Prof. Anchordoqui

Problems set # 5 **Physics 167** **Solutions**

1. *(i)* Three capacitors are connected to a 12.0 V battery as shown in Fig. 1 Their capacitances are $C_1 = 3.00\ \mu$F, $C_2 = 4.00\ \mu$F, and $C_3 = 2\mu$F. Find the equivalent capacitance of this set of capacitors. *(ii)* Find the charge on and the potential difference across each.

Solution *(i)* Using the rules for combining capacitors in series and in parallel, the circuit is reduced in steps as shown in Fig. 2. The equivalent capacitor is hown to be a 2.00 μF capacitor. *(ii)* From Fig. 2 (right panel) it follows that $Q_{ac} = C_{ac}(\Delta V)_{ac} = 2.00\ \mu\text{F}\ 12.0V = 24.0\ \mu$C. From Fig. 2 (middle panel) it follows that $Q_{ab} = Q_{bc} = Q_{ac} = 24\ \mu$C and so the charge on the 3.00μF capacitor is $Q_3 = 24.0\ \mu$C. Continuing to use Fig. 2 (middle panel) it follows that $(\Delta V)_{ab} = \frac{Q_{ab}}{C_{ab}} = \frac{24.0\ \mu\text{C}}{6.00\ \mu\text{F}} = 4.00$ V, and $(\Delta V)_3 = (\Delta V)_{bc} = \frac{Q_{bc}}{C_{bc}} = \frac{24.0\ \mu\text{C}}{3.00\ \mu\text{F}} = 8.00$ V. Finally, from Fig. 2 (left panel) it follows that $(\Delta V)_4 = (\Delta V)_2 = (\Delta V)_{ab} = 4.00$ V, $Q_4 = C_4(\Delta V)_4 = 4.00\ \mu\text{F}\ 4.00\ \text{V} = 16.0\ \mu$C, and $Q_2 = C_2(\Delta V)_2 = 2.00\ \mu\text{F}4.00\ \text{V} = 8.00\ \mu$C.

2. A dielectric rectangular slab has length s, width w, thickness d, and dielectric constant κ. The slab is inserted on the right hand side of a parallel-plate capacitor consisting of two conducting plates of width w, length L, and thickness d. The left hand side of the capacitor of length $L - s$ is empty, see Fig. 3. The capacitor is charged up such that the left hand side has surface charge densities $\pm\sigma_L$ on the facing surfaces of the top and bottom plates respectively and the right hand side has surface charge densities $\pm\sigma_R$ on the facing surfaces of the top and bottom plates respectively. The total charge on the entire top and bottom plates is $+Q$ and $-Q$ respectively. The charging battery is then removed from the circuit. Neglect all edge effects. *(i)* Find an expression for the magnitude of the electric field E_L on the left hand side in terms of σ_L, σ_R, κ, s, w, L, ϵ_0, and d as needed. *(ii)* Find an expression for the magnitude of the electric field E_R on the right hand side in terms of σ_L, σ_R, κ, s, w, L, ϵ_0, and d as needed. *(iii)* Find an expression that relates the surface charge densities σ_L and σ_R in terms of κ, s, w, L, ϵ_0, and d as needed. *(iv)* What is the total charge $+Q$ on the entire top plate? Express your answer in terms of σ_L, σ_R, κ, s, w, L, ϵ_0, and d as needed. *(v)* What is the capacitance of this system? Express your answer in terms of κ, s, w, L, ϵ_0, and d as needed. *(vi)* Suppose the dielectric is removed. What is the change in the stored potential energy of the capacitor? Express your answer in terms of Q, κ, s, w, L, ϵ_0, and d as needed.

Solution: *(i)* Using Gauss' law $E_L = \frac{\sigma_L}{\epsilon_0}$. *(ii)* Using Gauss law for dielectrics $E_R = \frac{\sigma_R}{\kappa\epsilon_0}$. *(iii)* The potential difference on the left side is $E_L d = \frac{\sigma_L d}{\epsilon_0}$. On the right hand side it is $E_R d = \frac{\sigma_R d}{\kappa\epsilon_0}$. Since these must be equal we must have $\sigma_R/\kappa = \sigma_L$. *(iv)* $Q = \sigma_L(L-s)w + \sigma_R s w$. *(v)* The potential difference is $E_L d = \frac{\sigma_L d}{\epsilon_0}$, so the capacitance is $C = \frac{Q}{|\Delta V|} = \frac{\sigma_L(L-s)w + \sigma_R s w}{\sigma_L d/\epsilon_0} = \frac{\epsilon_0 w}{d}\left[(L-s) + \frac{\sigma_R}{\sigma_L}s\right] = \frac{\epsilon_0 w}{d}\left[(L-s) + \kappa s\right]$. *(vi)* Since the battery has been removed, the charge on the capacitor does not change when we do this, and the change in the energy stored is $= \frac{1}{2}\frac{Q^2}{C_0} - \frac{1}{2}\frac{Q^2}{C} = \frac{Q^2}{2}\left(\frac{1}{C_0} - \frac{1}{C}\right) = \frac{Q^2}{2}\left\{\frac{d}{\epsilon_0 L w} - \frac{d}{\epsilon_0 w[(L-s)+\kappa s]}\right\} = \frac{Q^2 d}{2\epsilon_0 w}\left[\frac{1}{L} - \frac{1}{(L-s)+\kappa s}\right] = \frac{Q^2 d}{2\epsilon_0 w}\left\{\frac{(L-s)+\kappa s - L}{L[(L-s)+\kappa s]}\right\} = \frac{Q^2 d}{2\epsilon_0 w}\left\{\frac{(\kappa-1)s}{L[(L-s)+\kappa s]}\right\}$.

3. *(i)* Consider a plane-parallel capacitor completely filled with a dielectric material of dielectric constant κ. What is the capacitance of this system? *(ii)* A parallel-plate capacitor is constructed by filling the space between two square plates with blocks of three dielectric materials, as in Fig. 4. You may assume that $l \gg d$. Find an expression for the capacitance of the device in terms of the plate area A, d, κ_1, κ_2, and κ_3.

Solution: *(i)* The capacitance is $C = \frac{\kappa\epsilon_0 A}{d} = \kappa C_0$. *(ii)* The capacitor can be regarded as being consisted of three capacitors, $C_1 = \frac{\kappa_1\epsilon_0 A/2}{d}$, $C_2 = \frac{\kappa_2\epsilon_0 A/2}{d/2}$, and $C_3 = \frac{\kappa_3\epsilon_0 A/2}{d/2}$, with C_2 and C_3 connected in series, and the combination connected in parallel with C_1. Thus, the equivalent capacitance is $C = C_1 + \left(\frac{1}{C_2} + \frac{1}{C_3}\right)^{-1} = C_1 + \frac{C_2 C_3}{C_2+C_3} = \frac{\kappa_1\epsilon_0 A/2}{d} + \frac{\epsilon_0 A}{d}\left(\frac{\kappa_2\kappa_3}{\kappa_2+\kappa_3}\right) = \frac{\epsilon_0 A}{d}\left(\frac{\kappa_1}{2} + \frac{\kappa_2\kappa_3}{\kappa_2+\kappa_3}\right)$.

4. A model of a red blood cell portrays the cell as a spherical capacitor – a positively charged liquid sphere of surface area A, separated by a membrane of thickness t from the surrounding negatively charged fluid. Tiny electrodes introduced into the interior of the cell show a potential difference of 100 mV across the membrane. The membrane's thickness is estimated to be 100 nm and its dielectric constant to be 5.00. *(i)* If an average red blood cell has a mass of 1.00×10^{-12} kg, estimate the volume of the cell and thus find its surface area. The density of blood is $1,100$ kg/m^3. *(ii)* Estimate the capacitance of the cell. *(iii)* Calculate the charge on the surface of the membrane. How many electronic charges does this represent?

Solution *(i)* The volume is $V = \frac{m}{\rho} = \frac{1.00\times 10^{-12}\ \text{kg}}{1{,}100\ \text{kg/m}^3} = 9.09 \times 10^{-16}\ \text{m}^3$. Since $V = \frac{4}{3}\pi r^3$, the inner radius of the capacitor is $r = \left(\frac{3V}{4\pi}\right)^{1/3} = 6.01 \times 10^{-6}$ m and the surface area is $A = 4\pi r^2 = 4\pi\left(\frac{3V}{4\pi}\right)^{2/3} = 4\pi\left(\frac{3}{4\pi} 9.09 \times 10^{-16}\ \text{m}^3\right)^{2/3} = 4.54 \times 10^{-10}\ \text{m}^2$. *(ii)* The outer radius of the capacitor is $R = r + t = 6.11 \times 10^{-6}$ m, where $t = 100$ nm is the thickness of the membrane. The capacitance is $C = 4\pi\kappa\epsilon_0 \frac{Rr}{R-r} = 2.04 \times 10^{-13}$ F. *(iii)* $Q = C\,\Delta V = 2.04 \times 10^{-13}\ \text{F}\ 100 \times 10^{-3}\ \text{V} = 2.01 \times 10^{-14}$ C, and the number of electronic charges is $n = \frac{Q}{e} = \frac{2.01\times 10^{-14}\ \text{C}}{1.60\times 10^{-19}\ \text{C}} = 1.27 \times 10^5$.

5. A parallel plate capacitor has capacitance C. It is connected to a battery until is fully charged, and then disconnected. The plates are then pulled apart an extra distance d, during which the measured potential difference between them changed by a factor of 4. What is the volume of the dielectric necessary to fill the region between the plates? (Make sure that you give your answer only in terms of variables defined in the statement of this problem, fundamental constants and numbers).

Solution How in the world do we know the volume? We must be able to figure out the cross-sectional area and the distance between the plates. The first relationship we have is from knowing the capacitance: $C = \frac{\epsilon_0 A}{x}$ where x is the original distance between the plates. Make sure you don't use the more typical variable d here because that is used for the distance the plates are pulled apart. Next, the original voltage $V_0 = Ex$, which increases by a factor of 4 when the plates are moved apart by a distance d, that is, $4V_0 = E(x + d)$. From these two equations we can solve for x: $4V_0 = 4Ex = E(x + d) \Rightarrow x = d/3$. Now, we can use the capacitance to get the area, and multiply that by the distance between the plates (now $x + d$) to get the volume, i.e., volume

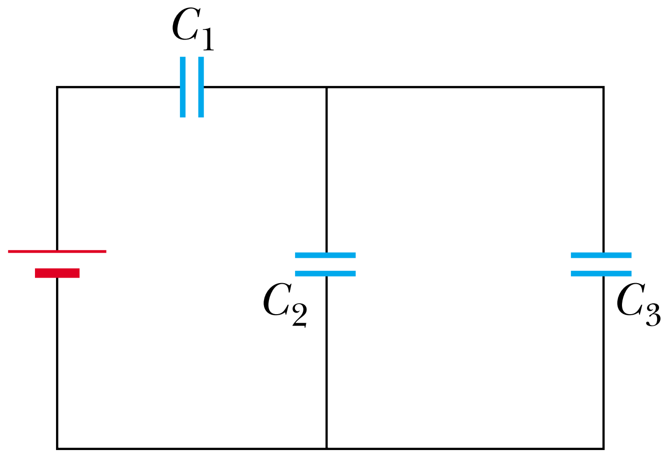

Figure 1: Problem 1.

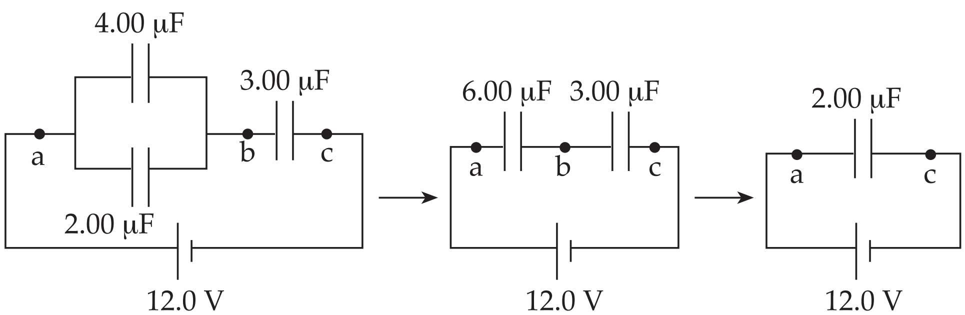

Figure 2: Solution of problem 1.

$= A(x+d) = \frac{xC}{\epsilon_0}(x+d) = \frac{dC}{3\epsilon_0}\left(\frac{d}{3}+d\right) = \frac{4d^2C}{9\epsilon_0}.$

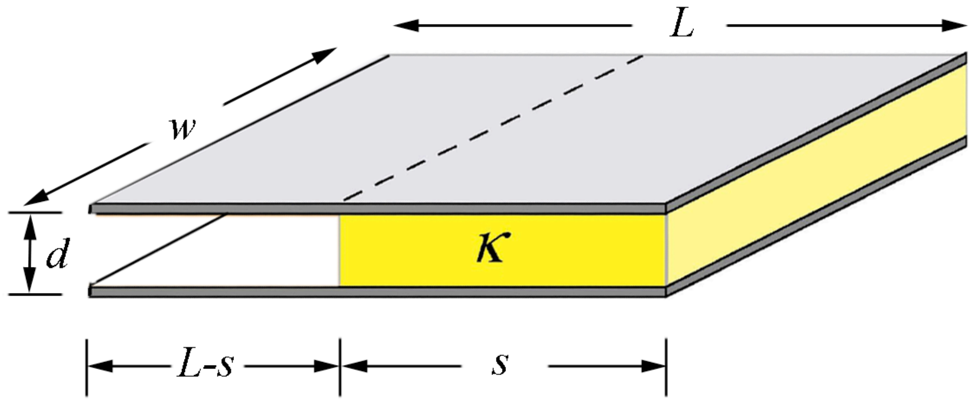

Figure 3: Problem 2.

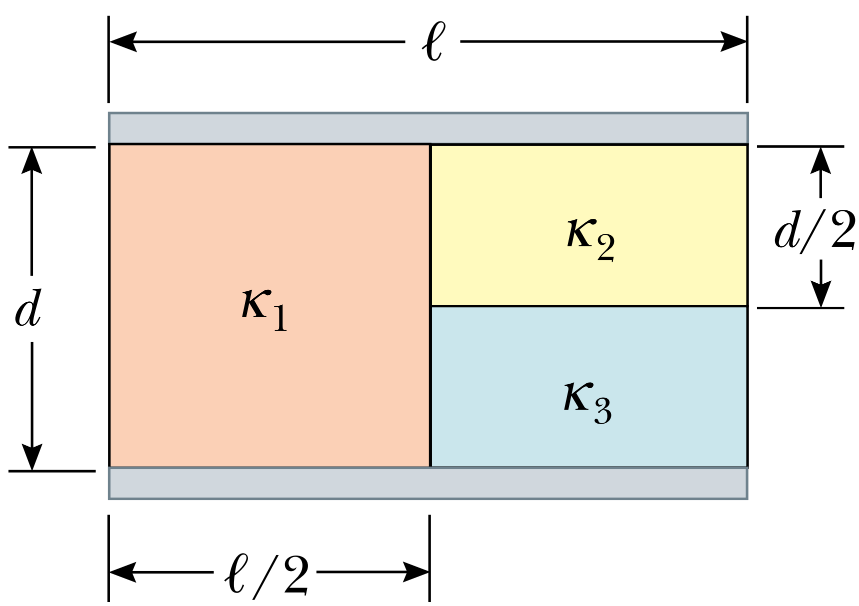

Figure 4: Problem 3.