MATH71000, Spring 2024 **Renato Ghini Bettiol**

Final Exam

Due: May 17, 2024

The exam contains 6 problems, each is worth 2.5 points, but the maximum grade is 10/10. Please indicate which 5 problems you would like to be graded, the problem with the lowest score will be discarded.

1. Consider the Lie group

$$\mathsf{G} = \left\{ \begin{pmatrix} x & y & z \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in \mathsf{GL}(3,\mathbb{R}) : x > 0,\, y, z \in \mathbb{R} \right\},$$

whose Lie algebra, endowed with the Lie bracket $[A,B] = AB - BA$ from $\mathfrak{gl}(3,\mathbb{R})$, is

$$\mathfrak{g} = \operatorname{span}\left\{ X = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, Y = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, Z = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right\} \subset \mathfrak{gl}(3,\mathbb{R}).$$

a) Compute the Lie brackets $[X,Y]$, $[Y,Z]$, $[X,Z]$ of the basis elements of $\mathfrak{g}$, and write them in terms of X, Y, Z.

b) Let g be the left-invariant Riemannian metric on G that at $\mathrm{Id} \in \mathsf{G}$ coincides with the inner product on $\mathfrak{g}$ for which $\{X, Y, Z\}$ is an orthonormal basis. Denote by $X, Y, Z \in \mathfrak{X}(\mathsf{G})$ the left-invariant vector fields corresponding to $X, Y, Z \in \mathfrak{g}$. Use the Koszul formula and a) to compute the the Levi-Civita connection ∇ of (G, g),

$$\begin{array}{ccc} \nabla_X X = & \nabla_X Y = & \nabla_X Z = \\ \nabla_Y X = & \nabla_Y Y = & \nabla_Y Z = \\ \nabla_Z X = & \nabla_Z Y = & \nabla_Z Z = \end{array}$$

(To avoid unnecessary computations, recall that $\nabla_A B - \nabla_B A = [A,B]$.)

c) Show that the curvature operator $R\colon \wedge^2 \mathfrak{g} \to \wedge^2 \mathfrak{g}$ is $R = -\mathrm{Id}$. Conclude that g is not a bi-invariant metric, and that (G, g) is isometric to hyperbolic 3-space.

d) On a matrix Lie group, such as G, the adjoint action is given by conjugation, i.e., $\mathrm{Ad}(A)B = ABA^{-1}$, for all $A \in \mathsf{G}$ and $B \in \mathfrak{g}$. Show that

$$\text{if } A = \begin{pmatrix} x & y & z \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \text{ then } \begin{array}{l} \mathrm{Ad}(A)X = X - yY - zZ, \\ \mathrm{Ad}(A)Y = xY, \\ \mathrm{Ad}(A)Z = xZ, \end{array}$$

and use the matrix that represents $\mathrm{Ad}(A)\colon \mathfrak{g} \to \mathfrak{g}$ in the basis $\{X, Y, Z\}$ to compute the eigenvalues of $\mathrm{Ad}(A)$. Conclude that G does not admit bi-invariant metrics.

2. Let (M^4, g) be a closed Riemannian 4-manifold. The conformal metric $\mathrm{h} = u^2\,\mathrm{g}$, where $u\colon M^4 \to \mathbb{R}$ is a positive smooth function, has scalar curvature given by:

$$\mathrm{scal}_{\mathrm{h}} = \big(-6\,\Delta u + u\,\mathrm{scal}_{\mathrm{g}} \big)\, u^{-3},$$

where $\Delta u = \mathrm{tr}\,\mathrm{Hess}\, u$. Suppose $\mathrm{scal}_{\mathrm{g}} \equiv \kappa$ and $\mathrm{scal}_{\mathrm{h}} \equiv \kappa$ are both constant and equal.

a) Prove that if $\kappa \neq 0$, then either $u \equiv 1$, or there exist points $p, q \in M$ such that $u(p) < 1 < u(q)$.

b) Prove that if $\kappa < 0$, then $\mathrm{h} = \mathrm{g}$. What happens if $\kappa = 0$?

3. Let K^2 be the Klein bottle, and recall that it is double-covered by the 2-torus T^2. Provide either a construction (just a brief outline of the curvature computations is fine) or a topological obstruction (quoting a theorem) as answer to the following questions:

a) Does $K^2 \times S^1$ admit a Riemannian metric with $\sec \leq 0$? How about $\sec < 0$?

b) Does $K^2 \times \mathbb{R}P^2$ admit a Riemannian metric with $\mathrm{Ric} > 0$? How about $\mathrm{Ric} \geq 0$?

c) Does $\mathbb{C}P^n$ admit a Riemannian metric with $\sec \leq 0$? How about $\sec \geq 10$?

d) Does $\mathbb{S}^3/\mathbb{Z}_3 \times \mathbb{S}^3/\mathbb{Z}_5$ admit a metric with $\sec > 0$? How about $\mathrm{scal} \equiv k > 0$?

4. Let (M, g) be a connected compact Riemannian manifold with $\sec_M \geq 0$ and compact boundary $\partial M \neq \emptyset$. Suppose ∂M is convex, i.e., for all $p \in \partial M$, the shape operator $(S_{\vec{n}})_p\colon T_p\partial M \to T_p\partial M$, given by $S_{\vec{n}}(X) = -\nabla_X \vec{n}$, is positive-definite, where $\vec{n}$ is the inward-pointing unit normal of ∂M.

a) Does ∂M have to be connected? If yes, give a proof; if no, give a counter-example.

b) Prove that $\pi_1(M, \partial M) = \{1\}$, hence the inclusion $\partial M \hookrightarrow M$ induces a surjective homomorphism $\pi_1(\partial M) \to \pi_1(M)$. You may use (without proof) the fact that minimizing length in a nontrivial free homotopy class of curves in M with endpoints in ∂M yields a geodesic in M with endpoints in ∂M.

c) Give two examples of the above situation, to show that $\pi_1(\partial M) \to \pi_1(M)$ may or may not be injective.

5. Recall from Problem 6 in the Midterm that if (M, g) is a complete noncompact manifold and $p \in M$, then there exists a unit speed geodesic $\gamma\colon \mathbb{R} \to M$ such that $\gamma(0) = p$ and $\mathrm{dist}(\gamma(t), \gamma(s)) = |t - s|$ for all $t, s \geq 0$. Assume, moreover, that (M^n, g) has $\mathrm{Ric} \geq 0$.

a) Fix $a > 0$, and use Bishop Volume Comparison to show that, for all $t > a$,

$$\frac{\mathrm{Vol}\big(B_{t+a}(\gamma(t))\big)}{\mathrm{Vol}\big(B_{t-a}(\gamma(t))\big)} \leq \frac{(t+a)^n}{(t-a)^n}$$

b) Show that $B_a(p) \subset B_{t+a}(\gamma(t)) \setminus B_{t-a}(\gamma(t))$, and conclude that, given $t > t_0 > a$,

$$\mathrm{Vol}\left(B_{t-a}(\gamma(t))\right) \geq c(n, t_0)\, \mathrm{Vol}(B_a(p))\, t,$$

where $c(n, t_0) = \inf\limits_{t \in [t_0, +\infty)} \frac{1}{t} \frac{(t-a)^n}{(t+a)^n - (t-a)^n} > 0$.

c) Show that $B_{\frac{r+a}{2} - a}\left(\gamma\left(\frac{r+a}{2}\right)\right) \subset B_r(p)$ and conclude that, for all $r > 2t_0 - a$,

$$c\, r \leq \mathrm{Vol}(B_r(p)) \leq C\, r^n,$$

where $c, C > 0$ are constants. In particular, (M^n, g) has infinite volume.

d) For each $1 \leq k \leq n$, give an example of a complete noncompact Riemannian manifold (M^n, g) with $\mathrm{Ric} \geq 0$ for which $\mathrm{Vol}(B_r(p)) = O(r^k)$ as $r \nearrow +\infty$.

6. Consider the unit sphere $\mathbb{S}^5 \subset \mathbb{R}^6$ and let $M = \mathbb{S}^5 \cap (\mathbb{R}^3 \oplus \{0\})$ and $N = \mathbb{S}^5 \cap (\{0\} \oplus \mathbb{R}^3)$, which are isometric copies of the unit sphere $\mathbb{S}^2$ sitting in $\mathbb{S}^5$.

a) Verify that M and N are totally geodesic in $\mathbb{S}^5$.

b) Given a unit vector $x \in \mathbb{R}^3$, identify $T_{(x,0)}M$ with a subspace of $T_{(x,0)}\mathbb{S}^5$, and let $T_{(x,0)}M^\perp \subset T_{(x,0)}\mathbb{S}^5$ be its orthogonal complement. If $v \in T_{(x,0)}M^\perp$ is a unit vector, find an explicit formula for the geodesic $\gamma(t) = \exp_{(x,0)} tv$ on $\mathbb{S}^5 \subset \mathbb{R}^6$.

c) Let $TM^\perp = \bigcup_{x \in M} T_{(x,0)}M^\perp$ be the normal bundle of $M \subset \mathbb{S}^5$, which is a trivial bundle $TM^\perp \cong M \times \mathbb{R}^3$. We write $(x, t, v) \in TM^\perp$, where $t \geq 0$ and $v \in T_{(x,0)}M^\perp$ is a unit vector. Show that the restriction of $f\colon TM^\perp \to \mathbb{S}^5$, $f(x, t, v) = \exp_{(x,0)} tv$, to $\{(x, t, v) \in TM^\perp : 0 \leq t < \frac{\pi}{2}\}$ is a diffeomorphism onto its image $\mathbb{S}^5 \setminus N$.

d) Show that, for each $(x, 0) \in M$, the map $\phi_x\colon \mathbb{S}^2 \subset T_{(x,0)}M^\perp \to N$ given by $\phi_x(v) = f(x, \frac{\pi}{2}, v)$ is an isometry.

e) Show that $M, N \subset \mathbb{S}^5$ are subsets at maximal distance $\mathrm{dist}_{\mathrm{g}}(M, N) = \pi/2$, i.e., the function $M \ni x \mapsto \mathrm{dist}_{\mathrm{g}}(x, N) = \inf\{\mathrm{dist}_{\mathrm{g}}(x, y) : y \in N\}$ is constant and equal to $\pi/2$, and for any subset $P \subset \mathbb{S}^5$, $P \not\subset N$, one has $\mathrm{dist}_{\mathrm{g}}(x, P) < \frac{\pi}{2}$ for some $x \in M$.