1. A particle of mass $m$ and charge $e$ is accelerated for a time by a uniform electric field to a velocity not necessarily small compared with $c$. (i) What is the momentum of the particle at the end of the acceleration time? (ii) What is the velocity of the particle at that time? (iii) The particle is unstable and decays with a lifetime $\tau$ in its rest frame. What lifetime would be measured by a stationary observer who observed the decay of the particle moving uniformly with the above velocity?

2. An electron (mass $m$, charge $e$) moves in a plane perpendicular to a uniform magnetic field. If energy loss by radiation is neglected the orbit is a circle of radius $R$. Let $E$ be the total electron energy, allowing for relativistic kinematics so that $E \gg mc^2$. (i) Express the needed field induction $B$ analytically in terms of the above parameters. Compute numerically, in Gauss, for the case where $R = 30$ m, $E = 2.5 \times 10^9$ eV. For this part of the problem you will have to recall some universal constants. (ii) Actually, the electron radiates electromagnetic energy because it is being accelerated by the $\vec{B}$ field. However, suppose that the energy loss per revolution, $\Delta E$, is small compared with $E$. Express the ration $\Delta E/E$ analytically in terms of the parameters. Then evaluate this ratio numerically for the particular values of $R$ and $E$ given above.

3. Consider an arbitrary plane electromagnetic wave propagating in vacuum in the $x$-direction. Let $A(x-ct)$ be the vector potential of the wave; there are no sources, so adopt a gauge in which the scalar potential is identically zero. Assume that the wave does not extend throughout all spaces, in particular $A = 0$ for sufficiently large values of $x - ct$. The wave strikes a particle with charge $e$ which is initially at rest and accelerates it to a velocity which may be relativistic. (i) Show that $A_x = 0$. (ii) Show that $\vec{p}_\perp = -eA$, where $\vec{p}_\perp$ is the particle momentum in the $yz$ plane.

4. (i) A classical electromagnetic wave satisfies the relations

$$\vec{E} \cdot \vec{B} = 0, \quad \vec{E}^2 = c^2 \vec{B}^2$$

between the electric field and magnetic fields. Show that these relations, if satisfied in any one Lorentz frame, are satisfied in all frames. (ii) If $\hat{K}$ is a unit three-vector in the direction of propagation of the wave, then according to classical electromagnetism, $\hat{K} \cdot \vec{E} = \hat{K} \cdot vecB = 0$. Show that this statement is also invariant under Lorentz transformation by showing its equivalence to the manifestly Lorentz invariant statement $n^\mu F_{\mu\nu} = 0$, where $n^\mu$ is a four vector oriented in the direction of propagation of the wave and $F_{\mu\nu}$ is the field strength tensor. Parts (i) and (ii) together show that what looks like a light wave in one frame looks like one in any frame. (iii) Consider an electromagnetic wave which in some frame has the form

$$E_x = cB_y = f(ct - z),$$

where

$$\lim_{z \to \pm \infty} f(z) = 0.$$
What would be the values of the fields in a different coordinate system moving with velocity $v$ in the $z$ direction relative to the frame in which the fields are given above? Give an expression for the energy momentum densities of the wave in the original frame and in the frame moving with velocity $v$, show that the total energy-momentum of the wave transforms as a four-vector under the transformation between the two frames. (Assume the extent of the wave in the $x−y$ plane os large but finite, so that its total energy and momentum are finite.)

5. (i) A muon at rest lives $10^{-6}$ s, and its mass is 100 MeV$/c^2$. How energetic must a muon be to reach the Earth surface if it is produced high in the atmosphere (say 10$^4$ m up)? (ii) Suppose to a zeroth approximation the Earth has a 1 Gauss magnetic field pointing in the direction of its axis, extending out to 10$^4$ cm. How much, and in what direction, is a muon of energy $E$ normally incident at the equator deflected by the field?

6. Maxwell’s equations of classical electrodynamics are, in vacuo,

$$\vec{\nabla} \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0,$$

$$\vec{\nabla} \cdot \vec{E} = \epsilon \rho,$$

$$\vec{\nabla} \times \vec{B} - \frac{\partial \vec{E}}{\partial t} = \epsilon \vec{j},$$

$$\vec{\nabla} \cdot \vec{B} = 0,$$

where the fields $\vec{E}$ and $\vec{B}$ are the components of a second rank tensor,

$$F^{\mu \nu} = \begin{pmatrix}
0 & -E_x & -E_y & -E_z \\
E_x & 0 & -B_z & B_y \\
E_y & B_z & 0 & -B_x \\
E_z & -B_y & B_x & 0
\end{pmatrix},$$

$j^\mu = (\rho, \vec{j})$ is a fourth-vector, and we are using Heaviside-Lorentz rationalized units. Show that Maxwell’s equations are equivalent to the following covariant equation for $A^\mu$

$$\Box^2 A^\mu - \partial^\mu (\partial_\nu A^\nu) = e j^\mu,$$

where $A^\mu = (\phi, \vec{A})$, the four-vector potential, is related to the electric and magnetic fields by

$$\vec{E} = -\frac{\partial \vec{A}}{\partial t} - \vec{\nabla} \phi, \quad \vec{B} = \vec{\nabla} \times \vec{A}.$$
Maxwell’s equations take the compact form
\[ \partial_{\mu} F^{\mu \nu} = \epsilon j^{\nu} \]
and the current conservation
\[ \partial_{\nu} e j^{\nu} = \epsilon \frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{e} j = 0, \]
follows as a natural compatibility. [Hint: \( \nabla \times (\nabla \times \mathbf{A}) = -\nabla^2 \mathbf{A} + \nabla (\nabla \cdot \mathbf{A}) \).]

7. Consider the static magnetic field given in rectangular coordinates by
\[ \mathbf{B} = B_0 (x \hat{x} - y \hat{y}) / a \]
(i) Show that this field obeys Maxwell’s equations in free space. (ii) Sketch the field lines and indicate where filamentary currents would be placed to approximate such a field. (iii) Calculate the magnetic flux per unit length in the \( \hat{z} \)-direction between the origin and the field line whose minimum distance from the origin is \( R \). (iv) If an observer is moving with a non-relativistic velocity \( \mathbf{v} = v \hat{z} \) at some location \( (x, y) \) what electric potential would he measure with respect to the origin? (v) If the magnetic field \( B_0(t) \) is slowly varying in time, what electric field would the stationary observer at location \( (x, y) \) measure?

8. A beam of relativistic particles with charge \( e > 0 \) is passed successively through two regions, each of length \( l \) which contain uniform magnetic and electric fields \( \mathbf{B} \) and \( \mathbf{E} \), as shown in the figure. The fields are adjusted so that the beam suffers fixed small deflections \( \theta_B \) and \( \theta_E \) \( (\theta_B \ll 1, \theta_E \ll 1) \) in the respective fields. (i) Show that the momentum \( p \) of the particle can be determined in terms of \( B, \theta_B, \) and \( l \). (ii) Show that by using both \( \mathbf{B} \) and \( \mathbf{E} \) fields, one can determine the velocity and mass of the particles on the beam.