1. The hamiltonian, $H = \alpha_i p_i + \beta m$, must be hermitian to give real eigenvalues. Thus, $H = H^\dagger = {p_i}^\dagger {\alpha_i}^\dagger + \beta^\dagger m$. From non-relativistic quantum mechanics we know that $p_i = {p_i}^\dagger$. In addition, $[\alpha_i, p_j] = 0$ because we impose that the $\alpha$, $\beta$ operators act on the spinor indices while $p_i$ act on the coordinates of the wave function itself. With these conditions we may proceed:

$$p_i {\alpha_i}^\dagger + \beta^\dagger m = \alpha_i p_i + \beta m \tag{1}$$

yields

$$ {\alpha_i}^\dagger p_i + \beta^\dagger m = a_i p_i + \beta m \tag{2}$$

${\alpha_i}^\dagger = \alpha_i$, $\beta^\dagger = \beta$. Thus $\alpha$, $\beta$ are hermitian operators.

To verify the other properties of the $\alpha$, and $\beta$ operators we compute

$$\begin{aligned} H^2 &= (\alpha_i p_i + \beta m)(\alpha_j p_j + \beta m) \\ &= \left( {\alpha_i}^2 {p_i}^2 + \frac{1}{2}(\alpha_i\alpha_j + \alpha_j\alpha_i) p_i p_j + (\alpha_i\beta + \beta\alpha_i) m + \beta^2 m^2 \right), \end{aligned} \tag{3}$$

where for the second term $i \neq j$. Then imposing the relativistic energy relation, $E^2 = |p|^2 + m^2$, we obtain the anti-commutation relation

$$\{b_i, b_j\} = 2\delta_{\mathrm{ij}} \mathbb{1}, \tag{4}$$

where $b_0 = \beta$, $b_i = \alpha_i$, and $\mathbb{1} \equiv n \times n$ idenitity matrix.

Using the anti-commutation relation and the fact that ${b_i}^2 = 1$ we are able to find the trace of any $b_i$

$$\mathrm{Tr}(b_i) = \mathrm{Tr}(b_j b_j b_i), \quad (i \neq j) \tag{5}$$

Recall that traces are invariant under permutations of the matrices and hence

$$\mathrm{Tr}(b_i) = \mathrm{Tr}(b_j b_i b_j) = \mathrm{Tr}(b_j(-b_j b_i)) = -\mathrm{Tr}(b_i). \tag{6}$$

which implies $\mathrm{Tr}(b_i) = 0$. In the second step of Eq. (6 we made use of $b_i b_j = -b_j b_i, i \neq j$

Now, using the anti-commutation relations, with the determinate leads to

$$|b_i b_j| = |-b_j b_i| \rightarrow |b_i||b_j| = (-1)^N |b_i||b_j| \tag{7}$$

where $N$ is the dimension of the $b_i$ matrix and $i \neq j$. Then $1 = (-1)^N$ which implies that $N$ is an even number.

Finally, making use of ${b_i}^2 = 1$ we can find the eigenvalues: $b_i u = \lambda u$ and $b_i b_i u = \lambda b_i u$, hence $u = \lambda^2 u$ and $\lambda = \pm 1$.

2. We need to find the $\lambda = +1/2$ helicity eigenspinor for an electron with momentum $\vec{p}^{\,'} = (p\sin\theta,\ 0,\ p\cos\theta)$. The helicity operator is given by $\frac{1}{2}\sigma \cdot \hat{p}$, and the positive eigenvalue $\frac{1}{2}$ corresponds to $u_1$ Dirac solution [see (1.5.98) in `http://arXiv.org/abs/0906.1271`]. Computing $\sigma \cdot \hat{p}$ we find that

$$\begin{aligned} \boldsymbol{\sigma} \cdot \hat{p} &= \sigma_1 \sin\theta + \sigma_2 \cos\theta \\ &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \sin\theta + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \cos\theta \\ &= \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}. \end{aligned} \tag{8}$$

This tells us that $\vec{p}^{\,'} = (p\sin\theta,\ 0,\ p\cos\theta)$ is obtained from $\vec{p} = (0,\ 0,\ p)$ by a rotation around the $y$-axis. A rotation through a finite angle $\theta$ around the $y$-axis has associated the unitary transformation $\cos(\theta/2) - i\sigma_2 \sin(\theta/2)$ [see (1.3.21) in `http://arXiv.org/abs/0906.1271`], so we can obtain the eigenspinor $u(\vec{p}^{\,'})$ by applying this unitary transformation to the eigenspinor $u_1(\vec{p})$

$$u(\vec{p}^{\,'}) = \sqrt{E+m} \left[ \begin{array}{c:c} \cos\left(\frac{\theta}{2}\right) - i\sigma_2 \sin\left(\frac{\theta}{2}\right) & 0 \\ \hdashline 0 & \cos\left(\frac{\theta}{2}\right) - i\sigma_2 \sin\left(\frac{\theta}{2}\right) \end{array} \right] \left[ \begin{array}{c} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ \hdashline \frac{p}{E+m} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \end{array} \right]. \tag{9}$$

This is the $\lambda = +1/2$ eigenspinor of the electron with momentum $\vec{p}^{\,'}$.

3. *(a) The Commutation of $H$ with $L$.* First, we check $[H, \mathbf{L}]$, where

$$H = \boldsymbol{\alpha} \cdot \mathbf{P} + \boldsymbol{\beta} m, \tag{10}$$

where

$$\vec{\alpha} \equiv \boldsymbol{\alpha} = \begin{pmatrix} 0 & \boldsymbol{\sigma} \\ \boldsymbol{\sigma} & 0 \end{pmatrix}, \qquad \boldsymbol{\beta} = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}, \tag{11}$$

and

$$\mathbf{L} = \mathbf{r} \times \mathbf{P} = (r_2 P_3 - r_3 P_2)\,\hat{\boldsymbol{x}} + (r_3 P_1 - r_1 P_3)\,\hat{\boldsymbol{y}} + (r_1 P_2 - r_2 P_1)\,\hat{\boldsymbol{z}}, \tag{12}$$

$[H, \mathbf{L}]$ can determined by evaluating explicit components:

$$\begin{aligned}
[H, L_1] &= [H, r_2 P_3 - r_3 P_2] \\
&= [\boldsymbol{\alpha} \cdot \mathbf{P} + \boldsymbol{\beta} m, r_2 P_3 - r_3 P_2] \\
&= [\boldsymbol{\alpha} \cdot \mathbf{P}, r_2 P_3 - r_3 P_2] + [\boldsymbol{\beta} m, r_2 P_3 - r_3 P_2] \\
&= [\boldsymbol{\alpha} \cdot \mathbf{P}, r_2 P_3 - r_3 P_2] \\
&= [\boldsymbol{\alpha} \cdot \mathbf{P}, r_2 P_3] - [\boldsymbol{\alpha} \cdot \mathbf{P}, r_3 P_2] \\
&= [\alpha_1 P_1 + \alpha_2 P_2 + \alpha_3 P_3, r_2 P_3] - [\alpha_1 P_1 + \alpha_2 P_2 + \alpha_3 P_3, r_2 P_3, r_3 P_2] \\
&= [\alpha_1 P_1, r_2 P_3] + [\alpha_2 P_2, r_2 P_3] + [\alpha_3 P_3, r_2 P_3] - [\alpha_1 P_1, r_3 P_2] - [\alpha_2 P_2, r_3 P_2] - [\alpha_3 P_3, r_3 P_2] \\
&= [\alpha_1 P_1, r_2]\, P_3 + r_2\, [\alpha_1 P_1, P_3] + [\alpha_2 P_2, r_2]\, P_3 + r_2\, [\alpha_2 P_2, P_3] + [\alpha_3 P_3, r_2]\, P_3 + r_2\, [\alpha_3 P_3, P_3] \\
&- [\alpha_1 P_1, r_3]\, P_2 - r_3\, [\alpha_1 P_1, P_2] - [\alpha_2 P_2, r_3]\, P_2 - r_3\, [\alpha_2 P_2, P_2] - [\alpha_3 P_3, r_3]\, P_2 - r_3\, [\alpha_3 P_3, P_2] \\
&= 0 + 0 - i\alpha_2 P_3 + 0 + 0 + 0 - 0 - 0 - 0 - 0 + i\alpha_3 P_2 - 0 \\
&= -i\,(\alpha_2 P_3 - \alpha_3 P_2)\,.
\end{aligned} \tag{13}$$

Two techniques were employed along the way: between the 7$^{\text{th}}$ and the 8$^{\text{th}}$ line of Eq. (13) we used $[A, BC]$=$[A, B]C + B[A, C]$, and between the 8$^{\text{th}}$ and the 9$^{\text{th}}$ we used $[P_j, x_i]$= $-[x_i, P_j]$= $-i\delta_{ij}$. The same strategy can be adopted to solve for $[H, L_2]$ and $[H, L_3]$:

$$[H, L_2] = i\,(\alpha_1 P_3 - \alpha_3 P_1) \tag{14}$$

$$[H, L_3] = -i\,(\alpha_1 P_2 - \alpha_2 P_1)\,. \tag{15}$$

Thus, $[H, \mathbf{L}]$ can be determined to be:

$$\begin{aligned}
[H, \mathbf{L}] &= [H, L_1]\,\hat{\boldsymbol{x}} + [H, L_2]\,\hat{\boldsymbol{y}} + [H, L_3]\,\hat{\boldsymbol{z}} \\
&= -i\,(\alpha_2 P_3 - \alpha_3 P_2)\,\hat{\boldsymbol{x}} + i\,(\alpha_1 P_3 - \alpha_3 P_1)\,\hat{\boldsymbol{y}} + -i\,(\alpha_1 P_2 - \alpha_2 P_1)\,\hat{\boldsymbol{z}} \\
&= -i\,(\boldsymbol{\alpha} \times \mathbf{P})\,.
\end{aligned} \tag{16}$$

So $\mathbf{L}$ is not conserved.

*(b) The Commutation of $H$ with $\boldsymbol{\Sigma}$.* Next, we check $[H, \boldsymbol{\Sigma}]$, where

$$\boldsymbol{\Sigma} = \frac{1}{2} \begin{pmatrix} \boldsymbol{\sigma} & 0 \\ 0 & \boldsymbol{\sigma} \end{pmatrix}. \tag{17}$$

Again, $[H,\boldsymbol{\Sigma}]$ can be determined via matrix multiplication and evaluation of explicit components:

$$\begin{aligned}
[H,\Sigma_1] &= [\boldsymbol{\alpha}\cdot\mathbf{P}+\boldsymbol{\beta}m,\Sigma_1] \\
&= [\boldsymbol{\alpha}\cdot\mathbf{P},\Sigma_1]+[\boldsymbol{\beta}m,\Sigma_1] \\
&= [\boldsymbol{\alpha}\cdot\mathbf{P},\Sigma_1] \\
&= (\boldsymbol{\alpha}\cdot\mathbf{P})\,\Sigma_1-\Sigma_1\,(\boldsymbol{\alpha}\cdot\mathbf{P}) \\
&= \frac{1}{2}\begin{pmatrix} 0 & \boldsymbol{\sigma}\cdot\mathbf{P} \\ \boldsymbol{\sigma}\cdot\mathbf{P} & 0 \end{pmatrix}\begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_1 \end{pmatrix}-\frac{1}{2}\begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_1 \end{pmatrix}\begin{pmatrix} 0 & \boldsymbol{\sigma}\cdot\mathbf{P} \\ \boldsymbol{\sigma}\cdot\mathbf{P} & 0 \end{pmatrix} \\
&= \frac{1}{2}\begin{pmatrix} 0 & (\boldsymbol{\sigma}\cdot\mathbf{P})\,\sigma_1 \\ (\boldsymbol{\sigma}\cdot\mathbf{P})\,\sigma_1 & 0 \end{pmatrix}-\frac{1}{2}\begin{pmatrix} 0 & \sigma_1\,(\boldsymbol{\sigma}\cdot\mathbf{P}) \\ \sigma_1\,(\boldsymbol{\sigma}\cdot\mathbf{P}) & 0 \end{pmatrix} \\
&= \frac{1}{2}\begin{pmatrix} 0 & [\boldsymbol{\sigma}\cdot\mathbf{P},\sigma_1] \\ [\boldsymbol{\sigma}\cdot\mathbf{P},\sigma_1] & 0 \end{pmatrix}.
\end{aligned} \tag{18}$$

Remember that

$$\sigma_1=\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_2=\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma_3=\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{19}$$

To proceed, multiply out $[\boldsymbol{\sigma}\cdot\mathbf{P},\sigma_1]$:

$$\begin{aligned}
[\boldsymbol{\sigma}\cdot\mathbf{P},\sigma_1] &= (\boldsymbol{\sigma}\cdot\mathbf{P})\,\sigma_1-\sigma_1\,(\boldsymbol{\sigma}\cdot\mathbf{P}) \\
&= (\sigma_1P_1+\sigma_2P_2+\sigma_3P_3)\,\sigma_1-\sigma_1\,(\sigma_1P_1+\sigma_2P_2+\sigma_3P_3) \\
&= \sigma_1P_1\sigma_1+\sigma_2P_2\sigma_1+\sigma_3P_3\sigma_1-\sigma_1\sigma_1P_1-\sigma_1\sigma_2P_2-\sigma_1\sigma_3P_3 \\
&= P_2\,(\sigma_2\sigma_1-\sigma_1\sigma_2)+P_3\,(\sigma_3\sigma_1-\sigma_1\sigma_3) \\
&= P_2\left[\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}-\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\right]+P_3\left[\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}-\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right] \\
&= P_2\left[\begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}-\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}\right]+P_3\left[\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}-\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\right] \\
&= P_2\begin{pmatrix} -2i & 0 \\ 0 & 2i \end{pmatrix}+P_3\begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix} \\
&= 2P_2\begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}-2iP_3\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \\
&= -2iP_2\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}+2iP_3\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \\
&= 2i\,(P_3\sigma_2-P_2\sigma_3).
\end{aligned} \tag{20}$$

Therefore,

$$\begin{aligned}
[H,\Sigma_1] &= \frac{1}{2}\begin{pmatrix} 0 & [\boldsymbol{\sigma}\cdot\mathbf{P},\sigma_1] \\ [\boldsymbol{\sigma}\cdot\mathbf{P},\sigma_1] & 0 \end{pmatrix} \\
&= \frac{1}{2}\begin{pmatrix} 0 & 2i\,(P_3\sigma_2-P_2\sigma_3) \\ 2i\,(P_3\sigma_2-P_2\sigma_3) & 0 \end{pmatrix} \\
&= i\begin{pmatrix} 0 & P_3\sigma_2 \\ P_3\sigma_2 & 0 \end{pmatrix}-i\begin{pmatrix} 0 & P_2\sigma_3 \\ P_2\sigma_3 & 0 \end{pmatrix} \\
&= iP_3\begin{pmatrix} 0 & \sigma_2 \\ \sigma_2 & 0 \end{pmatrix}-iP_2\begin{pmatrix} 0 & \sigma_3 \\ \sigma_3 & 0 \end{pmatrix} \\
&= i\,(P_3\alpha_2-P_2\alpha_3) \\
&= i\,(\alpha_2P_3-\alpha_3P_2).
\end{aligned} \tag{21}$$

The same strategy can be used to solve for $[H,\Sigma_2]$ and $[H,\Sigma_3]$:

$$[H,\Sigma_2] = -i\,(\alpha_1P_3-\alpha_3P_1) \tag{22}$$

$$[H,\Sigma_3] = i\,(\alpha_1P_2-\alpha_2P_1). \tag{23}$$

Thus, $[H,\mathbf{L}]$ can be determined to be:

$$\begin{aligned}[H,\boldsymbol{\Sigma}] &= [H,\Sigma_1]\,\hat{\boldsymbol{x}} + [H,\Sigma_2]\,\hat{\boldsymbol{y}} + [H,\Sigma_3]\,\hat{\boldsymbol{z}} \\ &= i\,(\alpha_2 P_3 - \alpha_3 P_2)\,\hat{\boldsymbol{x}} - i\,(\alpha_1 P_3 - \alpha_3 P_1)\,\hat{\boldsymbol{y}} + i\,(\alpha_1 P_2 - \alpha_2 P_1)\,\hat{\boldsymbol{z}} \\ &= i\,(\boldsymbol{\alpha}\times\mathbf{P})\,. \end{aligned} \tag{24}$$

So $\boldsymbol{\Sigma}$ is not conserved.

*(c) The Commutation of $H$ with $J$.* While $\mathbf{L}$ and $\boldsymbol{\Sigma}$ are not conserved, it is easy to see that the combination $\mathbf{L}+\boldsymbol{\Sigma}$ is conserved, namely

$$\begin{aligned}[H,\mathbf{L}+\boldsymbol{\Sigma}] &= [H,\mathbf{L}] + [H,\boldsymbol{\Sigma}] \\ &= -i\,(\boldsymbol{\alpha}\times\mathbf{P}) + i\,(\boldsymbol{\alpha}\times\mathbf{P}) \\ &= 0. \end{aligned} \tag{25}$$

Thus, Dirac equation provides a description of "intrinsic" angular momentum ($\equiv$ spin)-$\frac{1}{2}$ elementary particles.

4. *(a) Handedness.* To show that handedness is not a good quantum number, show that $\gamma^5$ does not commute with the Hamiltonian. Recall, the Hamiltonian is given by $H = (\vec{\alpha}\cdot\vec{p}) + \beta m$, where $\vec{\alpha}$ and $\beta$ are given in Eq. (11), and handedness is given by

$$\gamma^5 = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}. \tag{26}$$

The commutator of $H$ and $\gamma^5$ is

$$\begin{aligned}\left[H,\gamma^5\right] &= H\gamma^5 - \gamma^5 H \\ &= (\vec{\alpha}\cdot\vec{p} + \beta m)\,\gamma^5 - \gamma^5\,(\vec{\alpha}\cdot\vec{p} + \beta m) \\ &= \begin{pmatrix} 0 & \boldsymbol{\sigma}\cdot\vec{p} \\ \boldsymbol{\sigma}\cdot\vec{p} & 0 \end{pmatrix}\begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} + m\begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}\begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} - \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}\begin{pmatrix} 0 & \boldsymbol{\sigma}\cdot\vec{p} \\ \boldsymbol{\sigma}\cdot\vec{p} & 0 \end{pmatrix} - m\begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}\begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix} \\ &= \begin{pmatrix} \boldsymbol{\sigma}\cdot\vec{p} & 0 \\ 0 & \boldsymbol{\sigma}\cdot\vec{p} \end{pmatrix} + m\begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix} - \begin{pmatrix} \boldsymbol{\sigma}\cdot\vec{p} & 0 \\ 0 & \boldsymbol{\sigma}\cdot\vec{p} \end{pmatrix} - m\begin{pmatrix} 0 & -\mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} \\ &= m\begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix}. \end{aligned} \tag{27}$$

Therefore, if $m \neq 0$, handedness does not commute with the Hamiltonian, which means handedness cannot be a good quantum number for massive particles. It's also useful to note the following relationship which shows that handedness does not commute with the Hamiltonian,

$$\left[\vec{\alpha}\cdot\vec{p} + \vec{\beta}m, \gamma^5\right] = \left[\gamma^5, \vec{\alpha}\cdot\vec{p} - \vec{\beta}m\right]. \tag{28}$$

*(b)Helicity.* Now let's show that helicity, $h$, does commute with the Hamiltonian, which means that helicity is a conserved quantity. Recall the definition of helicity, $h = \hat{p}\cdot\vec{\Sigma}$. Choosing the $x_3$-axis along the direction of momentum $\vec{p}$, helicity reduces to

$$h = \frac{1}{2}\begin{pmatrix} \sigma_3 & 0 \\ 0 & \sigma_3 \end{pmatrix}. \tag{29}$$

Now to compute the commutator of $H$ and $h$,

$$\begin{aligned}[H,h] &= \frac{1}{2}(\vec{\alpha}\cdot\vec{p} + \beta m)\begin{pmatrix} \sigma_3 & 0 \\ 0 & \sigma_3 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} \sigma_3 & 0 \\ 0 & \sigma_3 \end{pmatrix}(\vec{\alpha}\cdot\vec{p} + \beta m) \\ &= \frac{1}{2}\begin{pmatrix} 0 & \sigma_3 p_3 \\ \sigma_3 p_3 & 0 \end{pmatrix}\begin{pmatrix} \sigma_3 & 0 \\ 0 & \sigma_3 \end{pmatrix} + \frac{m}{2}\begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}\begin{pmatrix} \sigma_3 & 0 \\ 0 & \sigma_3 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} \sigma_3 & 0 \\ 0 & \sigma_3 \end{pmatrix}\begin{pmatrix} 0 & \sigma_3 p_3 \\ \sigma_3 p_3 & 0 \end{pmatrix} \\ &- \frac{m}{2}\begin{pmatrix} \sigma_3 & 0 \\ 0 & \sigma_3 \end{pmatrix}\begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix} \\ &= \frac{1}{2}\begin{pmatrix} 0 & \sigma_3 p_3 \sigma_3 \\ \sigma_3 p_3 \sigma_3 & 0 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} 0 & \sigma_3\sigma_3 p_3 \\ \sigma_3\sigma_3 p_3 & 0 \end{pmatrix}. \end{aligned} \tag{30}$$

If $\sigma_3 p_3 \sigma_3 = \sigma_3 \sigma_3 p_3$, then helicity commutes with the Hamiltonian, and therefore helicity is a conserved quantity. Let's verify that $\sigma_3 p_3 \sigma_3 = \sigma_3 \sigma_3 p_3$,

$$\begin{aligned} \sigma_3 p_3 \sigma_3 &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} p_3 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\ &= \begin{pmatrix} p_3 & 0 \\ 0 & -p_3 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\ &= \begin{pmatrix} p_3 & 0 \\ 0 & p_3 \end{pmatrix} \end{aligned} \tag{31}$$

and

$$\begin{aligned} \sigma_3 \sigma_3 p_3 &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} p_3 \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} p_3 \\ &= \begin{pmatrix} p_3 & 0 \\ 0 & p_3 \end{pmatrix}. \end{aligned} \tag{32}$$

Finally, to show that helicity is frame dependent return to the original definition of helicity,

$$h = \hat{p} \cdot \vec{\Sigma}. \tag{33}$$

Since $\hat{p}$ is frame dependent and $\vec{\Sigma}$ is not, the net result is that $h$ is frame dependent. When the particle is overtaken, $\hat{p}' = -\hat{p}$, making

$$h' = -\hat{p} \cdot \vec{\Sigma}. \tag{34}$$

In other words, helicity is reversed by overtaking the particle.