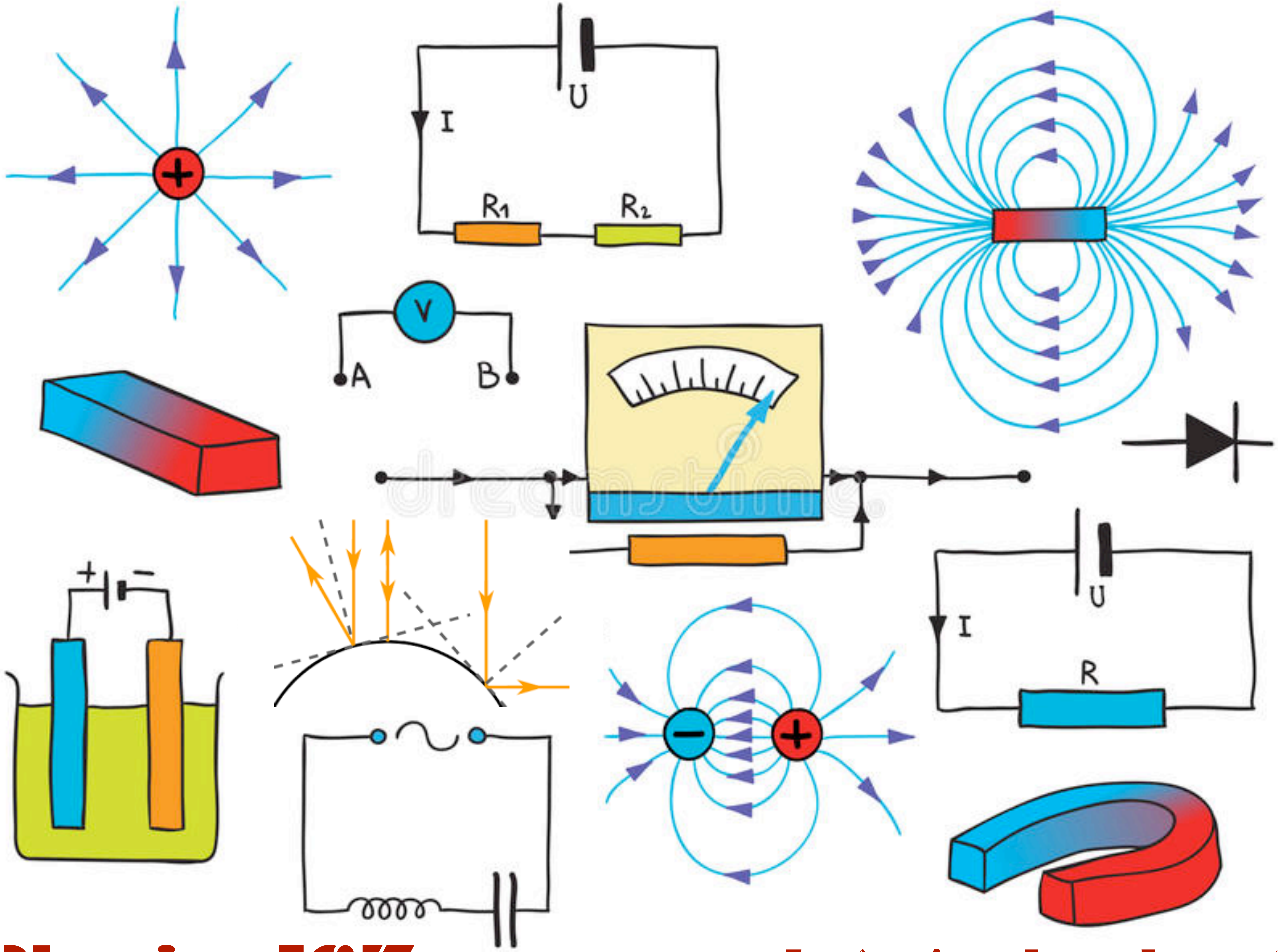


Physics 167

Luis Anchordoqui

Potential Energy and Conservative Forces

DEFINITION

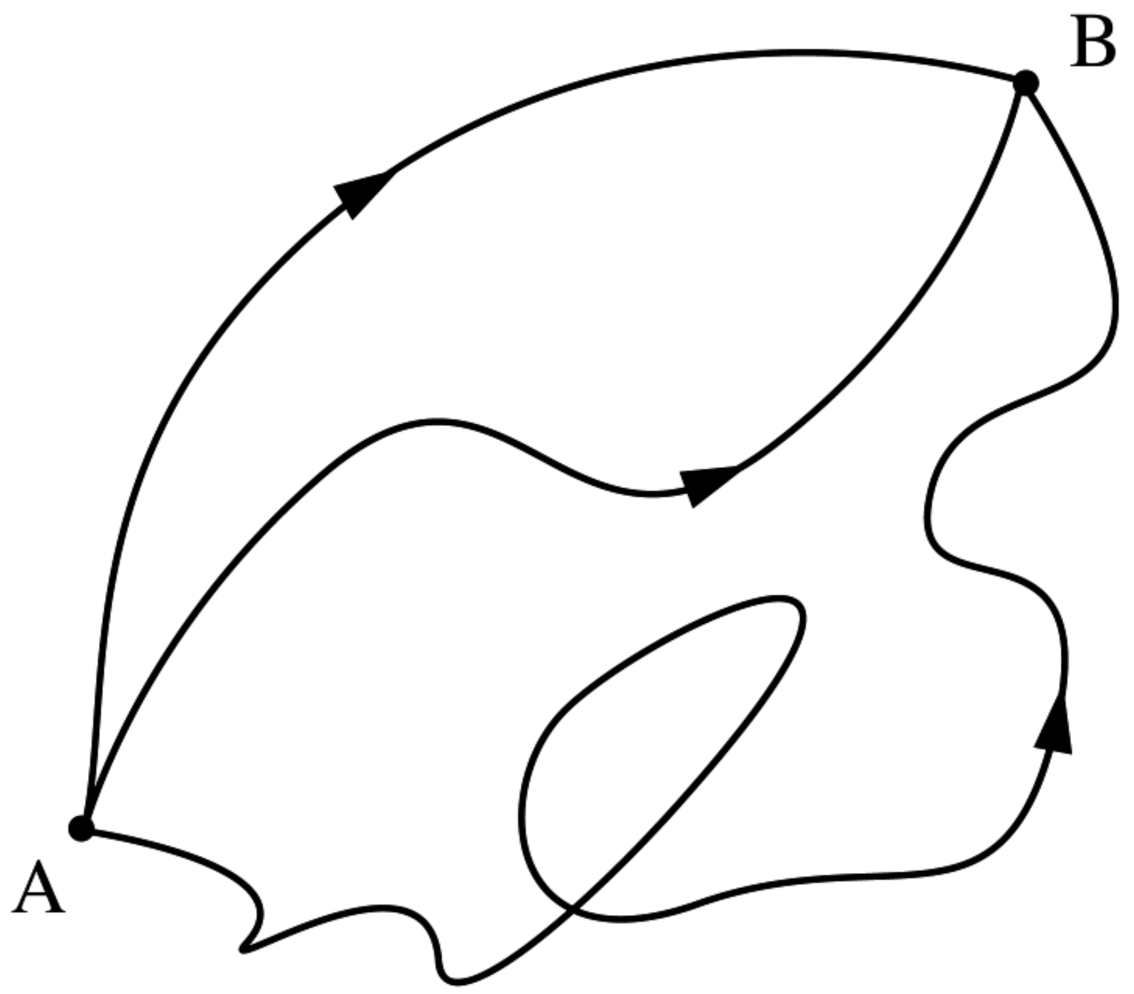
- A force is **conservative if** work done on a particle by force is **independent of path taken**

Alternative DEFINITION

Work done by a conservative force on a particle
when it moves around a closed path
returning to its initial position is zero

Conclusion Since work done by a conservative force $\vec{F}$ is **path-independent** we can define:

potential energy that depends only on **position** of particle



Potential Energy and Conservative Forces

➢ Work W done against conservative force gets stored as potential energy U

Convention

➢ For uniforme force $\vec{F} \parallel \Delta\vec{s}$ ☛ we define potential energy U such that

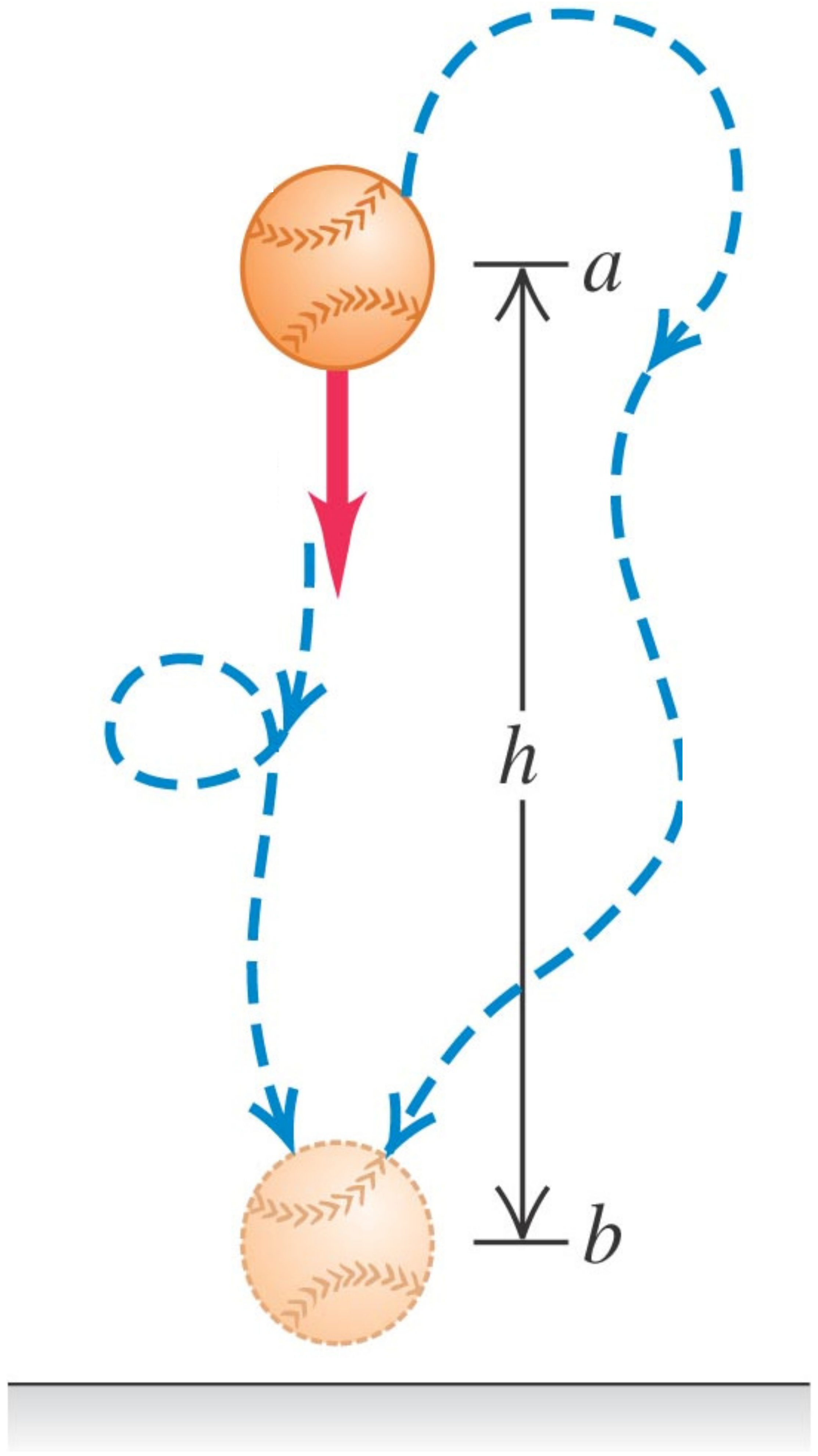
$$\Delta U = U_2 - U_1 = -W_{s_1 \to s_2} = -\vec{F} \cdot \Delta\vec{s} = -F(s_2 - s_1)$$

where U_1, U_2 are potential energy at position 1, 2

Gravitational force is a conservative force

Work $\equiv \Delta W_{12}$ $\Longrightarrow$ decrease in potential energy

Near the surface of the Earth ☛ $\vec{F}_{\text{gravity}} = m\vec{g}$



The work done by gravitational force is the same for any path from a to b

$$W_{a\to b} = -\Delta U = mgh$$

Electric Potential Energy

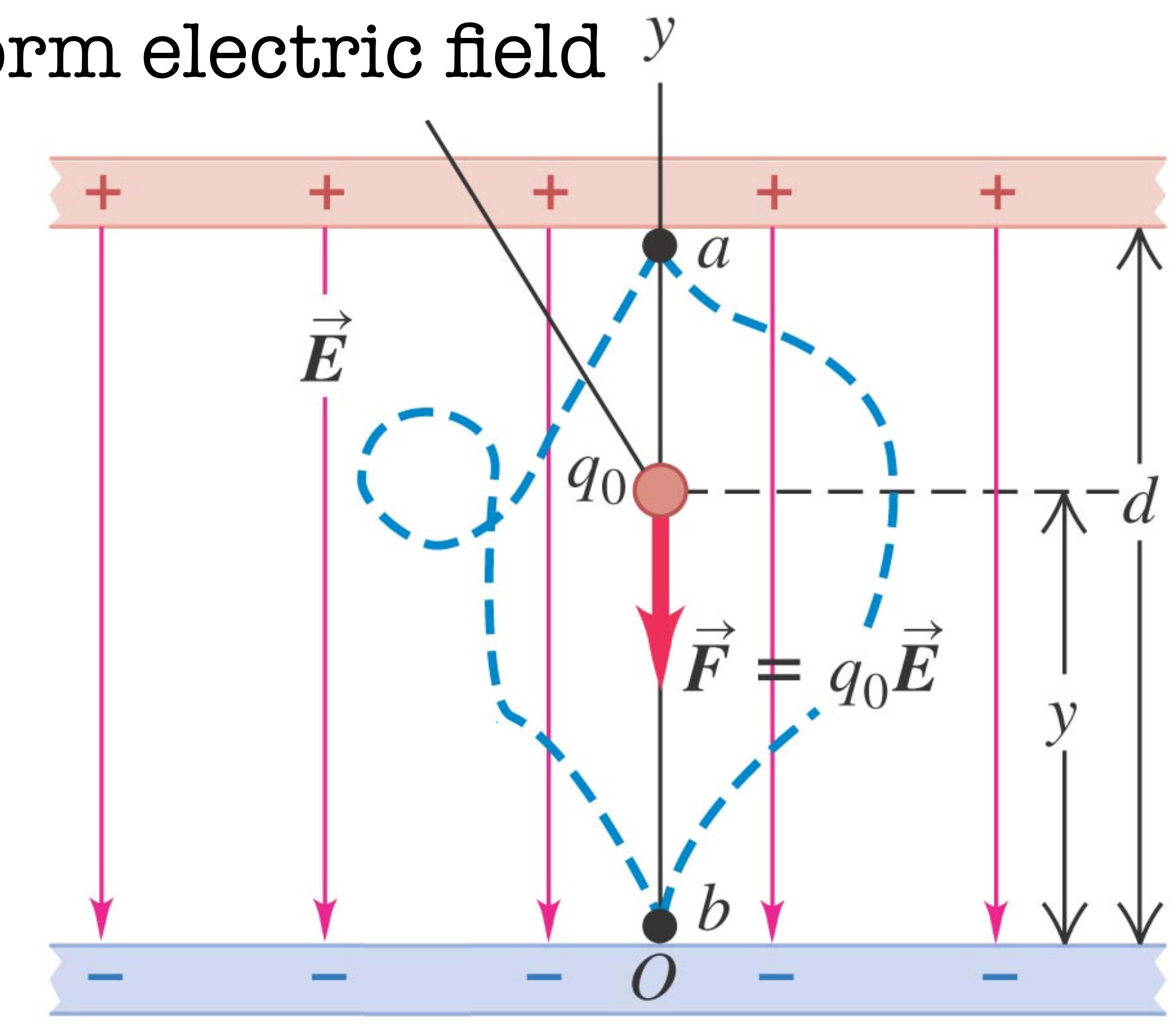
- ➢ When a charge particle moves in electric field, field exerts force that can do work on particle
- ➢ The work can be expressed in terms of electric potential energy
- ➢ **Electric potential energy depends only on position of charged particle in electric field**
- ➢ **Electric potential energy in uniform field**

$$W_{a \to b} = F \cdot d = q_0 E d$$

➢ **Electric field due to static charge distribution generales a conservative force**

$$\Delta W = -\Delta U \Rightarrow U = q_0 E \cdot y$$

Point charge moving in a uniform electric field

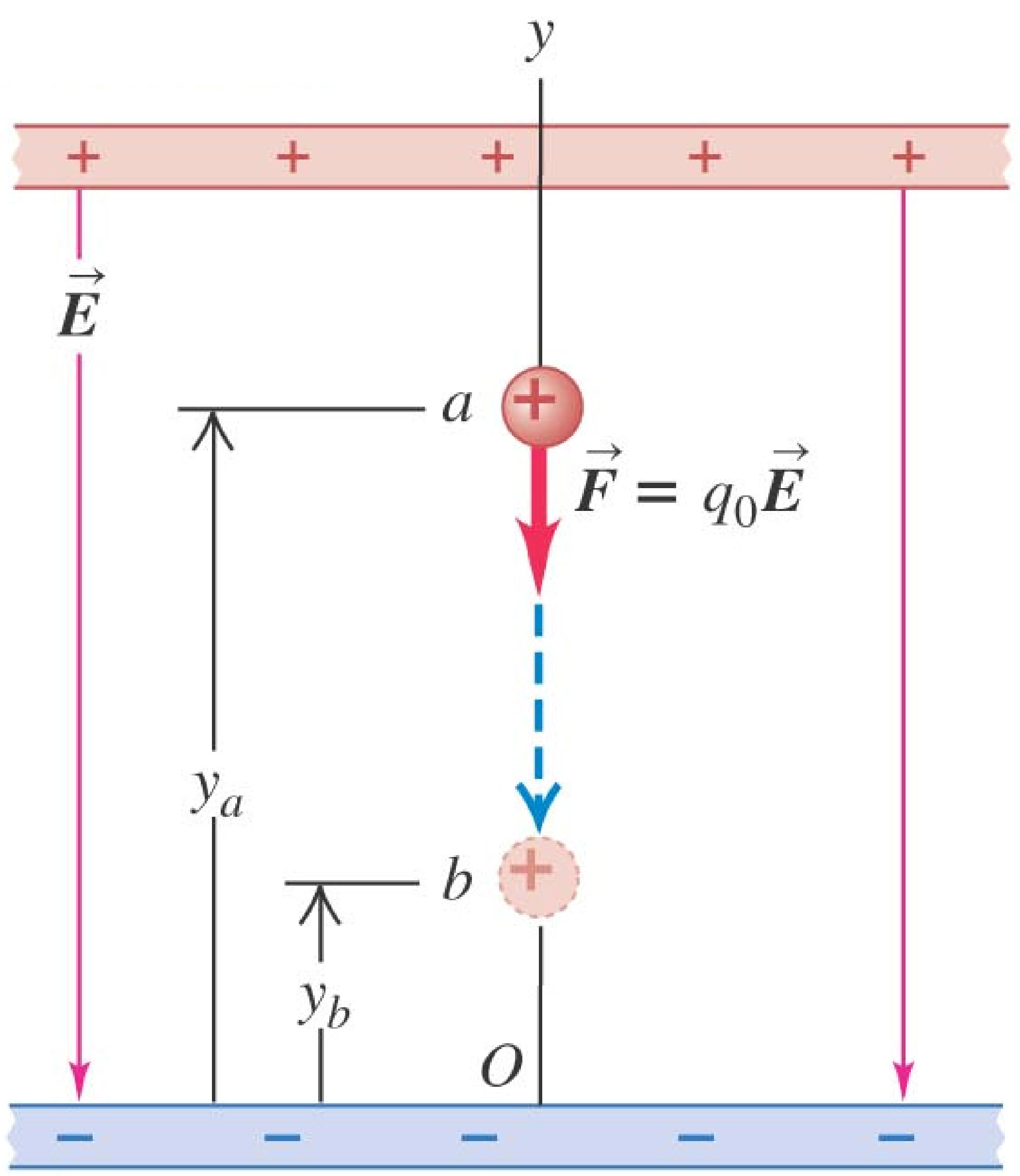


Test Charge Moving from Height y_a to y_b

$$W_{a\to b} = -\Delta U = -(U_b - U_a) = q_0 E(y_a - y_b)$$

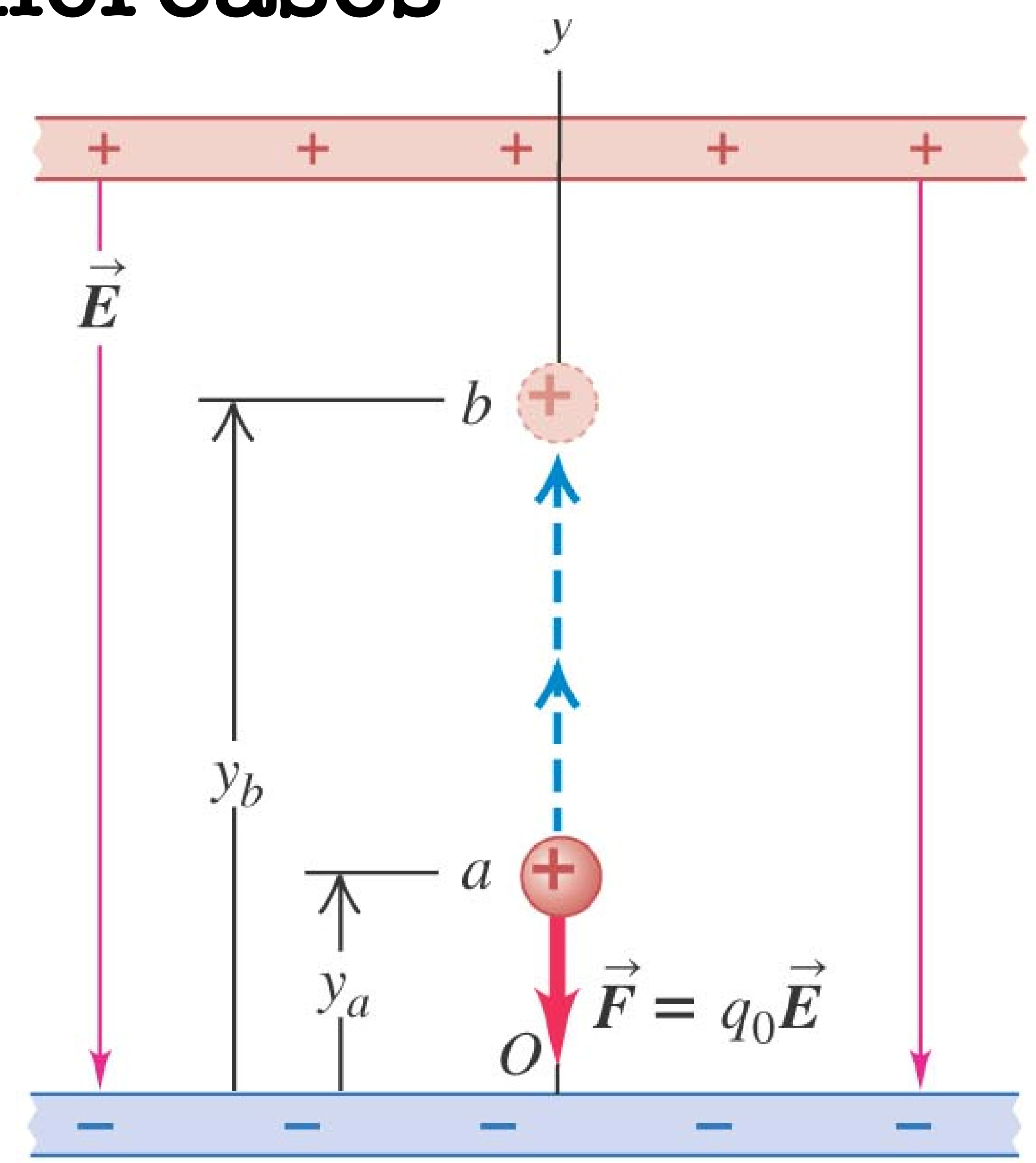
Positive charge moves in the direction of $\vec{E}$

- Field does **positive** work on charge
- U **decreases**



Positive charge moves opposite $\vec{E}$

- Field does **negative** work on charge
- U **increases**

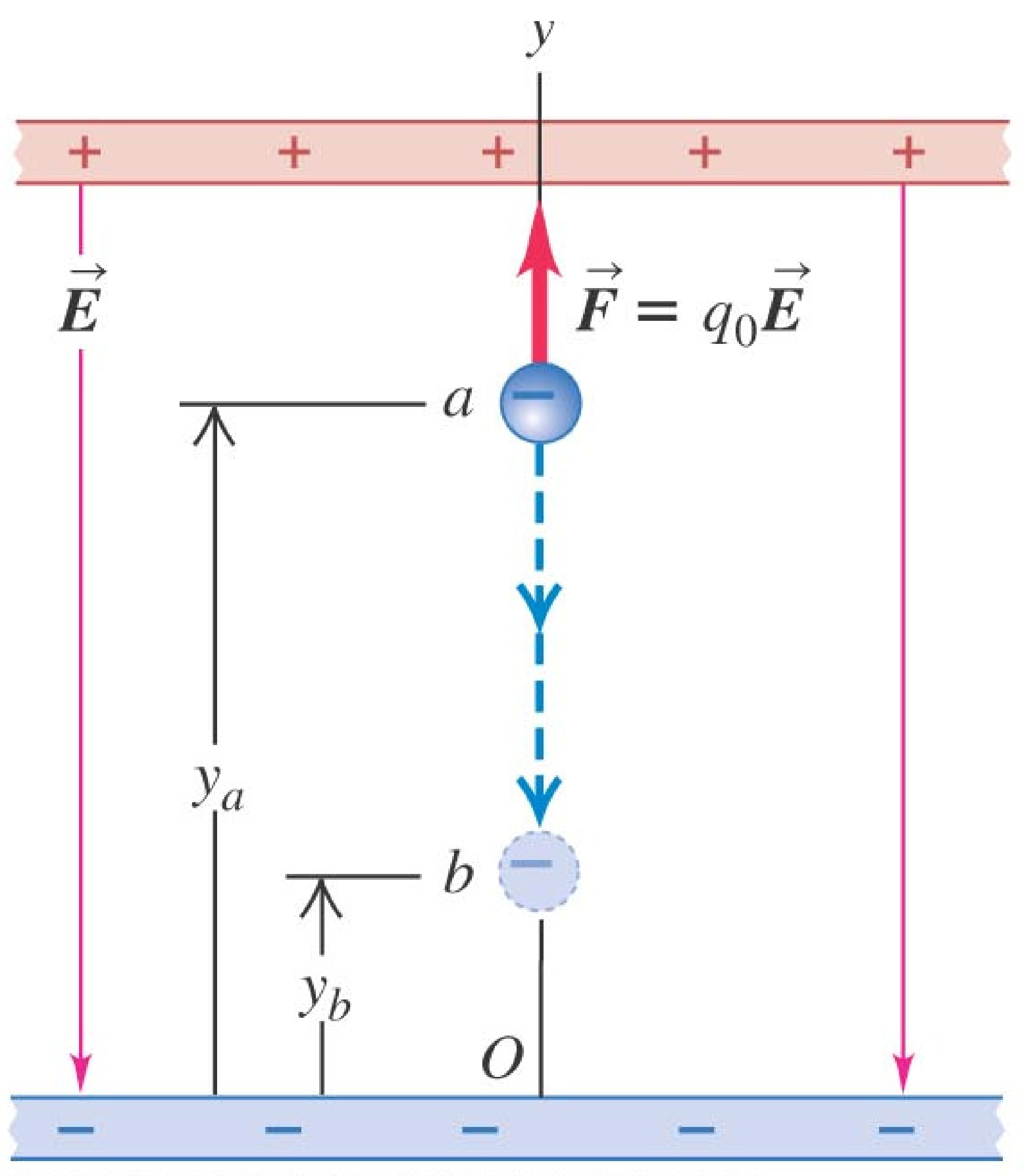


Independently of Whether Test Charge Is (+) or (-)

- U increases if q_0 moves in direction opposite to electric force
- U decreases if q_0 moves in same direction as $\vec{F} = q_0\vec{E}$

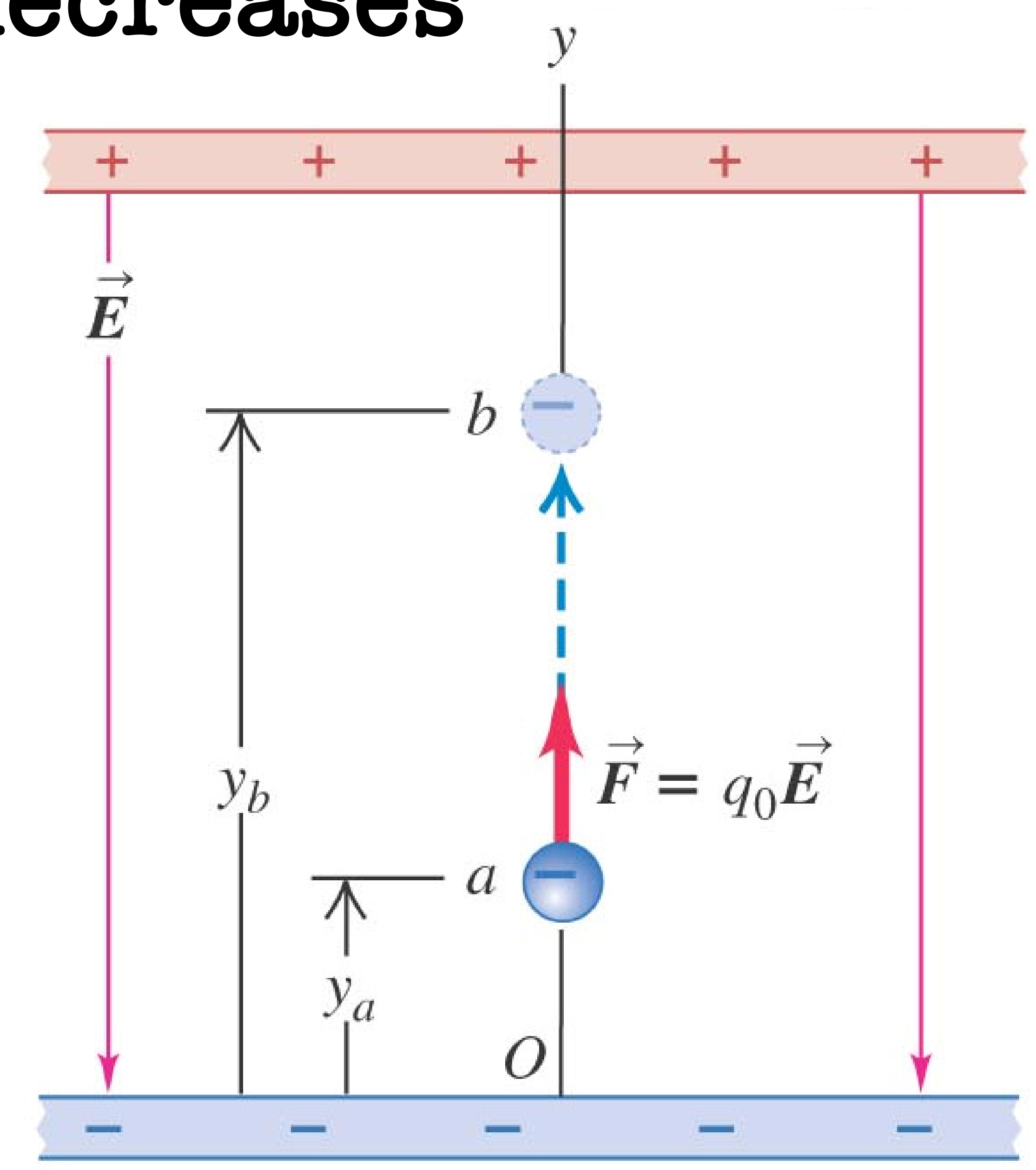
Negative charge moves in the direction of $\vec{E}$

- Field does **negative** work on charge
- U **increases**



Negative charge moves opposite $\vec{E}$

- Field does **positive** work on charge
- U **decreases**



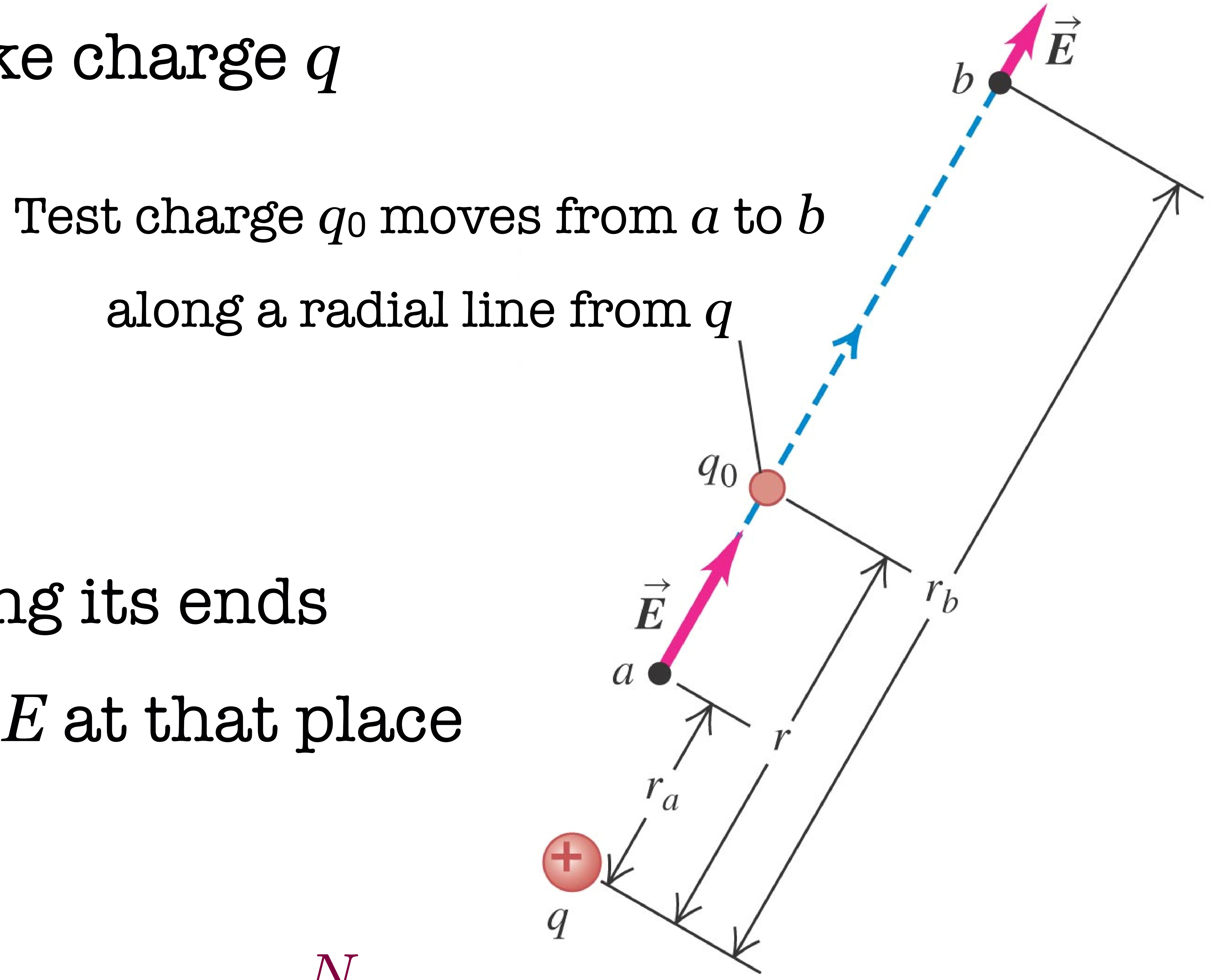
Mathematical Interlude: Telescoping Sum

➢ A sum in which subsequent terms cancel each other leaving only initial and final terms

$$
\begin{aligned}
S = &\sum_{i=1}^{n-1}(a_i - a_{i+1}) \\
=&(a_1 - a_2) + (a_2 - a_3) + \cdots + (a_{n-2} - a_{n-1}) + (a_{n-1} - a_n) \\
=&(a_1 - a_n)
\end{aligned}
$$

Electric Potential Energy of Two Point Charges

➢ A test charge (q_0) will move directly away from a like charge q



➢ Divide the chosen path into short segments,

each segment being represented by a vector connecting its ends

take scalar product of path-segment vector with field E at that place

add these products up for the whole path

$$\Delta U = -W_{a\to b} = -q_0 \sum E(r)\ \Delta r = -\frac{1}{4\pi\epsilon_0} q q_0 \sum_{i=1}^{N} \frac{\Delta r}{r_i^2}$$

$$\Delta r = \frac{b-a}{N} \qquad r_i = a + i\Delta r$$

Note that ☛ $$\frac{1}{r_i} - \frac{1}{r_i + \Delta r} = \frac{\Delta r}{r_i(r_i + \Delta r)} \leq \frac{\Delta r}{r_i^2}$$

and ☛ $$\frac{1}{r_i - \Delta r} - \frac{1}{r_i} = \frac{\Delta r}{r_i(r_i - \Delta r)} \geq \frac{\Delta r}{r_i^2}$$

$$\therefore \frac{1}{r_i} - \frac{1}{r_i + \Delta r} \leq \frac{\Delta r}{r_i^2} \leq \frac{1}{r_i - \Delta r} - \frac{1}{r_i}$$

Telescoping sums ☛ easy to calculate

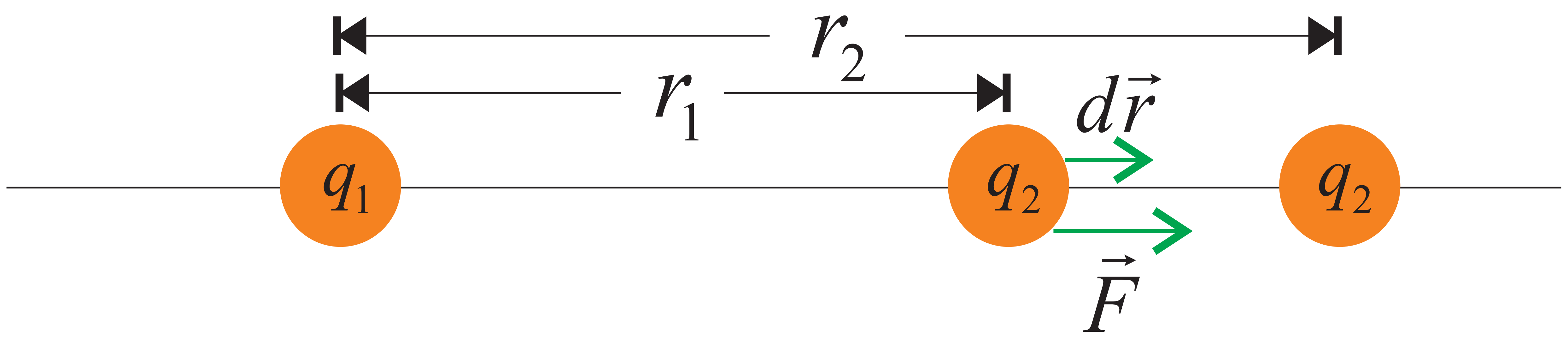
$$\underbrace{\sum_{i=1}^{N} \frac{1}{r_i} - \frac{1}{r_i + \Delta r}}_{\frac{1}{a+\Delta r} - \frac{1}{b+\Delta r}} \leq \sum_{i=1}^{N} \frac{\Delta r}{r_i^2} \leq \underbrace{\sum_{i=1}^{N} \frac{1}{r_i - \Delta r} - \frac{1}{r_i}}_{\frac{1}{a} - \frac{1}{b}}$$

$$\text{For } N \ll 1 \Rightarrow \Delta r \ll r \Rightarrow -\sum_{i=1}^{N} \frac{\Delta r}{r_i^2} = \frac{1}{b} - \frac{1}{a}$$

Electric Potential Energy of Two Point Charges

Summary

If charge q_2 moves from point 1 to 2



We have ☛ $$\Delta U = -\Delta W = \frac{1}{4\pi\epsilon_0} q_1 q_2 \left(\frac{1}{r_2} - \frac{1}{r_1} \right)$$

Note

① This result is generally true for 2-D and/or 3-D motion

② If q_2 moves away from q_1 then $r_2 > r_1$ we have

- ➢ If q_1, q_2 are of **same** sign then $\Delta U < 0, \quad \Delta W > 0$

($\Delta W =$ Work done by electric **repulsive** force)

- ➢ If q_1, q_2 are of **different** sign then $\Delta U > 0, \quad \Delta W < 0$

($\Delta W =$ Work done by electric **attractive** force)

③ If q_2 moves towards q_1 then $r_2 < r_1$ we have

- ➢ If q_1, q_2 are of **same** sign then $\Delta U > 0, \quad \Delta W < 0$

- ➢ If q_1, q_2 are of **different** sign then $\Delta U < 0, \quad \Delta W > 0$

Note

④ It is **difference** inpotential Energy that is important

Reference point

$$U(r = \infty) = 0$$

$$\therefore U_\infty - U_1 = \frac{1}{4\pi\epsilon_0} q_1 q_2 \Big(\underbrace{\frac{1}{r_2}}_{\infty} - \frac{1}{r_1}\Big)$$

$$U(r) = \frac{1}{4\pi\epsilon_0} \cdot \frac{q_1 q_2}{r}$$

➢ If q_1,q_2 **same** sign then $U(r) > 0$ for all r

➢ If q_1,q_2 **opposite** sign then $U(r) < 0$ for all r

⑤ Conservation of Mechanical Energy

➢ For a system of charges with no external force

$$E = \underbrace{K}_{\textbf{Kinetic Energy}} + \underbrace{U}_{\textbf{Potential Energy}} = \textbf{Constant}$$

or $\Delta E = \Delta K + \Delta U = 0$

Potential Energy of a System of Charges

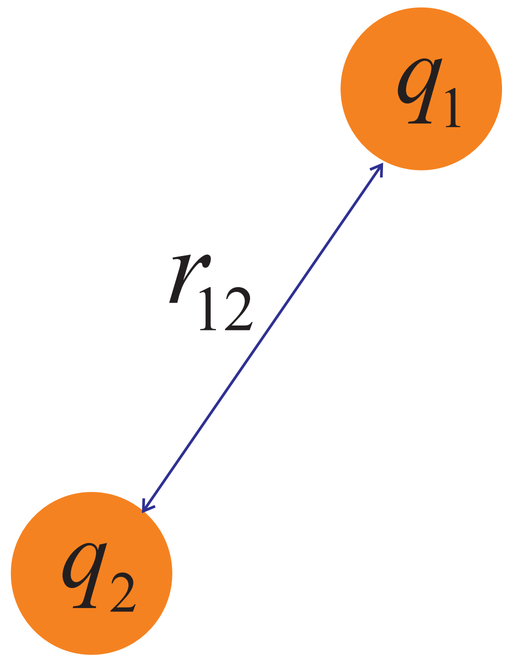
Example P.E. of 3 charges q_1, q_2, q_3

Start q_1, q_2, q_3 all at $r = \infty$, $U = 0$

Step 1

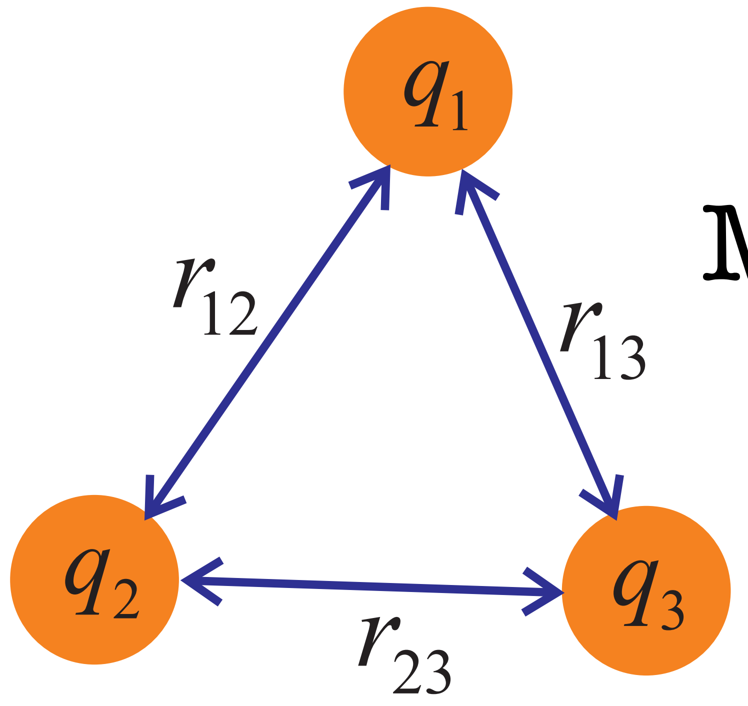
Move q_1 from ∞ to its position $\Rightarrow U = 0$

Step 2



Move q_2 from ∞ to new position $\Rightarrow U = \dfrac{1}{4\pi\epsilon_0}\dfrac{q_1 q_2}{r_{12}}$

Step 3



Move q_3 from ∞ to new position $\Rightarrow$ **Total P.E**

$$U = \frac{1}{4\pi\epsilon_0}\left[\frac{q_1 q_2}{r_{12}} + \frac{q_1 q_3}{r_{13}} + \frac{q_2 q_3}{r_{23}}\right]$$

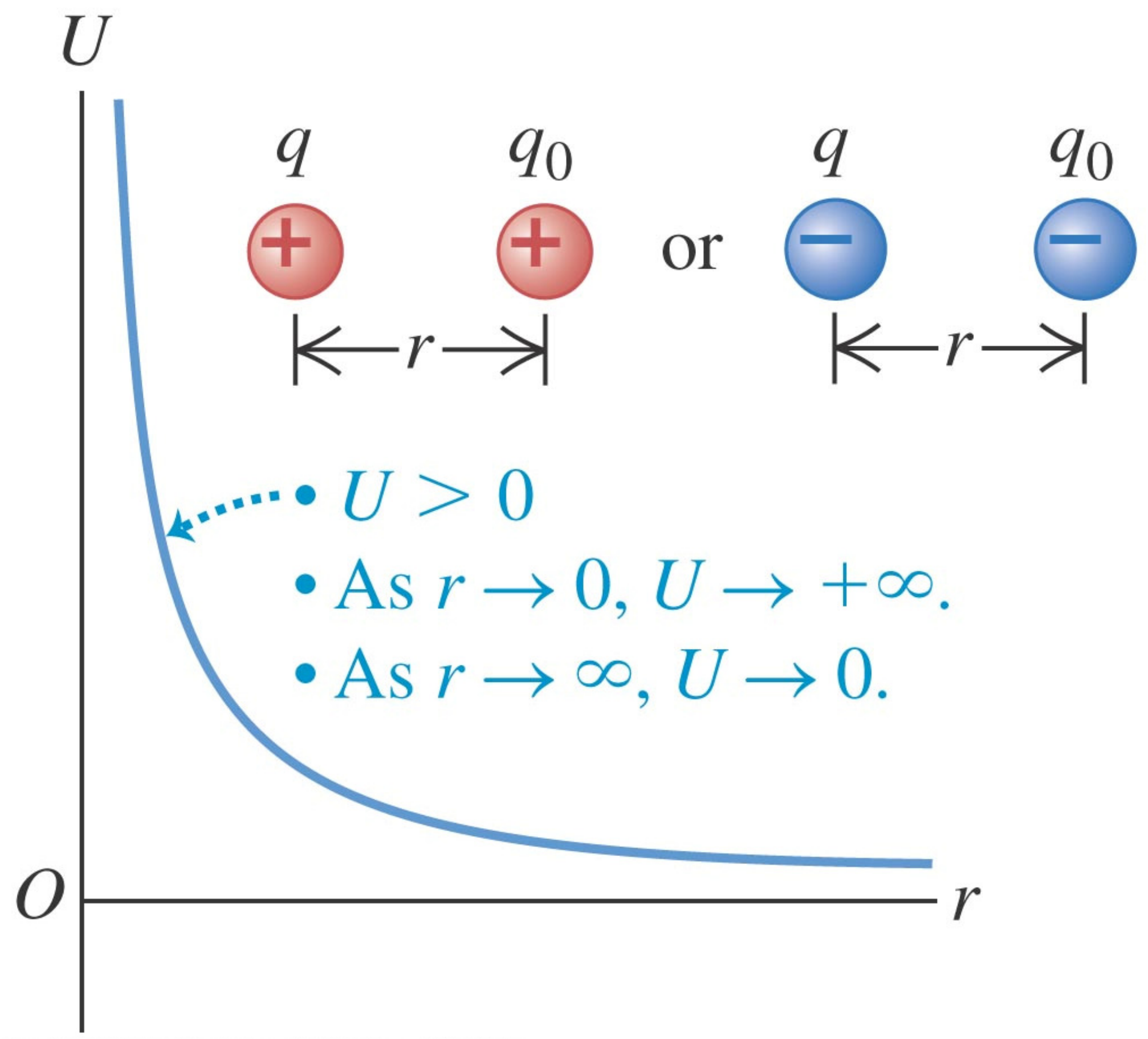
Step 4 What if there are 4 charges?

Summary of Electric Potential Energy

➢ Potential energy when charge q_0 is at distance r from q

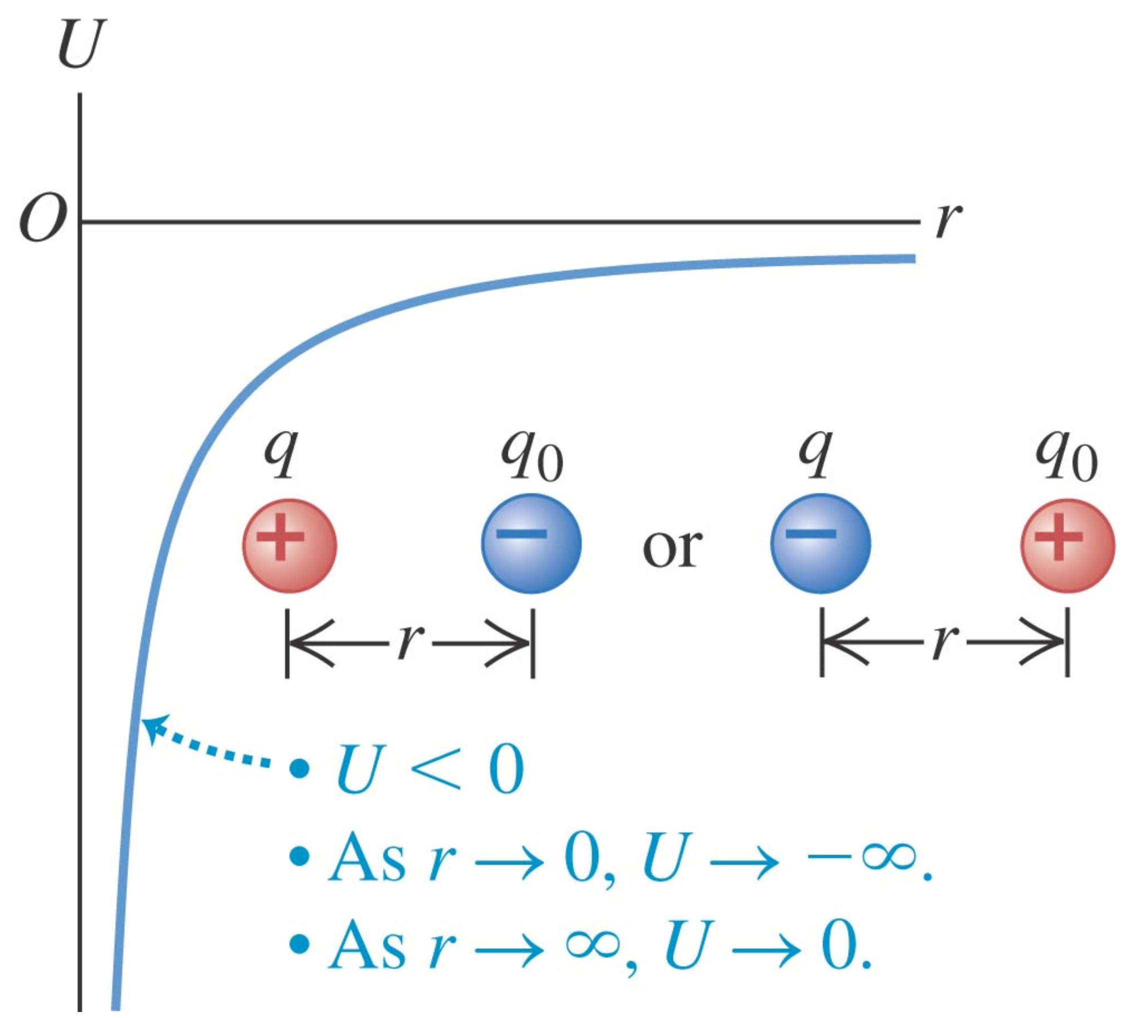
$$W_{a\to b} = \frac{qq_0}{4\pi\varepsilon_0}\left(\frac{1}{r_a} - \frac{1}{r_b}\right) = -\Delta U \quad \rightarrow \quad \boxed{U = \frac{qq_0}{4\pi\varepsilon_0 r}}$$

q and q_0 have the same sign



Graphically, U between like charges increase sharply to positive (repulsive) values as the charges become close

q and q_0 have opposite signs



Unlike charges have U becoming sharply negative as they become close (attractive)

Summary of Electric Potential Energy

- Potential energy is always relative to certain reference point where $U = 0$

  Location of this point is arbitrary

  $U = 0$ when q and q_0 are infinitely apart ($r \to \infty$)

- U is shared property of 2 charges, a consequence of interaction between them

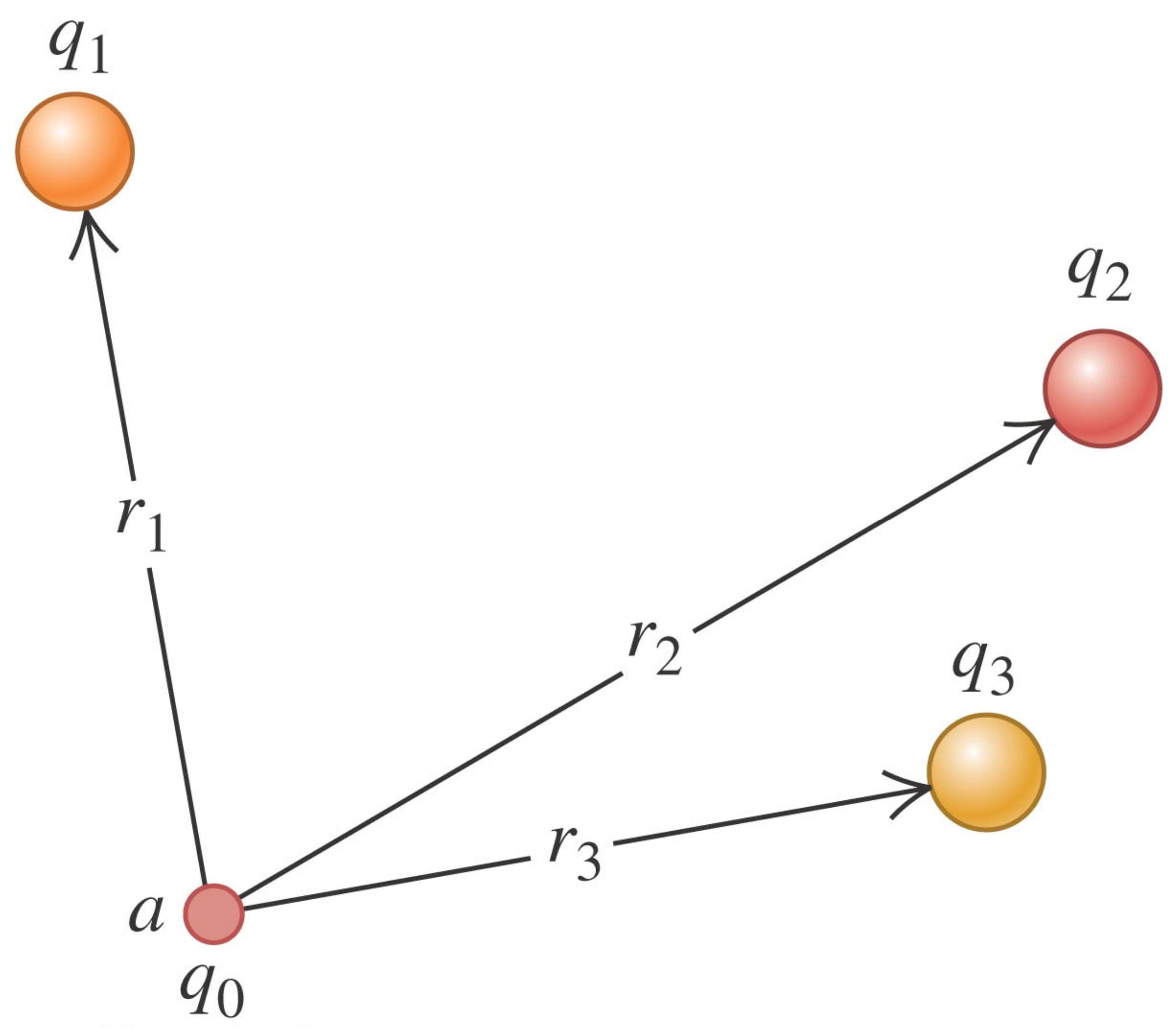
  If distance between 2 charges is changed from r_a to r_b, ΔU is same whether q is fixed and q_0 moved, or vice versa

Electric Potential Energy with Several Point Charges

- Potential energy associated with q_0 at " a" is **algebraic sum** of U associated with each par of charges

$$\boxed{U = \frac{q_0}{4\pi\varepsilon_0}\left(\frac{q_1}{r_1} + \frac{q_2}{r_2} + \frac{q_3}{r_3} + \quad\right) = \frac{q_0}{4\pi\varepsilon_0}\sum_i \frac{q_i}{r_i}}$$

$$\boxed{U = \frac{1}{4\pi\varepsilon_0}\sum_{i<j}\frac{q_i q_j}{r_{ij}}}$$



Electric Potential

Let q be charge at the center and consider its effect on test charge q_0

DEFINITION We define electric potential V so that

$$\Delta V = \frac{\Delta U}{q_0} = \frac{-\Delta W}{q_0}$$

($\therefore V$ is P.E. per unit charge)

- Similarly ☛ we take $V(r=\infty) = 0$
- Electric Potential is a scalar
- Unit ☛ Volt (V) = ***Joules/Coulomb***
- For a single point charge $V(r) = \dfrac{1}{4\pi\epsilon_0} \cdot \dfrac{q}{r}$
- Energy Unit ☛ $\Delta U = q\Delta V$ $\quad \mathrm{elecrton-volt\ (eV)} = \underbrace{1.6 \times 10^{-19}\ \mathrm{J}}_{\text{charge of electron/C}}$

Relation Between Electric Field E and Electric Potential V

➢ Consider uniform electric field

e.g. E between the parallel plates whose difference of potential is V_{ba}

➢ Work done by the electric field to move a positive charge q from point a to point b is equal to the negative of change in potential energy

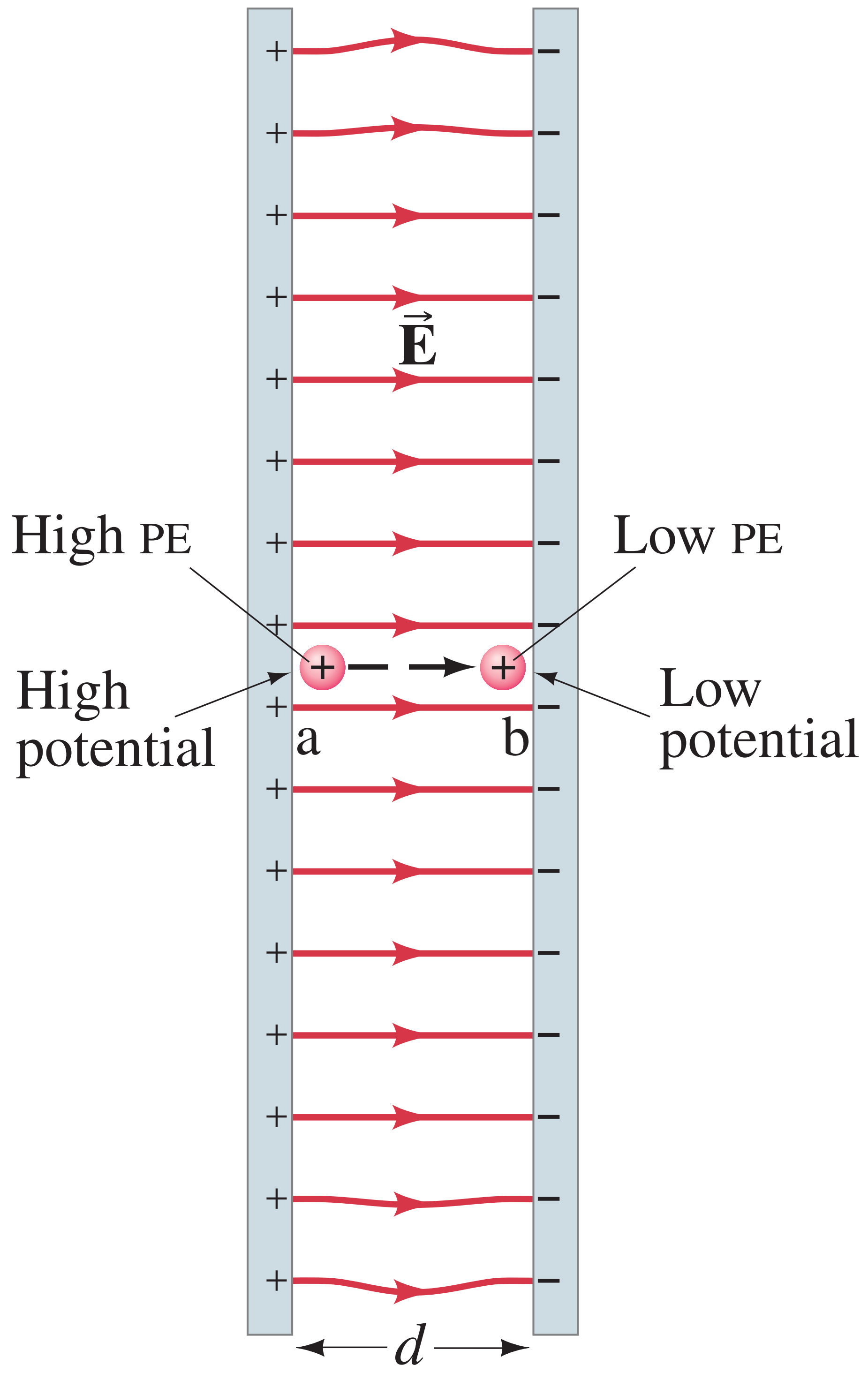
$$W = -q(V_b - V_a) = -qV_{ab}$$

➢ We can also write the work done as

$$W = Fd = qEd$$

➢ d ☛ distance (parallel to field lines) between points a and b

$$\therefore V_{ab} = -Ed \Leftrightarrow E = -\frac{V_{ab}}{d}$$



➢ In region where E is not uniform ☛ electric field in a given direction at any point in space is equal to rate at which the electric potential V decreases over distance in that direction ☛

$$E_x = -\frac{\Delta V}{\Delta x}$$

Example: Point Charge

$$E = -\frac{\Delta V}{\Delta r} = -\frac{1}{4\pi\epsilon_0} q \left(\frac{1}{r+\Delta r} - \frac{1}{r} \right) \frac{1}{\Delta r}$$

Now

$$\left(\frac{1}{r+\Delta r} - \frac{1}{r} \right) \frac{1}{\Delta r} = \frac{\Delta r}{(r+\Delta r)r} \frac{1}{\Delta r} = \frac{1}{r^2 + r\Delta r}$$

$$\therefore \Delta r \ll r \Rightarrow E = \frac{1}{4\pi\epsilon_0} \frac{q}{r^2}$$

➢ Determine the potential at a point 0.50 m

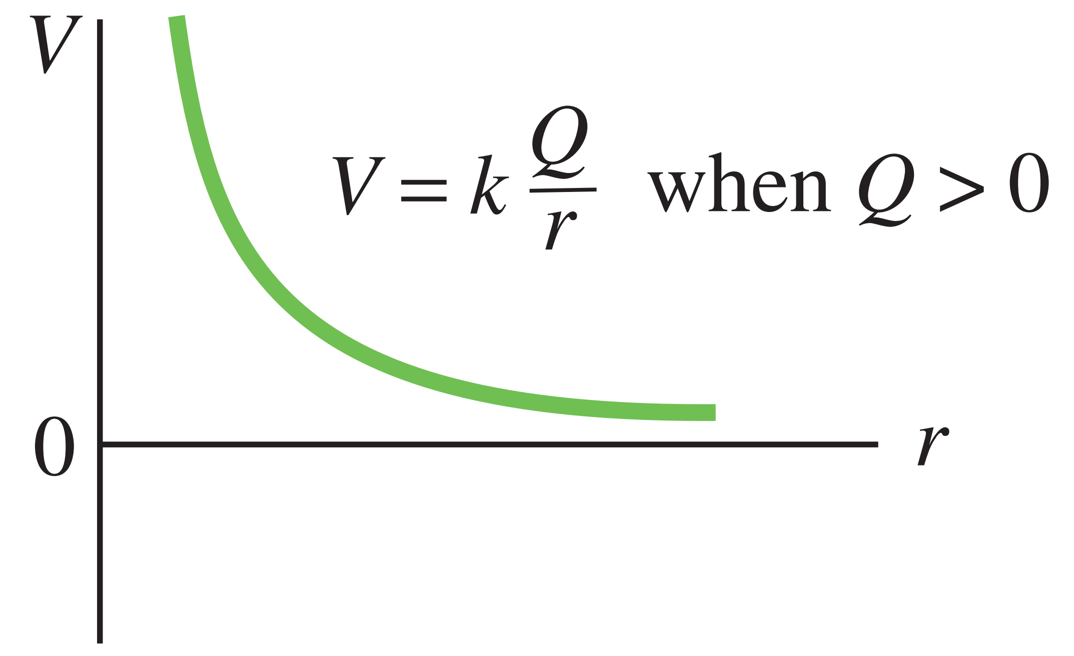
(a) from a 20 μC point charge

(b) from a -20 μC point charge

(a)

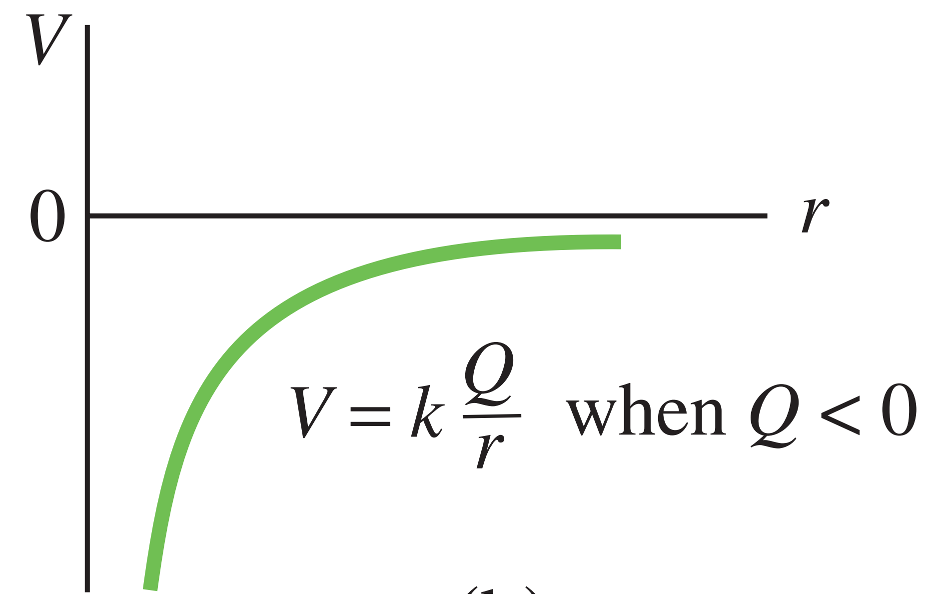
$$V = k\frac{Q}{r}$$

$$= \left(9.0 \times 10^{9}\,\mathrm{N \cdot m^2/C^2}\right)\left(\frac{20 \times 10^{-6}\,\mathrm{C}}{0.50\,\mathrm{m}}\right) = 3.6 \times 10^{5}\,\mathrm{V}$$

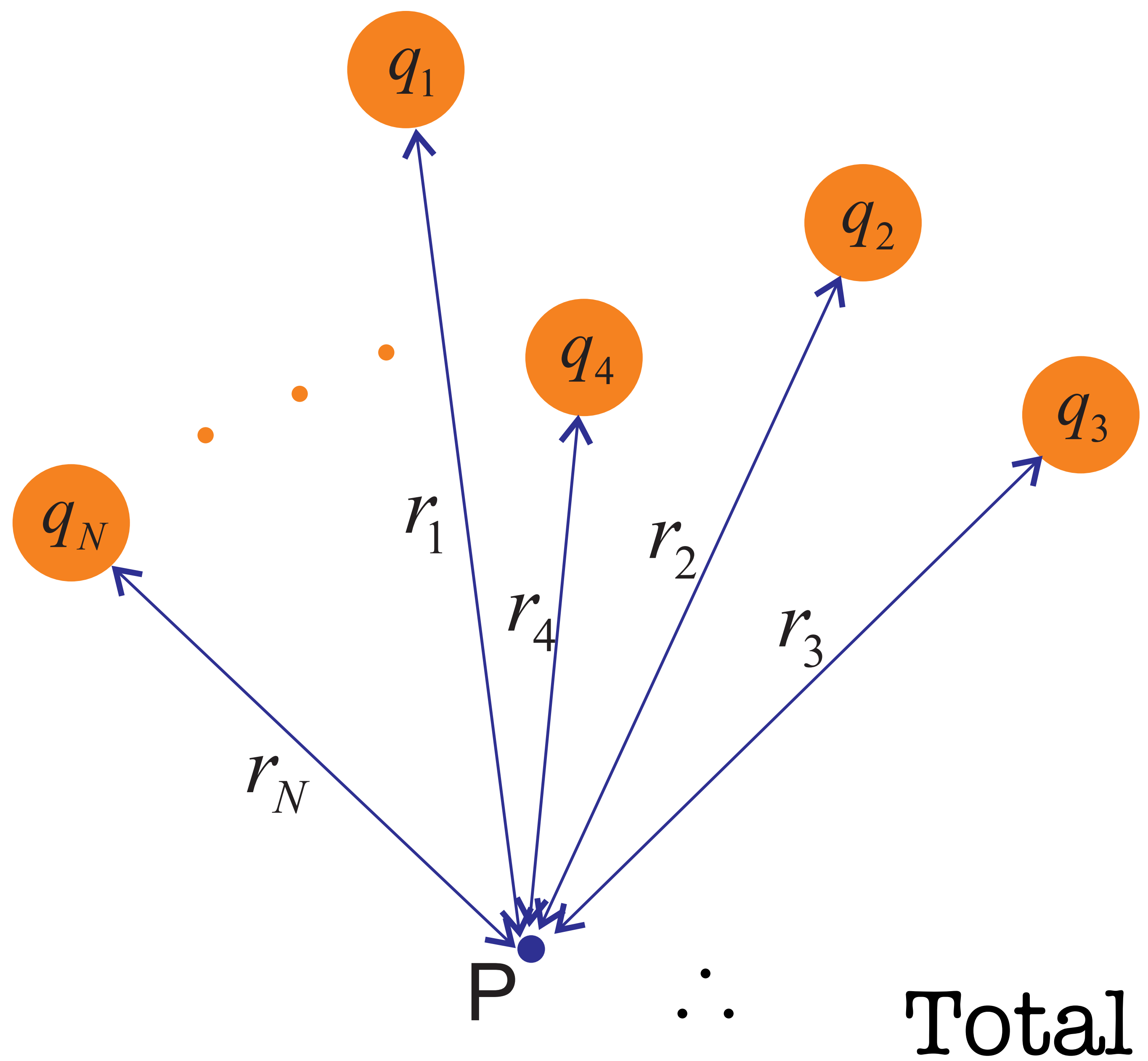


(b)

$$V = \left(9.0 \times 10^{9}\,\mathrm{N \cdot m^2/C^2}\right)\left(\frac{-20 \times 10^{-6}\,\mathrm{C}}{0.50\,\mathrm{m}}\right) = -3.6 \times 10^{5}\,\mathrm{V}.$$



Potential For A System of Charges



For a total of N point charges potential V at any point P can be derived from **principle of superposition**

Recall that potential due to q_1 at point P

$$V_1 = \frac{1}{4\pi\epsilon_0} \cdot \frac{q_1}{r_1}$$

$\therefore$ Total potential at point P due to N charges

principle of superposition

$$V = V_1 + V_2 + \cdots + V_N = \frac{1}{4\pi\epsilon_0}\left[\frac{q_1}{r_1} + \frac{q_2}{r_2} + \cdots \frac{q_N}{r_N}\right]$$

$$V = \frac{1}{4\pi\epsilon_0} \sum_{i=1}^{N} \frac{q_i}{r_i}$$

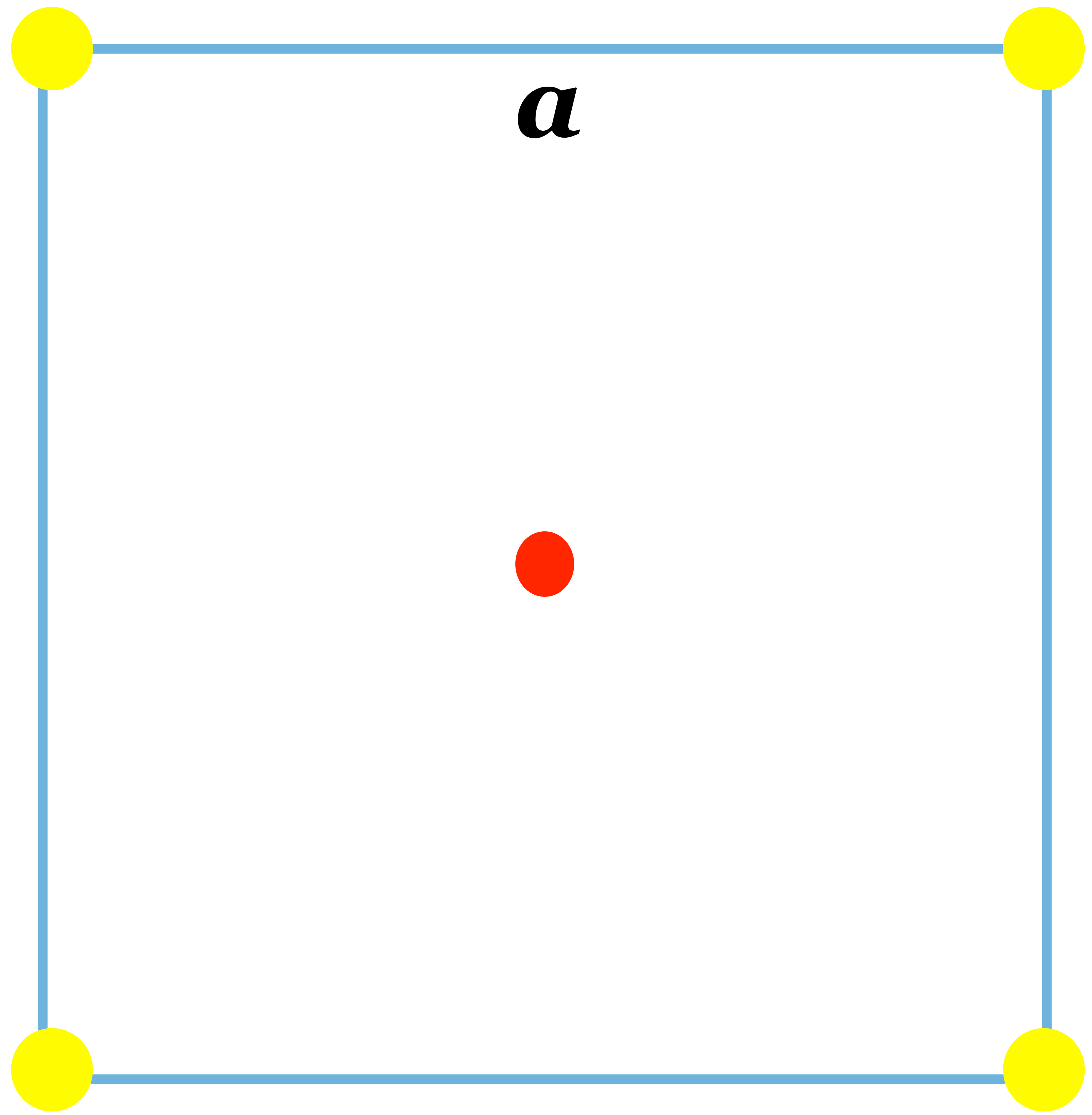
➢ For identical charges q are located at the four corners of a square with side lengh a

What is the electric potential at the center of the square?

$$V = \sum_j V_j = V_1 + V_2 + V_3 + V_4$$

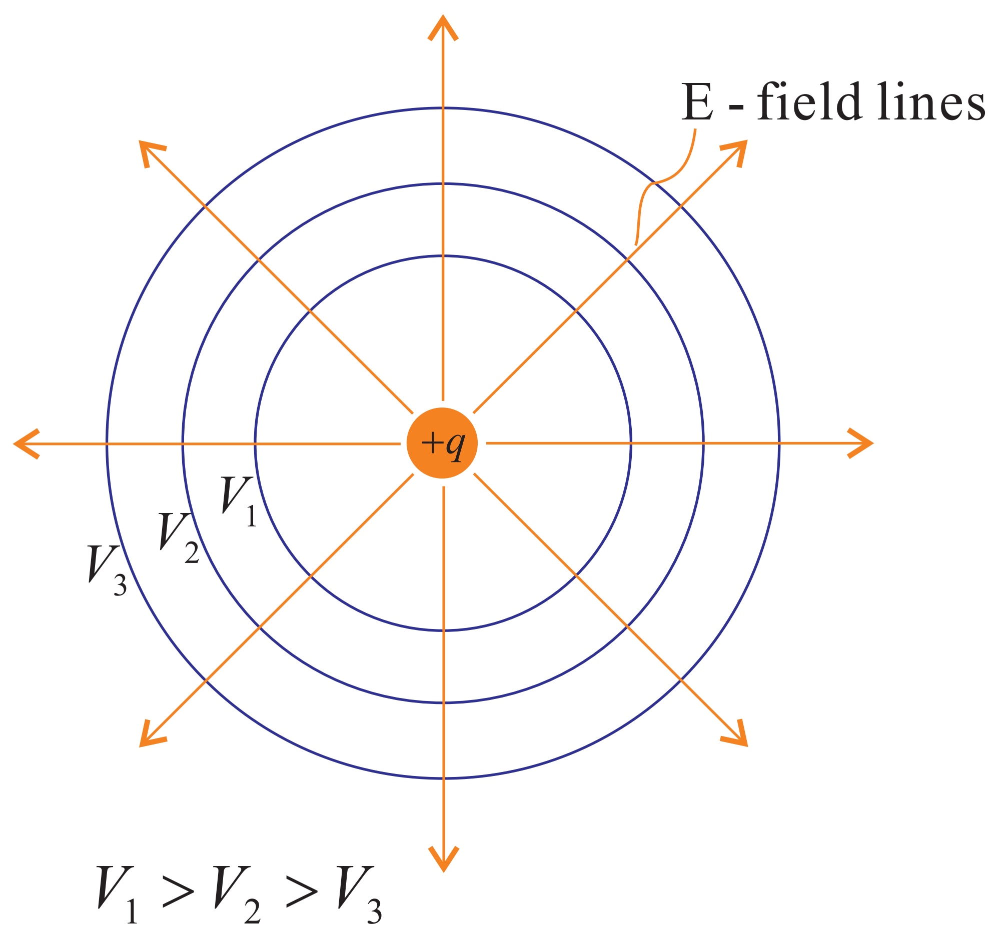
$$V_1 = k\frac{q}{r_1} = kq\frac{\sqrt{2}}{a}$$

$$V = 4\sqrt{2}k\frac{q}{a}$$



Equipotential Surfaces

➢ Equipotential surface is a surface on which **potential** is constant



For point charge $\Rightarrow \quad (\Delta V = 0)$

$$V(r) = \frac{1}{4\pi\epsilon_0} \cdot \frac{+q}{r} = const$$

$$\Rightarrow \quad r = const$$

$\Rightarrow$ Equipotential surface are $circles\,/\,spherical\ surface$

Note ☛

① A charge can move freely on an equipotential surface without any work done

② **Electric field lines** must be perpendicular to **equipotential surfaces**

Why?

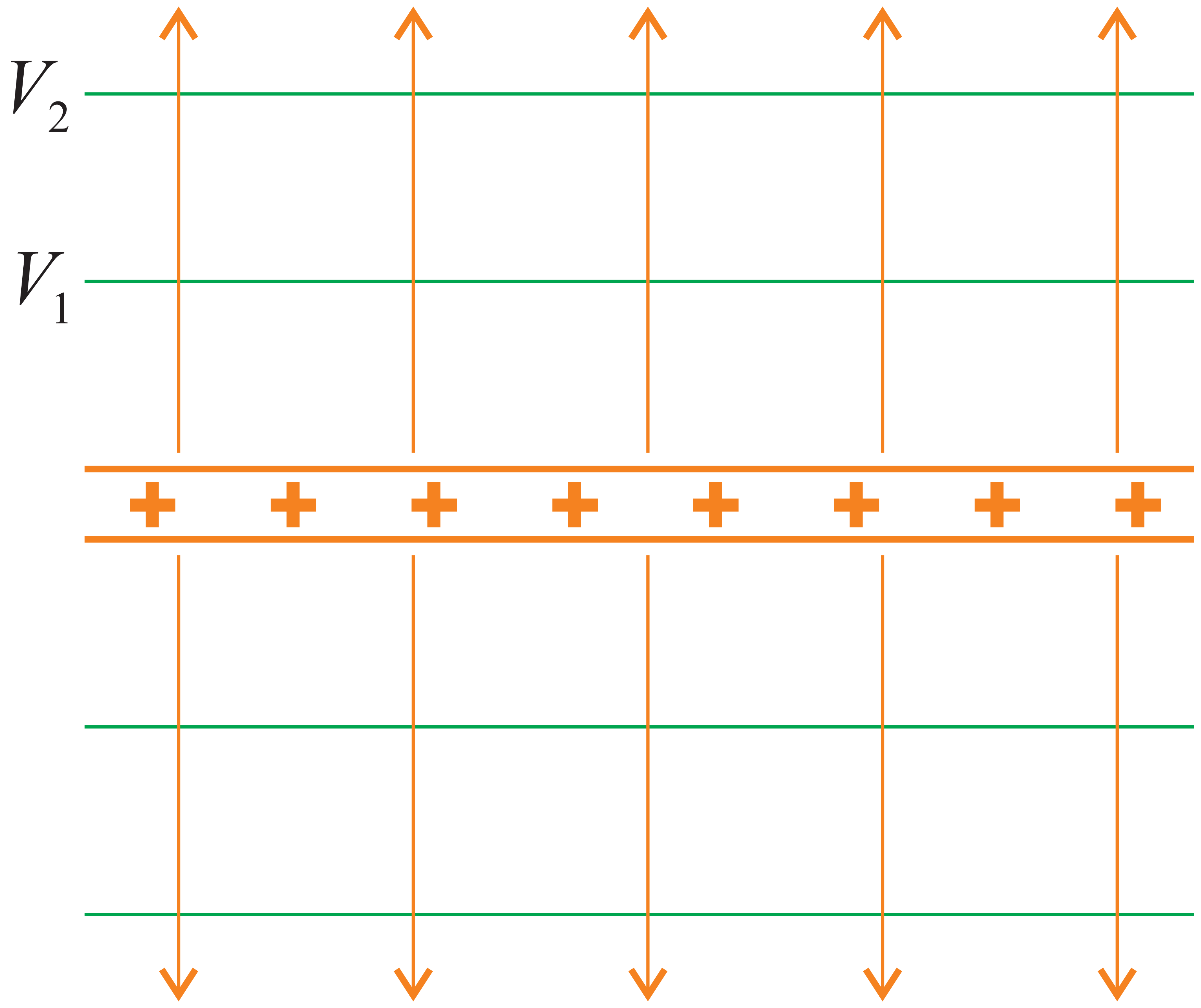
On an equipotential surface $V = constant$

$\Rightarrow \Delta V = 0 \Rightarrow \vec{E} \cdot \Delta\vec{d} = 0$ where $\Delta\vec{d}$ is **tangent** to equipotential surface

$\therefore \vec{E}$ must be **perpendicular** to equipotential surfaces

Example

Uniformly charged surface (infinite)



$V_1 > V_2$

Recall $$V = V_0 - \frac{\sigma}{2\epsilon_0}|z|$$

Potential at $z = 0$

Equipotential surface means

$$V = const \Rightarrow V_0 - \frac{\sigma}{2\epsilon_0}|z| = C$$

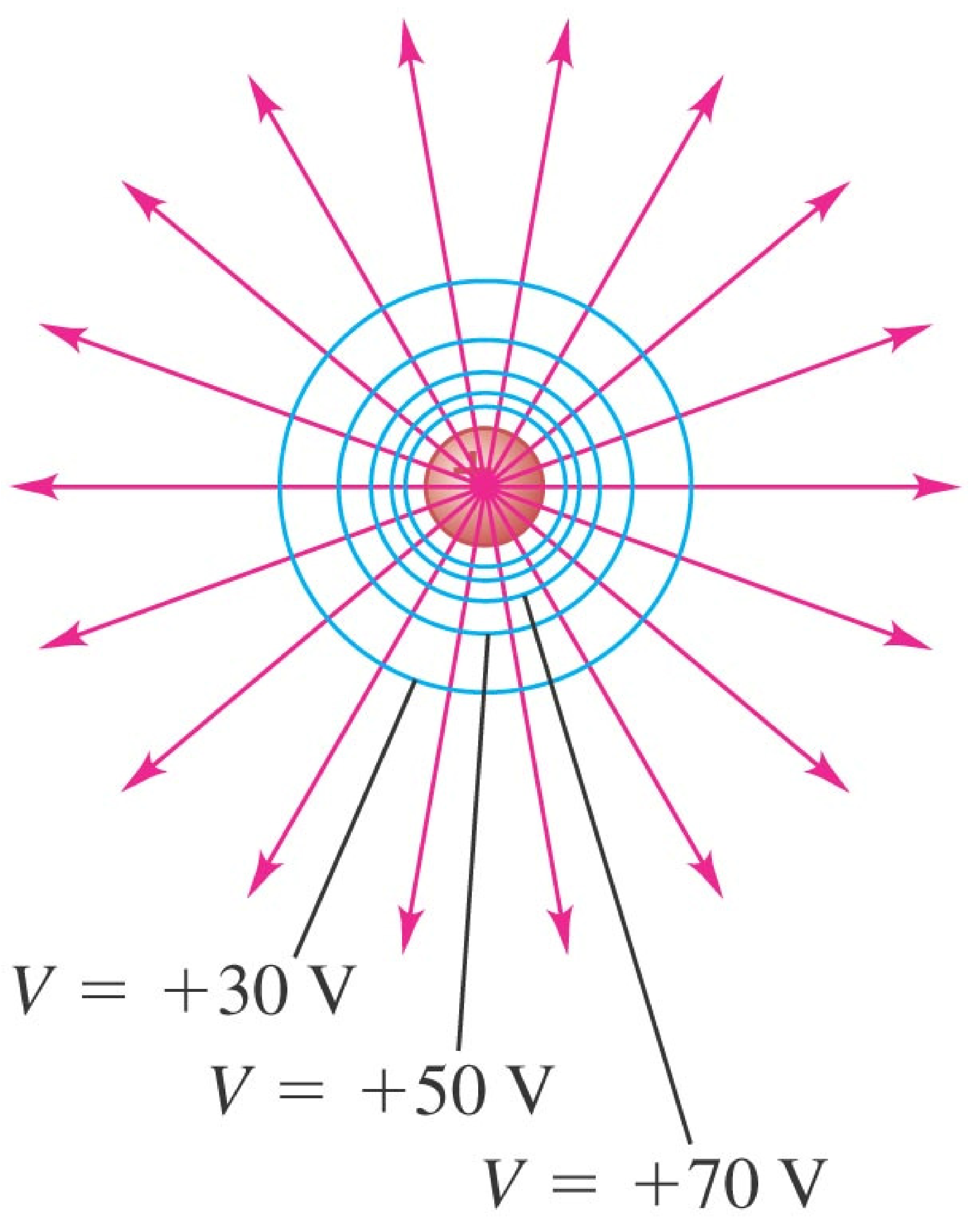
$$\Rightarrow |z| = constant$$

Important

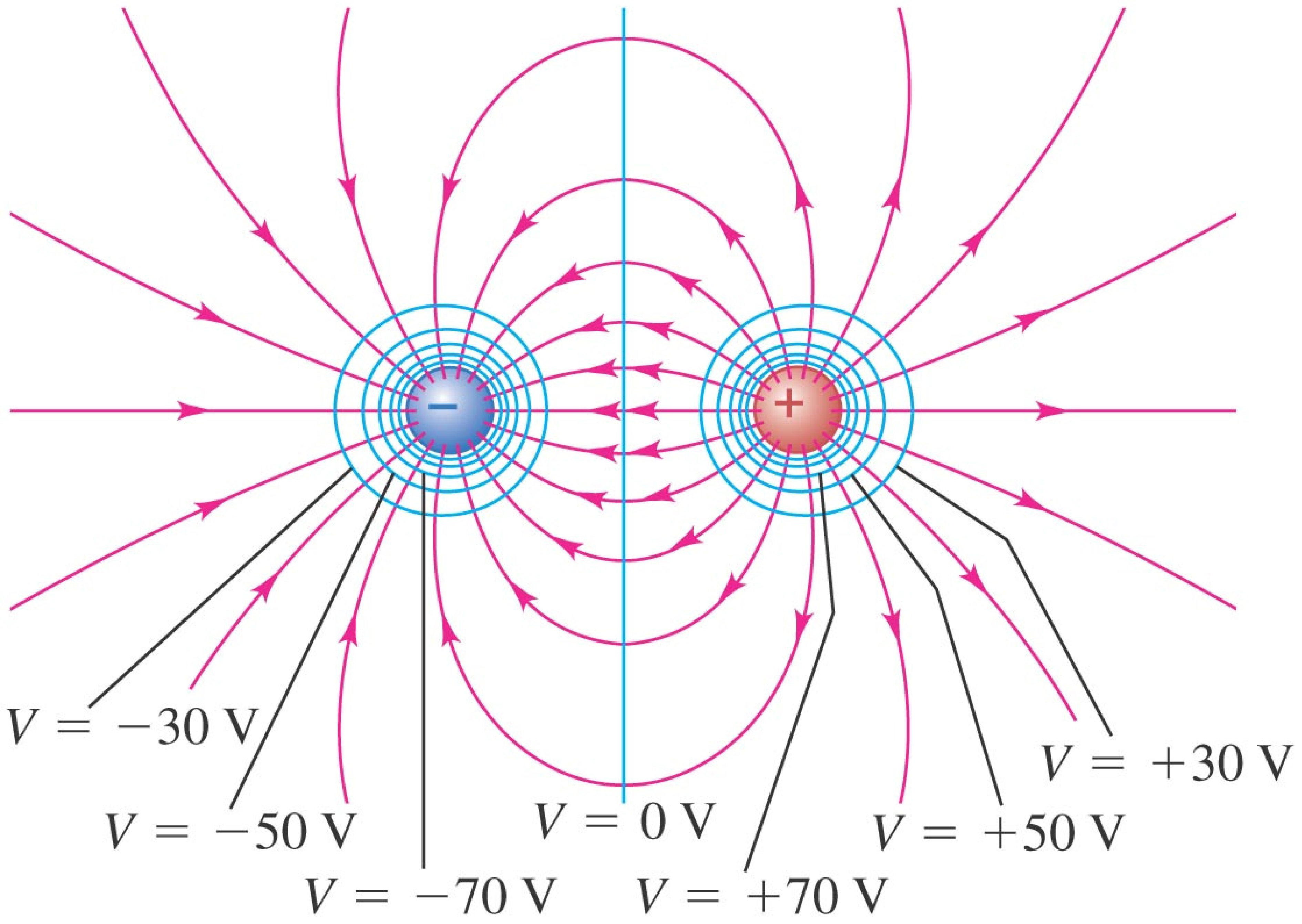
➢ E does not need to be constant over an equipotential surface

➢ Only V is constant

(a) A single positive charge



(b) An electric dipole

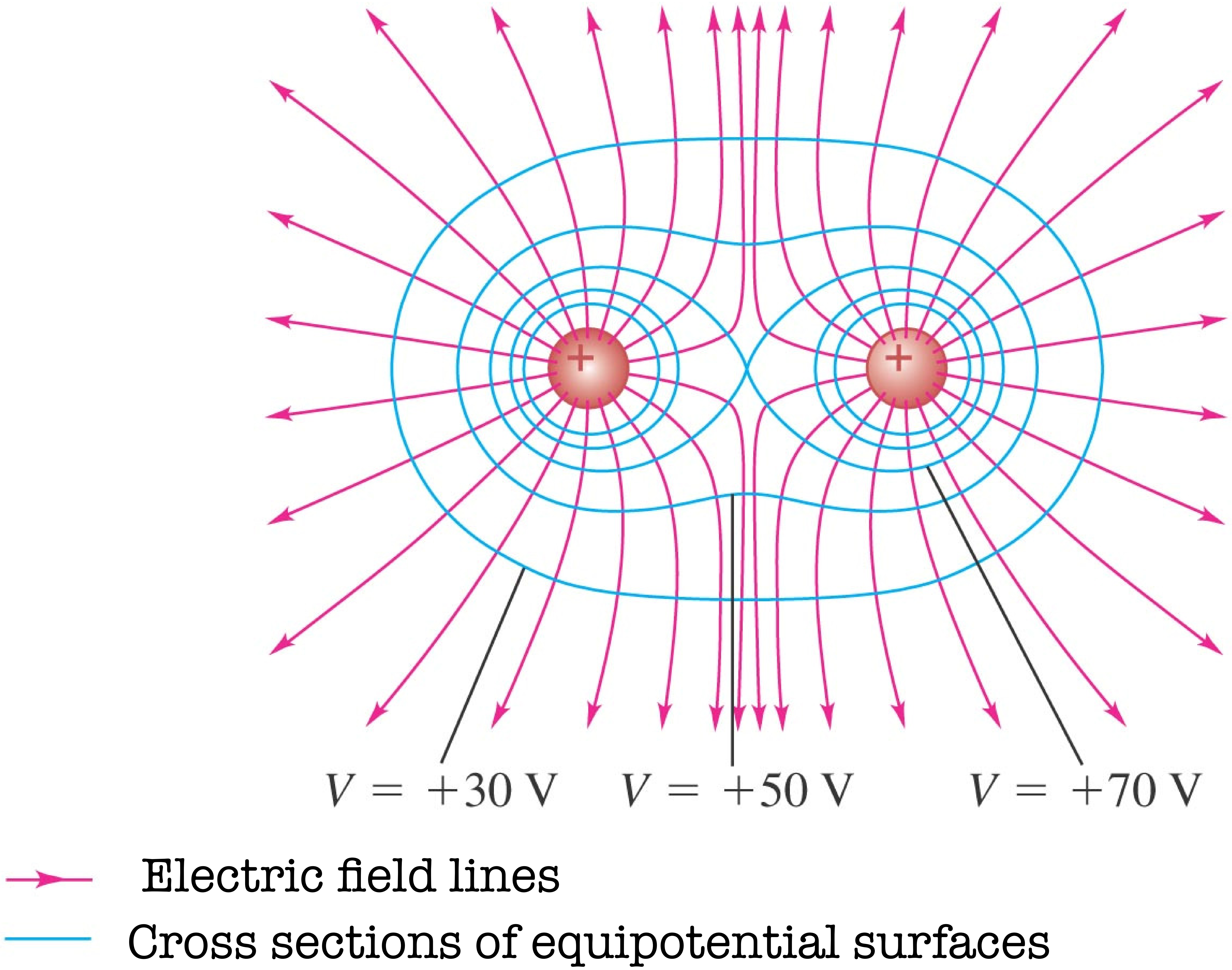


Electric field lines

Cross sections of equipotential surfaces

(c) Two equal positive charges

➢ E is not a constant → $E = 0$ in between two charges (at equal distance from each one), but not elsewhere within same equipotential surface



Equipotentials and Conductors

➢When all charges are at rest, surface of a conductor is always an equipotential surface

→ E outside a conductor $\perp$ to surface at each point

Demonstration

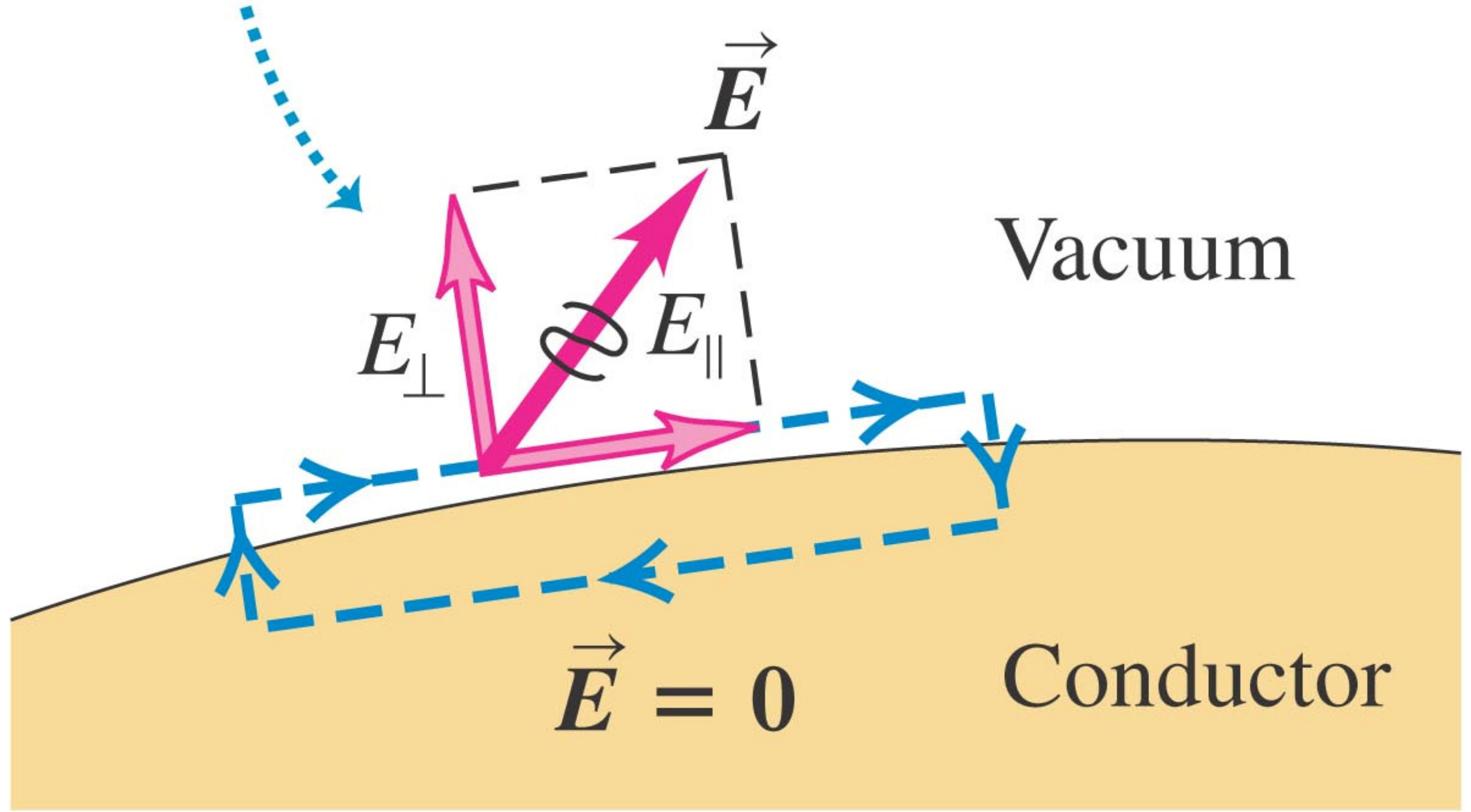
E = 0 (inside conductor) → E tangent to surface inside

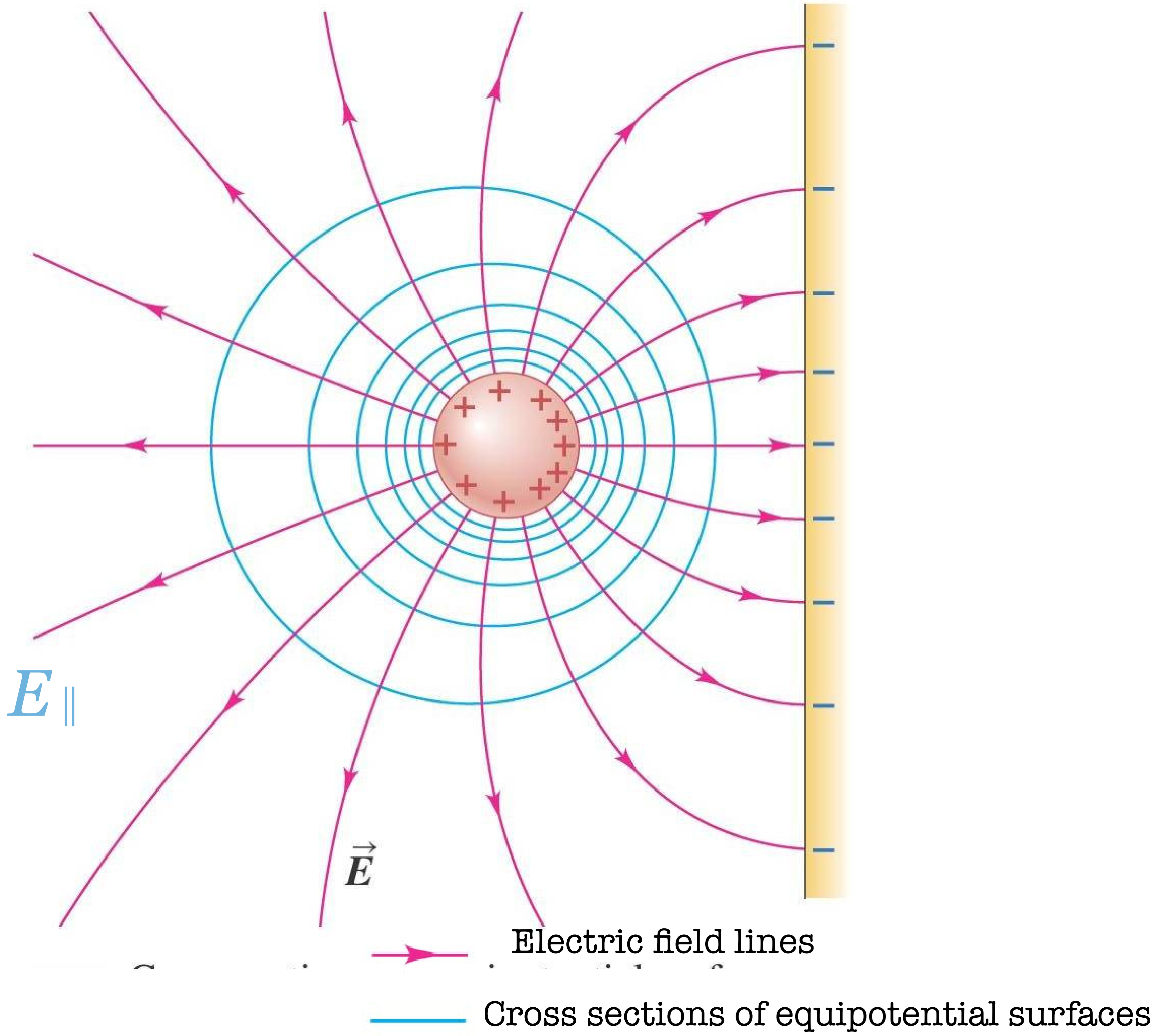
and out of conductor = 0 → otherwise charges would move

following rectangular path

An imposible electric field

If electric field just outside a conductor had a tangential component $E_{\parallel}$ a charge could move in a loop with net work done

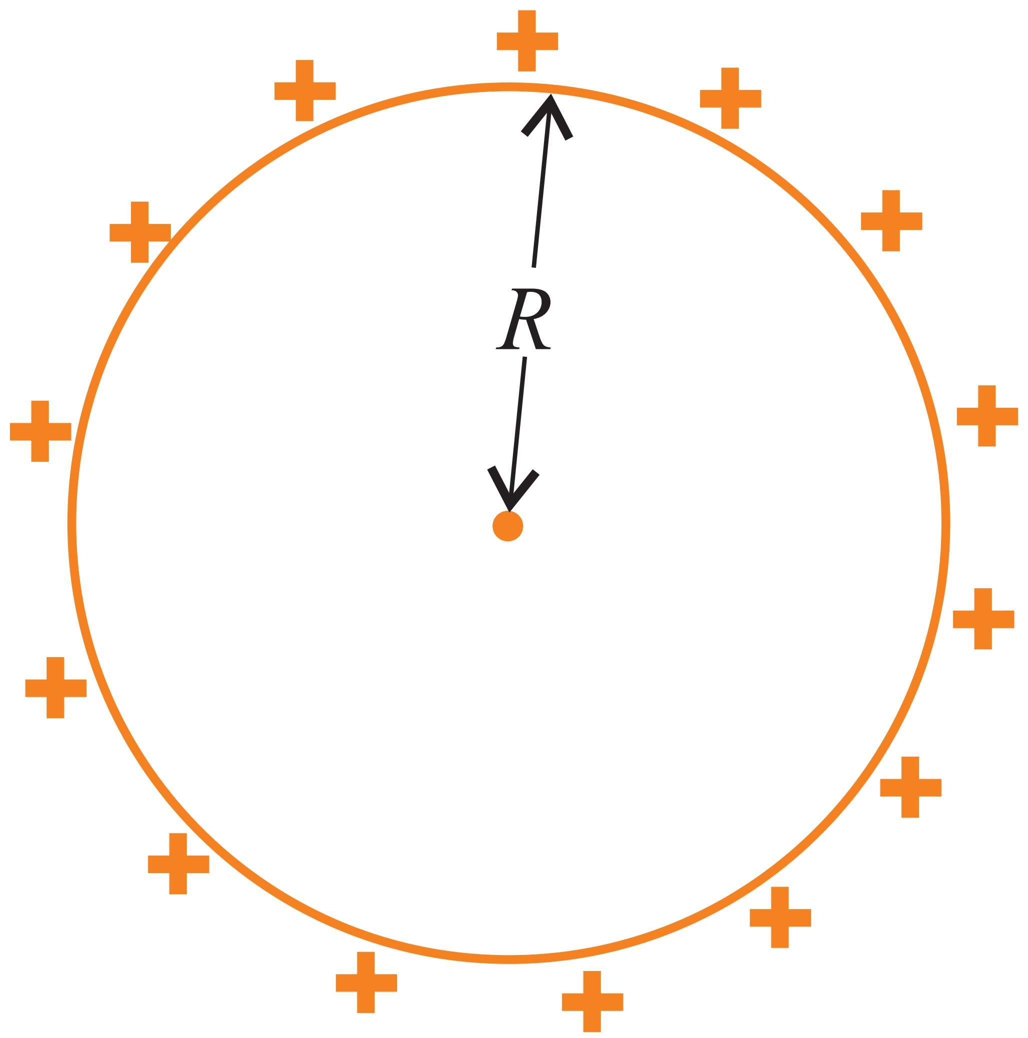




$\vec{E}$ $\perp$to conductor surface

Example

Isolated spherical charged conductors



Recall

① E-field inside = 0

② charge distributed on **outside** of conductors

(i) Inside conductor

$E = 0 \Rightarrow \Delta V = 0$ everywhere in conductor

$\Rightarrow V = constant$ everywhere in conductor

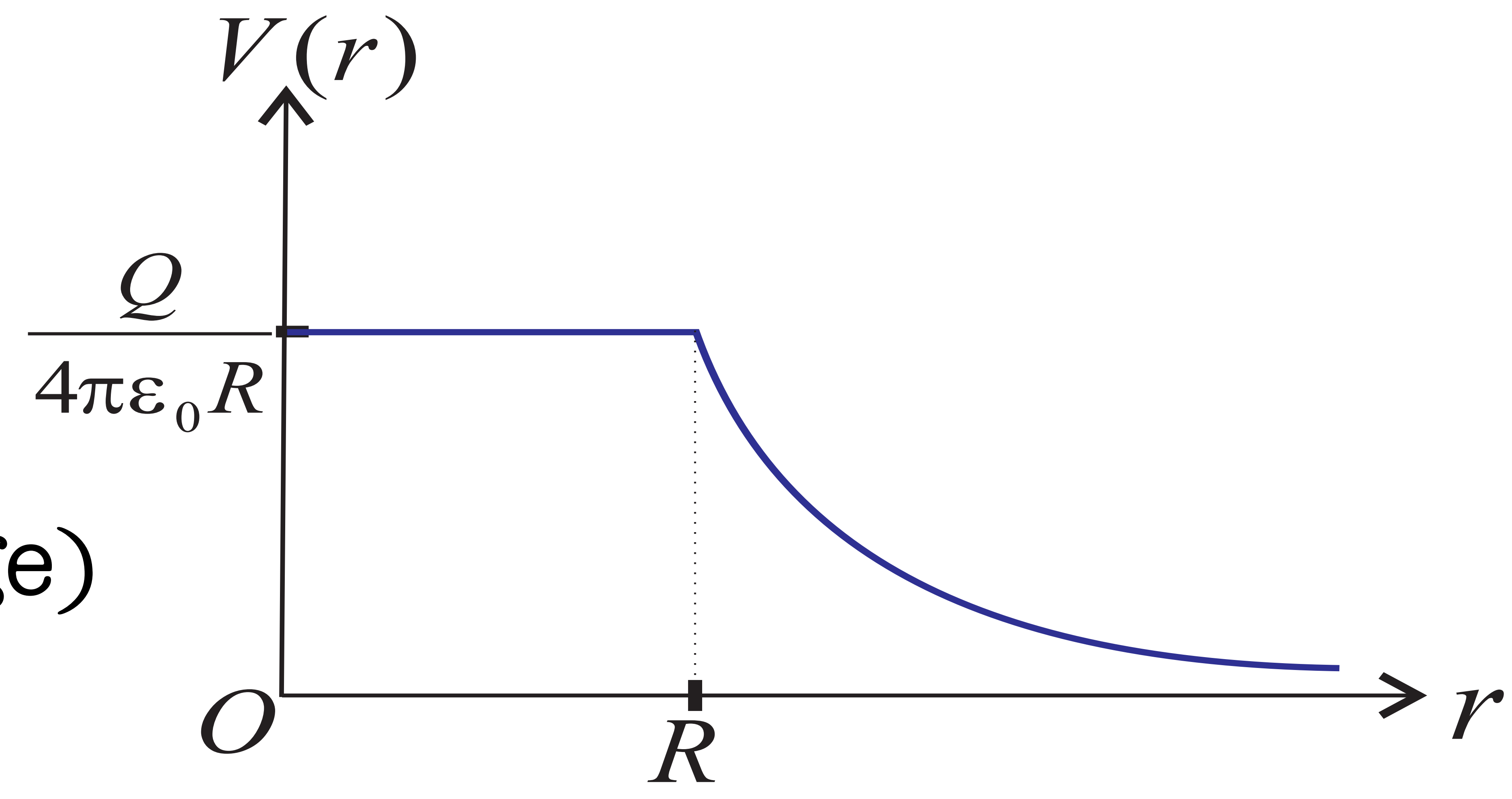
$\Rightarrow$ entire conductor is at same potential

(ii) Outside conductor

$$V = \frac{Q}{4\pi\epsilon_0 r}$$

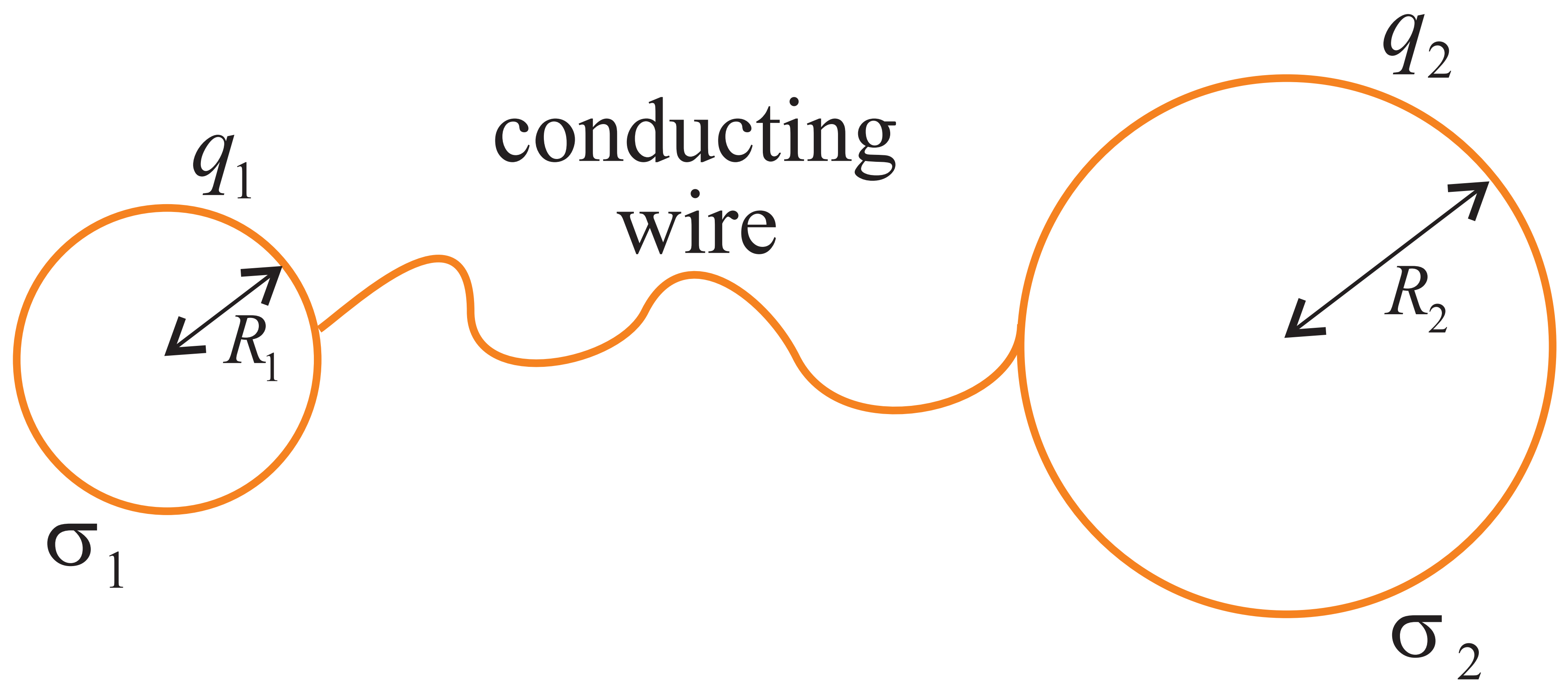
$\because$ Spherically symmetric (Just like a point charge)

BUT not true for conductors of arbitrary shape



Example Connected conducting spheres

Two conductors connected can be seen as a **single conductor**



$\therefore$ **Potential everywhere is identical**

Potential of radius R_1 sphere ☛

$$V_1 = \frac{q_1}{4\pi\epsilon_0 R_1}$$

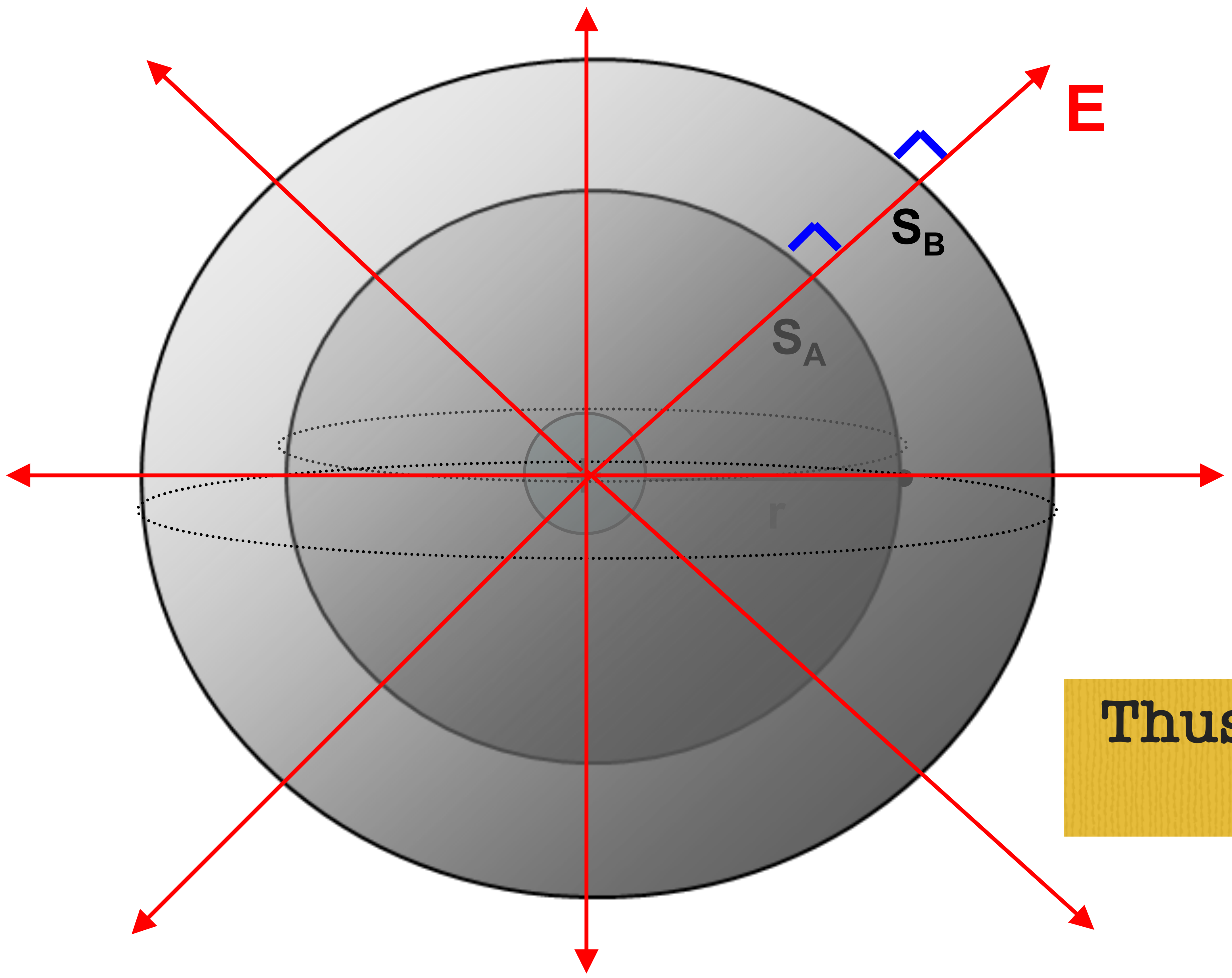
Potential of radius R_2 sphere ☛

$$V_2 = \frac{q_2}{4\pi\epsilon_0 R_2}$$

$$V_1 = V_2$$

$$\Rightarrow \frac{q_1}{R_1} = \frac{q_2}{R_2} \quad \Rightarrow \quad \frac{q_1}{q_2} = \frac{R_1}{R_2}$$

Since SA is closer to positive charge than S_B , S_A is at a higher potential than S_B



Thus, electric field lines point in direction of decreasing potential, i.e. they point from high potential to low potential

Work?

Net electric force does no work as a charge moves on an equipotential surface

Why?

We defined

$$V_B - V_A = \frac{-W_{AB}}{q}$$

But, if we are on an equipotential surface, then $V_A = V_B$ and $W_{AB} = 0$

or

In order for charge to feel a force along an equipotential surface, there must be a component of field along surface, but E is everywhere perpendicular to equipotential surface

Fields , Potentials, and Motion of Charges -Summary-

- ➢ Electric fields lines start on positive charges and end on negative ones
- ➢ Positive charges accelerate from regions of high potential toward low potential
- ➢ Negative charges accelerate from regions of low potential toward high potential
- ➢ Equipotential surfaces are surfaces of constant potential
- ➢ Electric field lines are perpendicular to an equipotential surface
- ➢ Electric field lines are perpendicular to the surface of a conductor, thus a conductor is an equipotential surface!
- ➢ Electric field lines point from regions of high potential toward low potential

Therefore, positive charges move in the same direction as electric field points, and negative charges move in opposite direction of electric field

- ➢ Electric force does no work as a charge moves on an equipotential surface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