Homework #5

DUE: MAY 10, 2024

All manifolds are assumed to be smooth, connected, and complete.

1. Prove that if (M^n, g) has $\mathrm{Ric} \geq (n-1)k$, where $k > 0$, and $\mathrm{Vol}(M^n, \mathrm{g}) > \frac{1}{2}\mathrm{Vol}\big(\mathbb{S}^n\big(\frac{1}{\sqrt{k}}\big)\big)$, then M is simply-connected. Give a counter-example if $\mathrm{Vol}(M^n, \mathrm{g}) = \frac{1}{2}\mathrm{Vol}\big(\mathbb{S}^n\big(\frac{1}{\sqrt{k}}\big)\big)$.

   HINT: Compute the volume of the universal covering of M.

   Endow the universal covering $\widetilde{M}^n$ of (M^n, g) with the pullback metric $\widetilde{\mathrm{g}}$, which also has $\mathrm{Ric} \geq (n-1)k$, where $k > 0$. By Myers' Theorem, $\widetilde{M}^n$ is closed, $\pi_1(M)$ is finite, and

   $$\mathrm{Vol}\big(\widetilde{M}^n, \widetilde{\mathrm{g}}\big) = |\pi_1(M)|\,\mathrm{Vol}(M^n, \mathrm{g}) > \frac{|\pi_1(M)|}{2}\,\mathrm{Vol}\big(\mathbb{S}^n\big(\tfrac{1}{\sqrt{k}}\big)\big).$$

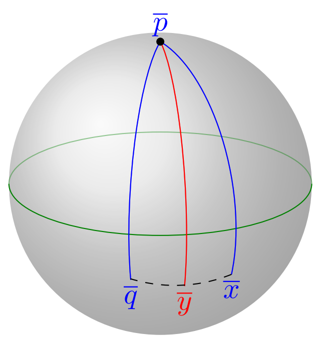
   On the other hand, by Bishop Volume Comparison, $\mathrm{Vol}\big(\widetilde{M}^n, \widetilde{\mathrm{g}}\big) \leq \mathrm{Vol}\big(\mathbb{S}^n\big(\frac{1}{\sqrt{k}}\big)\big)$. Thus, $|\pi_1(M)| = 1$, i.e., $M = \widetilde{M}$ is simply-connected.

   The conclusion becomes false if, instead, $\mathrm{Vol}(M^n, \mathrm{g}) = \frac{1}{2}\mathrm{Vol}\big(\mathbb{S}^n\big(\frac{1}{\sqrt{k}}\big)\big)$, as exemplified by $\mathbb{R}P^n\big(\frac{1}{\sqrt{k}}\big) = \mathbb{S}^n\big(\frac{1}{\sqrt{k}}\big)/\mathbb{Z}_2$ with the "round" metric of $\sec \equiv k$.

2. Suppose that (M^n, g) has $\sec \geq 1$ and $p, q \in M$ satisfy $\mathrm{dist}(p, q) = \mathrm{diam}(M, \mathrm{g}) > \frac{\pi}{2}$. Prove that q is the *unique* point at maximal distance from p, i.e., if $x \in M$ is such that $\mathrm{dist}(p, x) \geq \mathrm{dist}(p, q)$, then $x = q$.

   HINT: Use Toponogov's triangle comparison theorem with vertices p, q, x.

   If $x \in M$ is such that $\mathrm{dist}(p, x) \geq \mathrm{dist}(p, q) = \mathrm{diam}(M, \mathrm{g})$, then $\mathrm{dist}(p, x) = \mathrm{dist}(p, q)$. Suppose $x \neq q$ and consider the geodesic triangle in M with vertices p, q, x. Build a comparison triangle on the sphere $\mathbb{S}^2$ with $\sec \equiv 1$, with the vertex $\overline{p}$ at the north pole.



   Then, as the sides containing $\overline{p}$ have length equal to $\mathrm{diam}(M, \mathrm{g}) > \frac{\pi}{2}$, the vertices $\overline{q}, \overline{x}$ lie in the southern hemisphere of $\mathbb{S}^2$. Thus, any point $\overline{y}$ in the minimizing geodesic of $\mathbb{S}^2$ joining $\overline{q}$ and $\overline{x}$ is farther away from $\overline{p}$ than $\overline{q}, \overline{x}$. By Toponogov's triangle comparison theorem, the corresponding point y along the minimizing geodesic in M joining q and x would satisfy

$$\mathrm{dist}(p, y) \geq \mathrm{dist}(\overline{p}, \overline{y}) > \mathrm{dist}(\overline{p}, \overline{q}) = \mathrm{diam}(M, \mathrm{g}),$$

which is impossible, since $\mathrm{diam}(M, \mathrm{g})$ is the largest distance in (M, g). Thus, $x = q$.

3. About the Lie group $\mathsf{SU}(2) \cong \mathsf{Sp}(1) \cong \mathbb{S}^3$:

   a) Check that $\mathbb{S}^3 = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^2 + |z_2|^2 = 1\}$ endowed with the multiplication

   $$(z_1, z_2) \cdot (w_1, w_2) := \big(z_1 w_1 - \overline{z_2} w_2, w_1 z_2 + \overline{z_1} w_2\big)$$

   is a Lie group.

   b) Prove that $\varphi \colon \mathbb{S}^3 \to \mathsf{SU}(2)$, $\varphi(z_1, z_2) = \begin{pmatrix} z_1 & -\overline{z_2} \\ z_2 & \overline{z_1} \end{pmatrix}$, is a Lie group isomorphism.

   c) Find a Lie group isomorphism $\psi \colon \mathsf{Sp}(1) \to \mathbb{S}^3$, where

   $$\mathsf{Sp}(1) = \{a + b\,i + c\,j + d\,k \in \mathbb{H} : a^2 + b^2 + c^2 + d^2 = 1\}$$

   is the group of unit quaternions.

   a) The given multiplication is a polynomial map, hence smooth. Note that this choice of multiplication identifies $\mathbb{C}^2$ with $\mathbb{C} \oplus j\mathbb{C} \cong \mathbb{H}$ as a real algebra; e.g., $(i, 0) \cdot (0, 1) = (0, -i)$ in $\mathbb{C}^2$ corresponds to $i \cdot j = k$ in $\mathbb{H}$.

   b) Note that $\varphi \colon \mathbb{S}^3 \to \mathsf{SU}(2)$ is a smooth homomorphism, hence it is a Lie group homomorphism. Since $\ker \varphi = \{(1, 0)\}$ and $\varphi(\mathbb{S}^3) = \mathsf{SU}(2)$, it is an isomorphism.

   c) Let $\psi \colon \mathsf{Sp}(1) \to \mathbb{S}^3$ be given by $\psi(a + b\,i + c\,j + d\,k) = (a + b\,i, c - d\,i)$. Note that ψ is well-defined and it is a group homomorphism. Indeed, $\psi(1) = (1, 0)$, $\psi(i) = (i, 0)$, $\psi(j) = (0, 1)$, $\psi(k) = (0, -i)$ satisfy the same multiplication table in $\mathbb{S}^3$ as the quaternionic imaginary units $\{1, i, j, k\}$ in $\mathsf{Sp}(1)$, since $\mathbb{C} \oplus j\mathbb{C} \cong \mathbb{H}$ as real algebras: $(a + b\,i) + j(c - d\,i) = a + b\,i + c\,j + d\,k$. The homomorphism ψ is clearly smooth and injective, hence it is a Lie group isomorphism.

4. About the Lie algebra $\mathfrak{so}(3) \cong (\mathbb{R}^3, \times)$:

   a) Show that $\mathfrak{so}(3) = \{A \in \mathfrak{gl}(3, \mathbb{R}) : A^T + A = 0\}$ is a Lie algebra, with Lie bracket given by the matrix commutator

   b) Given $u = (u_1, u_2, u_3) \in \mathbb{R}^3$, let $A_u = \begin{pmatrix} 0 & -u_3 & u_2 \\ u_3 & 0 & -u_1 \\ -u_2 & u_1 & 0 \end{pmatrix}$. Show that $A_u v = u \times v$ for all $v \in \mathbb{R}^3$.

c) Show that $A_{u\times v} = [A_u, A_v]$. Conclude that the map $\phi\colon (\mathbb{R}^3, \times) \to \mathfrak{so}(3)$, given by $\phi(u) = A_u$, is a Lie algebra isomorphism.

a) The Lie bracket $[A, B] = AB - BA$ clearly satisfies the Jacobi identity. If $A^T = -A$ and $B^T = -B$, then $[A, B]^T = (AB - BA)^T = B^T A^T - A^T B^T = -[A, B]$, so $\mathfrak{so}(3)$ is closed under this bracket and is hence a Lie algebra.

b)

$$A_u v = \begin{pmatrix} 0 & -u_3 & u_2 \\ u_3 & 0 & -u_1 \\ -u_2 & u_1 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} u_2 v_3 - u_3 v_2 \\ u_3 v_1 - u_1 v_3 \\ u_1 v_2 - u_2 v_1 \end{pmatrix} = u \times v$$

c)

$$[A_u, A_v] = A_u A_v - A_v A_u =$$
$$= \begin{pmatrix} 0 & -u_1 v_2 + u_2 v_1 & u_3 v_1 - u_1 v_3 \\ u_1 v_2 - u_2 v_1 & 0 & -u_2 v_3 + u_3 v_2 \\ -u_3 v_1 + u_1 v_3 & u_2 v_3 - u_3 v_2 & 0 \end{pmatrix} = A_{u\times v}$$

The map $\phi\colon (\mathbb{R}^3, \times) \to \mathfrak{so}(3)$, $\phi(u) = A_u$, is clearly linear, and, by the above, satisfies $[\phi(u), \phi(v)] = \phi(u \times v)$ for all $u, v \in \mathbb{R}^3$, hence it is a Lie algebra homomorphism. Since $\ker \phi = \{0\}$ and $\phi(\mathbb{R}^3) = \mathfrak{so}(3)$, it is a Lie algebra isomorphism.

5. Prove that $\mathsf{SU}(2)$ is the universal covering of $\mathsf{SO}(3)$ via the following steps:

a) Decompose quaternions into real and imaginary part $q \in \mathbb{H} \cong \operatorname{Re}\mathbb{H} \oplus \operatorname{Im}\mathbb{H}$, where $\operatorname{Im} q = b\,i + c\,j + d\,k \in \operatorname{Im}\mathbb{H}$ is identified with the vector $\operatorname{Im} q = (b, c, d) \in \mathbb{R}^3$, so that quaternion multiplication can be written as:

$$\operatorname{Re}(\operatorname{Re} q_1 + \operatorname{Im} q_1)(\operatorname{Re} q_2 + \operatorname{Im} q_2) = \operatorname{Re} q_1 \operatorname{Re} q_2 - \langle \operatorname{Im} q_1, \operatorname{Im} q_2 \rangle$$
$$\operatorname{Im}(\operatorname{Re} q_1 + \operatorname{Im} q_1)(\operatorname{Re} q_2 + \operatorname{Im} q_2) = \operatorname{Re} q_2 \operatorname{Im} q_1 + \operatorname{Re} q_1 \operatorname{Im} q_2 + \operatorname{Im} q_1 \times \operatorname{Im} q_2$$

If $q \in \mathsf{Sp}(1)$, let $u = (u_1, u_2, u_3) \in \mathbb{S}^2 \subset \mathbb{R}^3 \cong \operatorname{Im}\mathbb{H}$ be a unit vector and $\theta \in [0, 2\pi]$, such that $\operatorname{Re} q = \cos\theta$ and $\operatorname{Im} q = (\sin\theta)u$. Prove that $T_q\colon \mathbb{R}^3 \to \mathbb{R}^3$ given by quaternionic conjugation $T_q(v) = q\,v\,q^{-1}$ is the orthogonal linear transformation

$$T_q = e^{2\theta A_u} \in \mathsf{SO}(3), \quad \text{where} \quad A_u = \begin{pmatrix} 0 & -u_3 & u_2 \\ u_3 & 0 & -u_1 \\ -u_2 & u_1 & 0 \end{pmatrix} \in \mathfrak{so}(3).$$

HINT: Compute $\frac{\mathrm{d}}{\mathrm{d}t} T_{\cos(t\theta)+\sin(t\theta)u}(v)\big|_{t=0}$ using the Leibniz rule.

b) Prove that $\varphi\colon \mathsf{Sp}(1) \to \mathsf{SO}(3)$, $\varphi(q) = T_q$, is a double covering map. In particular, $\pi_1(\mathsf{SO}(3)) \cong \mathbb{Z}_2$

a) If $q \in \mathsf{Sp}(1)$, we have $q^{-1} = \overline{q}$, in particular $\operatorname{Im} q^{-1} = -\operatorname{Im} q$. Thus, the formulas for quaternionic multiplication imply that if $\operatorname{Re} v = 0$, then $\operatorname{Re} q\, v\, q^{-1} = 0$, so quaternionic conjugation by q defines a linear transformation T_q of $\operatorname{Im} \mathbb{H} \cong \mathbb{R}^3$. This linear transformation is orthogonal because $|T_q(v)| = |q||v||q|^{-1} = |v|$ for all $v \in \operatorname{Im} \mathbb{H} \cong \mathbb{R}^3$. Thus, $T_q \in \mathsf{SO}(3)$ for all $q \in \mathsf{Sp}(1)$.

   Let $\mathbf{q}(t) = \cos(t\theta) + \sin(t\theta)\, u$, $t \in \mathbb{R}$, so that $\mathbf{q}(0) = 1$ and $\mathbf{q}(1) = q$. Note that $\mathbf{q}(t)$ is a 1-parameter subgroup of $\mathsf{Sp}(1)$ and $\mathbf{q}'(0) = \theta u \in T_1\mathsf{Sp}(1) \cong \mathfrak{sp}(1)$, so $\mathbf{q}(t) \in \mathsf{Sp}(1)$ is the 1-parameter subgroup $\mathbf{q}(t) = \exp t\theta u$. By the Leibniz rule:

$$
\begin{aligned}
\tfrac{\mathrm{d}}{\mathrm{d}t} T_{\mathbf{q}(t)}(v)\big|_{t=0} &= \tfrac{\mathrm{d}}{\mathrm{d}t}\mathbf{q}(t)\, v\, \mathbf{q}(t)^{-1}\big|_{t=0} \\
&= \tfrac{\mathrm{d}}{\mathrm{d}t}\mathbf{q}(t)\big|_{t=0}\, v\, \mathbf{q}(0)^{-1} + \mathbf{q}(0) \tfrac{\mathrm{d}}{\mathrm{d}t} v\big|_{t=0} \mathbf{q}(0)^{-1} + \mathbf{q}(0)\, v\, \tfrac{\mathrm{d}}{\mathrm{d}t}\mathbf{q}(t)^{-1}\big|_{t=0} \\
&= \theta u\, v \ + 0 + v\,(-\theta u) \\
&= \theta(u\, v - v\, u) \\
&= \theta(u \times v - v \times u) \\
&= 2\theta\, u \times v \\
&= 2\theta\, A_u v.
\end{aligned}
$$

   Clearly, $t \mapsto T_{\mathbf{q}(t)}$ is a 1-parameter subgroup of $\mathsf{SO}(3)$. By the above, $T_{\mathbf{q}(t)}$ is the 1-parameter subgroup $T_{\mathbf{q}(t)} = \exp t(2\theta A_u)$. Setting $t = 1$, we have $T_q = e^{2\theta\, A_u}$.

b) The map $\varphi \colon \mathsf{Sp}(1) \to \mathsf{SO}(3)$, $\varphi(q) = T_q$, is clearly a continuous group homomorphism, hence a Lie group homomorphism. Moreover, setting $\theta = 1$ in the above computation, we find $\mathrm{d}\varphi(1)u = 2A_u$ for all $u \in \mathbb{S}^2 \subset \mathbb{R}^3 \cong \mathfrak{sp}(1)$. Thus, $\mathrm{d}\varphi(1)$ is an isomorphism and hence φ is a covering map (by Prop. 1.24 in Chapter 1).

   Moreover, if $\varphi(q) = e^{2\theta A_u} = \mathrm{Id}$, since $u \in \mathbb{S}^2$, we must have $\theta = 0$ or $\theta = \pi$, i.e., $\ker \varphi = \{1, -1\} \cong \mathbb{Z}_2$. Thus, φ is a double covering. Since $\mathsf{Sp}(1) \cong \mathbb{S}^3$ is simply-connected, it is the universal covering of $\mathsf{SO}(3)$. In particular, $\pi_1(\mathsf{SO}(3)) \cong \mathbb{Z}_2$