Math 176

Workbook

Author: Yiannis Petridis

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Homework Assignments

1. Verify the statement by showing that the derivative of the right side is equal to the integrand of the left side.

$$\int \left(4x^3 - \frac{1}{x^2}\right) dx = x^4 + \frac{1}{x} + C, \quad \int \frac{x^2-1}{x^{3/2}}\, dx = \frac{2(x^2+3)}{3\sqrt{x}} + C.$$

2. Evaluate the indefinite integrals and check the result by differentiation.

$$\int x(x^2+3)\, dx, \quad \int \frac{x^2+x+1}{\sqrt{x}}\, dx, \quad \int (2\sin x - 5e^x)\, dx.$$

3. A baseball is thrown upward from ground level with a velocity of 10 meters per second. Determine its maximum height.

4. You are driving along the highway, when you notice a skunk 400 feet ahead. You immediately apply the brakes, your car starts to slow down and stops in 10 seconds. You speed decreases while the car slows down, although not at a uniform rate. In the table you are recording your speed in feet per second every two seconds after you brake.

t sec	speed
0	100
2	80
4	50
6	25
8	10
10	0

Give an overestimate and an underestimate of the distance that your car traveled before it stopped.

Decide whether the car stopped before getting to the skunk, whether the skunk was hit by the car, or whether the data is inconclusive.

5. Sketch the region whose area is given by each of the following definite integrals. Then use geometric reasoning to evaluate the integral.

(a) $\int_0^5 (5-x)\, dx$, (b) $\int_0^2 (2x+5)\, dx$, (c) $\int_{-3}^3 \sqrt{9-x^2}\, dx$, (d) $\int_{3/2}^3 \sqrt{9-x^2}\, dx$.

6. Consider the integral $\int_1^3 x\, dx$. Show that the following two formulas are true for the left-hand sum and the right-hand sum with n (equal) subintervals:

$$\text{LHS}(n) = \sum_{i=0}^{n-1} \left(1 + \frac{2i}{n}\right) \frac{2}{n}, \quad \text{RHS}(n) = \sum_{i=1}^{n} \left(1 + \frac{2i}{n}\right) \frac{2}{n}$$

Compute simplified expressions for these using the formula $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$. Show that

$$\lim_{n\to\infty} \text{LHS}(n) = \lim_{n\to\infty} \text{RHS}(n) = 4.$$

7. Use the Fundamental Theorem of Calculus to evaluate the definite integrals.

$$\text{(a)} \int_{-1}^{0} (x-1)\,dx, \quad \text{(b)} \int_{-1}^{1} (t^2-2)\,dt, \quad \text{(c)} \int_{-2}^{-1} (u-u^{-1})\,du,$$

$$\text{(d)} \int_{0}^{\pi} (1+\sin x)\,dx, \quad \text{(e)} \int_{-\pi/3}^{\pi/3} 4\sec\theta\tan\theta\,d\theta.$$

8. Integrate to find F as a function of x and demonstrate the Second Fundamental Theorem of Calculus by differentiating $F(x)$.

$$\text{(a) } F(x) = \int_{8}^{x} \sqrt[3]{t}\,dt, \quad \text{(b) } F(x) = \int_{-1}^{x} e^t\,dt.$$

9. Use the Second Fundamental Theorem of Calculus to find $F'(x)$. Remark: With your current knowledge you cannot find $F(x)$.

$$\text{(a) } F(x) = \int_{0}^{x} t\cos t\,dt, \quad \text{(b) } F(x) = \int_{1}^{x} \frac{t^4}{t^2+1}\,dt.$$

10. Evaluate the integrals.

$$\text{(a)} \int x^3\cos(x^4+2)\,dx, \quad \text{(b)} \int \frac{x}{\sqrt{1-4x^2}}\,dx, \quad \text{(c)} \int_{1}^{e} \frac{\ln x}{x}\,dx,$$

$$\text{(d)} \int \frac{\cos(\pi/x)}{x^2}\,dx, \quad \text{(e)} \int_{e}^{e^4} \frac{dx}{x\sqrt{\ln x}}, \quad \text{(f)} \int \frac{(1+\sqrt{x})^9}{\sqrt{x}}\,dx,$$

$$\text{(g)} \int \frac{\arctan x}{1+x^2}\,dx, \quad \text{(h)} \int \cos x\cos(\sin x)\,dx, \quad \text{(i)} \int \frac{\sin x}{1+\cos^2 x}\,dx,$$

$$\text{(j)} \int \sec^3 x\tan x\,dx, \quad \text{(k)} \int \frac{1+4x}{\sqrt{1+x+2x^2}}\,dx, \quad \text{(l)} \int \tan x\ln(\cos x)\,dx,$$

$$\text{(m)} \int \frac{x+3}{x^2+4}\,dx, \quad \text{(n)} \int \frac{5x+9}{x^2+2x+3}\,dx, \quad \text{(o)} \int \frac{x}{x^4+16}\,dx.$$

11. Find the area bounded between the line $y = 2x$ and the parabola $y = x^2 - 4x$. Sketch the region of integration.

12. Find the area bounded between the line $y = x - 1$ and the parabola $y^2 = 2x + 6$. Sketch the region of integration.

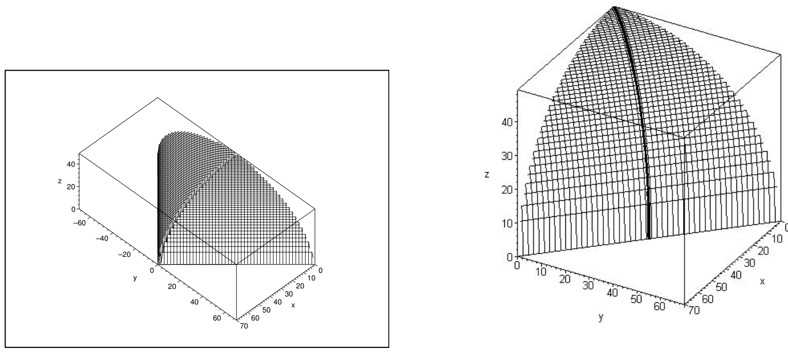


Figure 1: The bent pyramid and one side of it

13. The archeologist Flinders Petrie (1853–1942) visited the Bent Pyramid in Dahshur, Egypt. The pyramid looks as in the figure: the horizontal slices are squares. The equation of the intersection of the two slanted sides is

$$y = \frac{1}{\sqrt{2}}\sqrt{70^2 - x^2}.$$

Find the volume of the Bent Pyramid.

Hint: Write first an approximation for the volume of a horizontal slice, then a Riemann sum and finally convert to an integral.

14. Humpty Dumpty decided to eat a donut. We get this donut by rotating around the

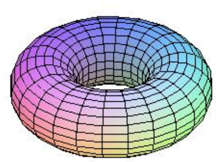


Figure 2: The donut

y-axis the circle with equation

$$(x-2)^2 + y^2 = 1.$$

Find the volume of the donut using both the method of cylindrical shells and the method of washers.

Hint: Write approximations of the volume of the slices (shells or washers), then write Riemann sums and finally convert to an integral.

15. When the ellipse $\frac{x^2}{4} + y^2 = 1$ is rotated around the x-axis we get a solid called ellipsoid. Compute its volume.

16. An oil slick from a tanker lies on the surface of the ocean. Its density at a distance r feet from the center of it, where the leak occured, is given by

$$\rho(r) = \frac{50}{1+r} \text{ kg/feet}^2.$$

If the slick extends to a distance of 10000 feet, write a Riemann sum that approximates the total mass of oil in the slick. Then convert the sum into an integral and find the exact value of the mass of oil.

17. Find the arclength of the curve $y = (e^x + e^{-x})/2$ from $x = 0$ to $x = 1$.

18. Evaluate the integrals. Do not use the integral tables, unless you are verifying your results.

$$\text{(a)} \int x \sin(6x)\,dx, \quad \text{(b)} \int x^2 e^x\,dx, \quad \text{(c)} \int \frac{x}{e^x}\,dx, \quad \text{(d)} \int \ln(3x)\,dx,$$

$$\text{(e)} \int x^2 \cos x\,dx, \quad \text{(f)} \int \theta \sec\theta \tan\theta\,d\theta, \quad \text{(g)} \int \arctan x\,dx,$$

$$\text{(h)} \int e^{2x} \sin x\,dx \quad \text{(choose } u \text{ in two ways)}, \quad \text{(i)} \int t\ln(t+1)\,dt, \quad \text{(j)} \int_0^1 \ln(1+x^2)\,dx,$$

$$\text{(k)} \int \frac{x^3}{\sqrt{4+x^2}}\,dx \quad \text{(use a substitution and do it also by parts)}.$$

19. Use the integral tables to evaluate

$$\text{(a)} \int \frac{1}{y^2+4y+5}\,dy, \quad \text{(b)} \int \frac{1}{\sin^2(2\theta)}\,d\theta, \quad \text{(c)} \int (2x^3+3x+4)\cos(2x)\,dx,$$

$$\text{(d)} \int \frac{dz}{z^2+z}, \quad \text{(e)} \int ze^{2z^2}\cos(2z^2)\,dz, \quad \text{(f)} \int \tan^4 x\,dx,$$

$$\text{(g)} \int x^4 e^{3x}\,dx, \quad \text{(h)} \int \frac{x^2+1}{x^2-3x+2}\,dx, \quad \text{(i)} \int \frac{5x+6}{x^2+4}\,dx.$$

20. Evaluate the integrals. Do not use the integral tables, unless you are verifying your results.

$$\text{(a)} \int (x^3-1)^4 x^2\,dx, \quad \text{(b)} \int xe^{x^2+1}dx, \quad \text{(c)} \int x^2 e^{2x}\,dx, \quad \text{(d)} \int x\sqrt{1-x}\,dx,$$

$$\text{(e)} \int (\ln x)^2\,dx, \quad \text{(f)} \int \ln(x^2)\,dx, \quad \text{(g)} \int \arcsin x\,dx, \quad \text{(h)} \int \frac{(2x-1)e^{x^2}}{e^x}\,dx,$$

(i) $\int x \arcsin x\,dx$ (use the substitution $u = \arcsin x$ and the trig. identity $2\sin u \cos u = \sin(2u)$)

21. Evaluate the integrals. Use the integral tables when necessary.

(a) $\int \cos^3(2x)\sin(2x)\,dx$ (b) $\int \sin^3 t \cos^3 t\,dt$, (c) $\int \sin^2 t\cos^3 t\,dt$, (d) $\int \sin^2 t \cos^4 t\,dt$.

22. Find the pattern of the following sequences, then write a formula for the n-term of the sequence and compute the limit $\lim a_n$.

$$1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots$$

$$1, \frac{1}{2}, \frac{1}{6}, \frac{1}{24}, \frac{1}{120}, \ldots$$

$$\frac{1}{4}, \frac{4}{9}, \frac{9}{16}, \frac{16}{25}, \frac{25}{36}, \ldots$$

23. Determine whether the following series converge or diverge. If they converge, find their limit.

$$\sum_{n=0}^{\infty} 1000\cdot(1.055)^n, \quad \sum_{n=0}^{\infty} 0.9^n, \quad \sum_{n=1}^{\infty} \frac{6}{n(n+3)}.$$

24. Use the integral test to decide whether the following series converge or diverge.

$$\sum_{n=2}^{\infty} \frac{1}{n\ln n}, \quad \sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}.$$

25. Find the second degree Taylor polynomial of $\sec x$ centered at $\pi/4$.

26. Find the power series expansion of

$$f(x) = \int_0^x \frac{\sin t}{t}\,dt$$

centered at 0.

Solutions

1.

$$\left(x^4+\frac{1}{x}+c\right)'=(x^4)'+(x^{-1})'+C'=4x^3-1x^{-2}+0=4x^3-\frac{1}{x^2}$$

$$\left(\frac{2(x^2+3)}{3\sqrt{x}}\right)'=\left(\frac{2x^2}{3\sqrt{x}}+\frac{6}{3\sqrt{x}}\right)'=\left(\frac{2}{3}x^{3/2}+2x^{-1/2}\right)'$$
$$=\frac{2}{3}\frac{3}{2}x^{1/2}-2\frac{1}{2}x^{-3/2}=x^{1/2}-x^{-3/2}=\frac{x^2-1}{x^{3/2}}.$$

2.

$$\int x(x^2+3)\,dx=\int x^3+3x\,dx=\frac{x^4}{4}+3\frac{x^2}{2}+C$$

$$\int\frac{x^2+x+1}{\sqrt{x}}\,dx=\int\frac{x^2}{\sqrt{x}}+\frac{x}{\sqrt{x}}+\frac{1}{\sqrt{x}}\,dx=\int x^{3/2}+x^{1/2}+x^{-1/2}\,dx$$
$$=\frac{x^{5/2}}{5/2}+\frac{x^{3/2}}{3/2}+\frac{x^{1/2}}{1/2}+C=\frac{2}{5}x^{5/2}+\frac{2}{3}x^{3/2}+2x^{1/2}+C.$$

$$\int(2\sin x-5e^x)\,dx=2\int\sin x\,dx-5\int e^x\,dx=2(-\cos x)-5e^x+C.$$

We check the answers by differentiation:

$$\left(\frac{x^4}{4}+3\frac{x^2}{2}+C\right)'=\frac{(x^4)'}{4}+3\frac{(x^2)'}{2}+C'=\frac{4x^3}{4}+3\frac{2x}{2}+0=x^3+3x=x(x^2+3).$$

$$\left(\frac{2}{5}x^{5/2}+\frac{2}{3}x^{3/2}+2x^{1/2}+C\right)'=\frac{2}{5}(x^{5/2})'+\frac{2}{3}(x^{3/2})'+2\frac{1}{2}x^{-1/2}+C'=\frac{2}{5}\frac{5}{2}x^{3/2}+\frac{2}{3}\frac{3}{2}x^{1/2}+x^{-1/2}$$
$$=x^{3/2}+x^{1/2}+x^{-1/2}=\frac{x^2+x+1}{x^{1/2}}$$

$$(-2\cos x-5e^x+C)'=-2(\cos x)'-5(e^x)'+C'=-2(-\sin x)-5e^x=2\sin x-5e^x.$$

3. The acceleration of gravity is -10m/sec^2.

$$a(t)=-10\Longrightarrow v(t)=\int a(t)\,dt=\int -10\,dt=-10t+c,$$

We plug the initial velocity 10m/sec. We get $v(0)=10=-10\cdot 0+c=c$. So $c=10$ and $v(t)=-10t+10$.

$$v(t)=-10t+10\Longrightarrow s(t)=\int v(t)\,dt=\int(-10t+10)\,dt=-5t^2+10t+c.$$

We plug the initial height $s(0)=0$, since the ball is thrown from ground level. We get $s(0)=0=-5\cdot 0^2+10\cdot 0+c=c$. So $c=0$ and $s(t)=-5t^2+10t$. The maximum height occurs when the velocity is 0.

$$v(t)=0\Longrightarrow -10t+10=0\Longrightarrow t=1.$$

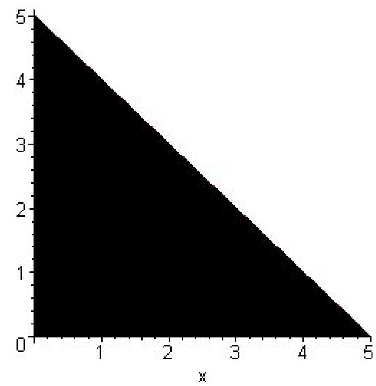


Figure 3: Graph for (a)

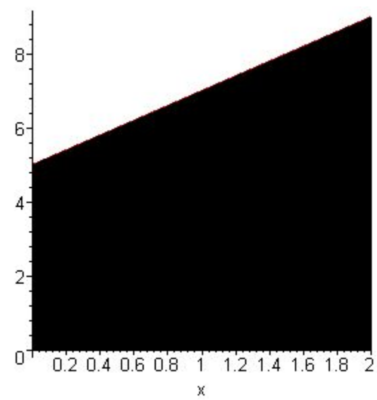


Figure 4: Graph for (b)

We plug the time when the ball reaches the maximum height into the height function to get the maximum height:

$$s(1) = -5 \cdot 1^2 + 10 \cdot 1 = -5 + 10 = 5$$

meters.

4.

$$\text{LHS}(5) = 100 \cdot 2 + 80 \cdot 2 + 50 \cdot 2 + 25 \cdot 2 + 10 \cdot 2 = 530\text{ft}.$$

$$\text{RHS}(5) = 80 \cdot 2 + 50 \cdot 2 + 25 \cdot 2 + 10 \cdot 2 + 0 \cdot 2 = 330\text{ft}.$$

The actual distance travelled is between the left-hand sum and the right-hand sum. So the car travels between 330 and 530 feet. If the skunk is 400 feet away, the results are inconclusive.

5. (a) The integral represents the area of a right-triangle with base 5 and height 5. The area is $5 \cdot 5/2 = 25/2$. So

$$\int_0^5 (5 - x)\, dx = \frac{25}{2}.$$

(b) The integral represents the area of a trapezoid. Its parallel sides have lengths 5 and 9. They are 2 units apart. The area of a trapezoid with bases b and B and height

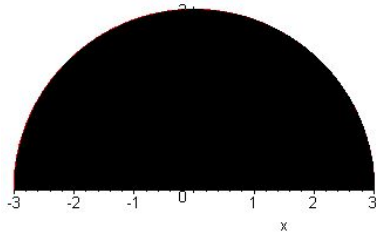


Figure 5: Graph for (c)

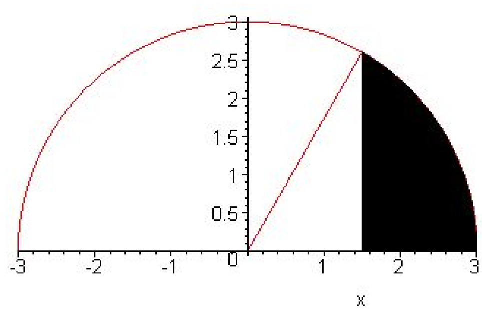


Figure 6: Graph for (d)

h is $A = (b+B)h/2$. This gives

$$\int_0^2 (2x+5)\,dx = \frac{5+9}{2}\cdot 2 = 14.$$

Another way is to split the area into a rectangle and a triangle on top of it by drawing the horizontal line at height 5. The rectangle had dimensions 2 and 5 and area 10. The right-triangle has base 2 and height $9-5=4$. Its area is $2\cdot 4/2 = 4$. The total area is $10+4=14$.

(c) We notice that the function to integrate has graph a half-circle of radius 3:

$$y = \sqrt{9-x^2} \Longrightarrow y^2 = 9-x^2 \Longrightarrow x^2+y^2 = 9.$$

The area of the whole circle of radius R is πR^2. So the area of the half-circle of radius 3 is $\pi\cdot 3^3/2 = 9\pi/2$.

$$\int_{-3}^{3} \sqrt{9-x^2}\,dx = \frac{9\pi}{2}$$

(d) We need to find the point A on the circle with $x = 3/2$. Using the right triangle we get

$$\cos\theta = \frac{3/2}{3} = \frac{1}{2} \Longrightarrow \theta = \frac{\pi}{3}.$$

We compute the area of the sector spanned by an angle $\theta = \pi/3$. The whole circle is 2π radians and has area $\pi R^2 = \pi 3^2 = 9\pi$. So the sector has area

$$9\pi\cdot\frac{\pi/3}{2\pi} = \frac{9\pi}{6} = \frac{3\pi}{2}.$$

However this is not the area of the shaded region. We need to subtract from it the area of the triangle. This is a right triangle with angle $\theta = \pi/3$ and base $3/2$. Trigonometry gives its height, i.e. the y-coordinate of the point A:

$$\tan\theta = \frac{y}{3/2} = \tan\left(\frac{\pi}{3}\right) = \sqrt{3} \Longrightarrow y = \frac{3}{2}\sqrt{3}.$$

The area of the triangle is

$$\frac{1}{2}\text{base}\cdot\text{height} = \frac{1}{2}\frac{3}{2}\frac{3}{2}\sqrt{3} = \frac{9\sqrt{3}}{8}.$$

This gives

$$\int_{3/2}^{3} \sqrt{9-x^2}\,dx = \frac{3\pi}{2} - \frac{9\sqrt{3}}{8}.$$

6. $\Delta x = (3-1)/n = 2/n$. The first point $x_0 = 1$, the next point $x_1 = 1+\Delta x = 1+2/n$. The next point is $x_2 = x_1 + 2/n = 1+2/n+2/n = 1+2\cdot 2/n$. The formula for the points on the x-axis is $x_i = 1+2i/n$.

$$\text{LHS}(n) = \sum_{i=0}^{n-1} f(x_i)\Delta x = \sum_{i=0}^{n-1} x_i\frac{2}{n} = \sum_{i=0}^{n-1}\left(1+\frac{2i}{n}\right)\frac{2}{n},$$

$$\text{RHS}(n) = \sum_{i=1}^{n} f(x_i)\Delta x = \sum_{i=1}^{n} x_i \frac{2}{n} = \sum_{i=1}^{n} \left(1 + \frac{2i}{n}\right)\frac{2}{n}.$$

We continue with the evaluation of the sums using the formula $\sum_{i=1}^{n} i = n(n+1)/2$.

$$\text{RHS}(n) = \sum_{i=1}^{n} 1 \cdot \frac{2}{n} + \sum_{i=1}^{n} \frac{2i}{n}\frac{2}{n} = n \cdot \frac{2}{n} + \frac{4}{n^2}\sum_{i=1}^{n} i$$

since the first sum adds the number $2/n$ n-times.

$$\text{RHS}(n) = 2 + \frac{4}{n^2}\frac{n(n+1)}{2} = 2 + \frac{2(n+1)}{n} \overset{n\to\infty}{\longrightarrow} 2 + 2 = 4.$$

$$\text{LHS}(n) = \sum_{i=0}^{n-1} 1 \cdot \frac{2}{n} + \sum_{i=0}^{n-1} \frac{2i}{n}\frac{2}{n} = n \cdot \frac{2}{n} + \frac{4}{n^2}\sum_{i=0}^{n-1} i$$

since the first sum adds the number $2/n$ n-times. The last sum is the sum of the first $n-1$ numbers and is given by $(n-1)n/2$, using the formula for the sum of the first n numbers with $n-1$ in the place of n.

$$\text{LHS}(n) = 2 + \frac{4}{n^2}\frac{(n-1)n}{2} = 2 + \frac{2(n-1)}{n} \overset{n\to\infty}{\longrightarrow} 2 + 2 = 4.$$

7.

(a) $\int_{-1}^{0} (x-1)\,dx = \left[x^2/2 - x\right]_{-1}^{0} = (0^2/0-0) - \big((-1)^2/2 - (-1)\big) = 0-(1/2+1) = -3/2.$

(b) $\int_{-1}^{1} (t^2-2)\,dt = \left[\frac{t^3}{3} - 2t\right]_{-1}^{1} = \left(\frac{1}{3} - 2\right) - \left(\frac{(-1)^3}{3} - 2(-1)\right) = \frac{1}{3} - 2 + \frac{1}{3} - 2 = \frac{2}{3} - 4 = -\frac{10}{3}.$

(c) $\int_{-2}^{-1} (u-u^{-1})\,du = \left[\frac{u^2}{2} - \ln|u|\right]_{-2}^{-1} = \left(\frac{(-1)^2}{2} - \ln|-1|\right) - \left(\frac{(-2)^2}{2} - \ln|-2|\right)$

$$= \frac{1}{2} - 0 - 2 + \ln 2 = -\frac{3}{2} + \ln 2.$$

(d) $\int_{0}^{\pi} (1+\sin(x))\,dx = [x - \cos x]_0^{\pi} = (\pi - \cos\pi) - (0 - \cos 0) = \pi - (-1) + 1 = \pi + 2.$

(e) $\int_{-\pi/3}^{\pi/3} 4\sec\theta\tan\theta\,d\theta = [4\sec\theta]_{-\pi/3}^{\pi/3} = 4\sec(\pi/3) - 4\sec(-\pi/3) = \frac{4}{\cos(\pi/3)} - \frac{4}{\cos(-\pi/3)}$

$$= \frac{4}{1/2} - \frac{4}{1/2} = 0.$$

8.

(a) $F(x) = \int_8^x \sqrt[3]{t}\,dt = \int_8^x t^{1/3}\,dt = \left[\frac{t^{4/3}}{4/3}\right]_8^x = \left[\frac{3t^{4/3}}{4}\right]_8^x = \frac{3x^{4/3}}{4} - \frac{3\cdot 8^{4/3}}{4}.$

$$F'(x) = \frac{3}{4}\frac{4}{3}x^{4/3-1} - 0 = x^{1/3} = \sqrt[3]{x}.$$

This what the second fundamental theorem of calculus gives.

$$\text{(b) } F(x) = \int_{-1}^{x} e^t\,dt = \left[e^t\right]_{-1}^{x} = e^x - e^{-1}.$$

$$F'(x) = e^x - 0 = e^x$$

as the second fundamental theorem of calculus gives.

9.

$$\text{(a) } F'(x) = x\cos x, \quad \text{(b) } F'(x) = \frac{x^4}{x^2+1}.$$

10. (a) Substitute $u = x^4 + 2, du = 4x^3\,dx$.

$$\int x^3\cos(x^4+2)\,dx = \int \cos(u)\frac{1}{4}\,du = \frac{1}{4}\int \cos(u)\,du = \frac{1}{4}\sin(u)+c = \frac{1}{4}\sin(x^4+2)+c.$$

(b) Substitute $u = 1 - 4x^2, du = -8x\,dx, x\,dx = -du/8$.

$$\int \frac{x}{\sqrt{1-4x^2}}\,dx = -\frac{1}{8}\int \frac{1}{\sqrt{u}}\,du = -\frac{1}{8}\int u^{-1/2}\,du = -\frac{1}{8}2u^{1/2}+c = -\frac{1}{4}\sqrt{1-4x^2}+c.$$

(c) Substitute $u = \ln(x), du = (1/x)\,dx$. We also change the limits of integration:

$$x = e \Longrightarrow u = \ln(e) = 1, \quad x = 1 \Longrightarrow u = \ln(1) = 0.$$

$$\int_1^e \frac{\ln(x)}{x}\,dx = \int_0^1 u\,du = \left[\frac{u^2}{2}\right]_0^1 = \frac{1}{2}.$$

(d) Substitute $u = \pi/x = \pi x^{-1}$, $du = -\pi x^{-2}\,dx = -(\pi/x^2)\,dx$.

$$\int \frac{\cos(\pi/x)}{x^2}\,dx = \int \cos(u)\frac{1}{-\pi}\,du = -\frac{1}{\pi}\int \cos(u)\,du = -\frac{1}{\pi}\sin(u)+c = -\frac{1}{\pi}\sin(\pi/x)+c.$$

(e) Substitute $u = \ln(x)$, $du = (1/x)\,dx$. We change the limits of integration:

$$x = e \Longrightarrow u = \ln(e) = 1, \quad x = e^4 \Longrightarrow u = \ln(e^4) = 4.$$

$$\int_e^{e^4} \frac{dx}{x\sqrt{\ln(x)}} = \int_1^4 \frac{du}{\sqrt{u}} = \int_1^4 u^{-1/2}\,du = \left[2u^{1/2}\right]_1^4 = 2\cdot 4^{1/2} - 2\cdot 1^{1/2} = 4 - 2 = 2.$$

(f) Substitute $u = 1 + \sqrt{x}$, $du = (1/2)x^{-1/2}\,dx$, $2du = dx/\sqrt{x}$.

$$\int \frac{(1+\sqrt{x})^9}{\sqrt{x}}\,dx = \int u^9 2du = 2\frac{u^{10}}{10} + c = \frac{2}{10}(1+\sqrt{x})^{10} + c.$$

(g) Substitute $u = \arctan(x)$, $du = \dfrac{1}{1+x^2}\,dx$.

$$\int \frac{\arctan(x)}{1+x^2}\,dx = \int u\,du = \frac{u^2}{2} + c = \frac{(\arctan x)^2}{2} + c.$$

(h) Substitute $u = \sin(x)$, $du = \cos(x)\,dx$.

$$\int \cos(x)\cos(\sin(x))\,dx = \int \cos(\sin(x))\cos(x)\,dx = \int \cos(u)\,du = \sin(u)+c = \sin(\sin(x))+c.$$

(i) Substitute $u = \cos(x)$, $du = -\sin(x)\,dx$.

$$\int \frac{\sin(x)}{1+\cos^2(x)}\,dx = \int \frac{-1}{1+u^2}\,du = -\arctan(u)+c = -\arctan(\cos(x))+c.$$

(j) Substitute $u = \sec(x)$, $du = \sec(x)\tan(x)\,dx$.

$$\int \sec^3(x)\tan(x)\,dx = \int \sec^2(x)\sec(x)\tan(x)\,dx = \int u^2\,du = \frac{u^3}{3}+c = \frac{\sec^3(x)}{3}+c.$$

(k) Substitute $u = 1+x+2x^2$, $du = (1+4x)\,dx$.

$$\int \frac{1+4x}{\sqrt{1+x+2x^2}}\,dx = \int \frac{du}{\sqrt{u}} = \int u^{-1/2}\,du = 2u^{1/2}+c = 2(1+x+2x^2)^{1/2}+c.$$

(l) Substitute $u = \cos(x)$, $du = -\sin(x)\,dx$.

$$\int \tan(x)\ln(\cos(x))\,dx = \int \frac{\sin(x)}{\cos(x)}\ln(\cos(x))\,dx = \int \frac{\ln(\cos(x))}{\cos(x)}\sin(x)\,dx = \int \frac{\ln(u)}{u}(-du).$$

Now we do another substitution $w = \ln(u)$, $dw = du/u$ and get

$$\int \tan x\ln(\cos x)\,dx = -\int \frac{\ln u}{u}\,du = -\int w\,dw = -\frac{w^2}{2}+c = -\frac{(\ln u)^2}{2}+c = -\frac{(\ln(\cos x))^2}{2}+c.$$

$$\text{(m)}\quad \int \frac{x+3}{x^2+4}\,dx = \int \frac{x}{x^2+4}\,dx + \int \frac{3}{x^2+4}\,dx.$$

We perform different substitutions to the two integrals. For the first we set $u = x^2+4$, $du = 2x\,dx$, while for the second we set $w = x/2$, $dw = dx/2$.

$$\int \frac{x+3}{x^2+4}\,dx = \int \frac{1}{u}\frac{1}{2}du+3\int \frac{2dw}{(2w)^2+4} = \frac{1}{2}\int \frac{1}{u}du+6\int \frac{dw}{4w^2+4} = \frac{1}{2}\ln|u|+\frac{6}{4}\int \frac{dw}{w^2+1}$$

$$= \frac{1}{2}\ln(x^2+4)+\frac{3}{2}\arctan(w)+c = \frac{1}{2}\ln(x^2+4)+\frac{3}{2}\arctan(x/2)+c.$$

$$\text{(n)}\quad \int \frac{5x+9}{x^2+2x+3}\,dx = \int \frac{5x+5+4}{(x+1)^2+2}\,dx = \int \frac{5(x+1)+4}{(x+1)^2+2}\,dx.$$

We substitute $u = x+1$, $du = dx$:

$$\int \frac{5x+9}{x^2+2x+3}\,dx = \int \frac{5u+4}{u^2+2}\,du = \int \frac{5u}{u^2+2}\,du + \int \frac{4}{u^2+2}\,du$$

$$= \frac{5}{2}\int \frac{2u}{u^2+2}\,du + \int \frac{4}{2(u^2/2+1)}\,du = \frac{5}{2}\ln(u^2+2) + 2\int \frac{1}{(u/\sqrt{2})^2+1}\,du$$

since $(\ln(u^2+2))' = \dfrac{2u}{u^2+1}$. For the second integral we substitute $w = u/\sqrt{2}$ to get

$$\int \frac{5x+9}{x^2+2x+3}\,dx = \frac{5}{2}\ln(u^2+2)+2\int \frac{\sqrt{2}}{w^2+1}\,dw = \frac{5}{2}\ln(u^2+2)+2\sqrt{2}\arctan(w)+c$$

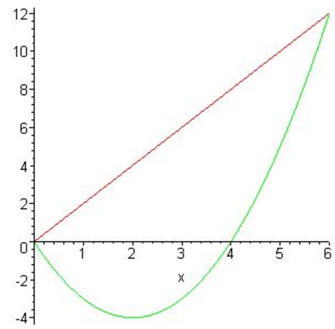


Figure 7: The region in problem 2

$$= \frac{5}{2}\ln((x+1)^2+2) + 2\sqrt{2}\arctan\left(\frac{x+1}{\sqrt{2}}\right) + c.$$

(o) Substitute $u = x^2$, $du = 2x\,dx$.

$$\int \frac{x}{x^4+16}\,dx = \int \frac{du/2}{u^2+16} = \frac{1}{2}\int \frac{du}{u^2+16} = \frac{1}{8}\arctan(u/4)+c = \frac{1}{8}\arctan(x^2/4)+c.$$

11. We need to find the points of intersection of $y = x^2 - 4x$ and $y = 2x$. Setting $2x = x^2 - 4x$ we get $x^2 - 6x = 0 \Longrightarrow x(x-6) = 0 \Longrightarrow x = 0$ or $x = 6$. So the points of intersection are $(0,0)$ and $(6,12)$. From the figure we see that $y = 2x$ is above the parabola $y = x^2 - 4x$. So the area is

$$\int_0^6 (y_u - y_l)\,dx = \int_0^6 2x - (x^2 - 4x)\,dx = \int_0^6 (6x - x^2)\,dx = \left[\frac{6x^2}{2} - \frac{x^3}{3}\right]_0^6 = \frac{6\cdot 36}{2} - \frac{216}{3} = 36.$$

12. We need to find the points of intersection of $y = x - 1$ and $y^2 = 2x + 6$. Setting $y^2 = 2(y+1) + 6$ we get

$$y^2 - 2y - 8 = 0 \Longrightarrow y = \frac{2 \pm \sqrt{4 - 4(-8)}}{2} = \frac{2 \pm \sqrt{36}}{2} = \frac{2 \pm 6}{2} = 4 \quad \text{or} \quad -2.$$

So the points of intersection are $(5,4)$ and $(-1,-2)$. The area we get by integrating along the y-axis. We solve the given equations for x:

$$y = x - 1 \Longrightarrow x = y + 1, \quad y^2 = 2x + 6 \Longrightarrow x = \frac{y^2 - 6}{2}.$$

$$\int_{-2}^4 (x_r - x_l)\,dy = \int_{-2}^4 (y+1) - (y^2-6)/2\,dy = \int_{-2}^4 -\frac{y^2}{2} + y + 4\,dy = \left[-\frac{y^3}{6} + \frac{y^2}{2} + 4y\right]_{-2}^4$$

$$= -\frac{64}{6} + 8 + 16 - (4/3 + 2 - 8) = 18.$$

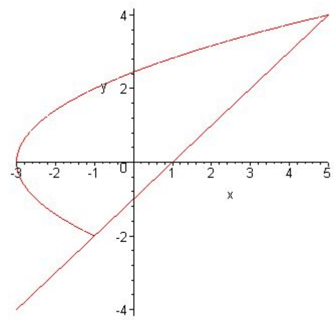


Figure 8: The region in problem 2

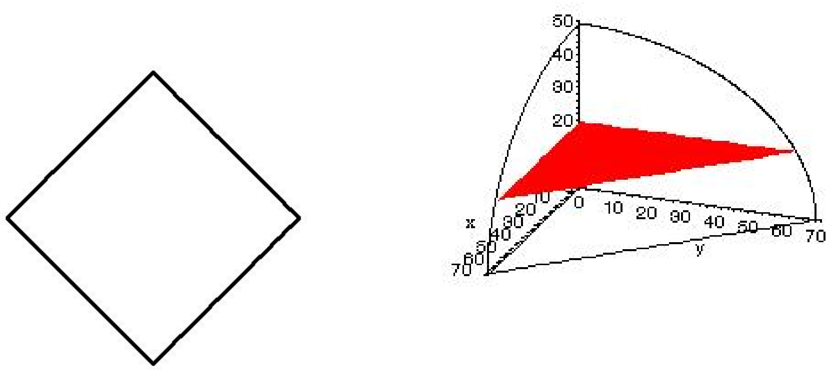


Figure 9: A cross section of the bent pyramid

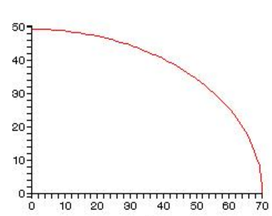


Figure 10: The height as a function of the half-diagonal

13. If the half-diagonal of a square is x, then the side is $\sqrt{2}x$ and the area of the square is $2x^2$. We form a Riemann sum by slicing the bent puramid horizontally. A slice of thickness Δy at height y has volume approximately the area of the base times Δy. This gives $\Delta V \approx 2x^2 \Delta y$. The relation of x and y at height y is given by

$$y = \frac{1}{\sqrt{2}}\sqrt{70^2 - x^2} \Longrightarrow y^2 = \frac{1}{2}(70^2 - x^2)$$

$$\Longrightarrow 2y^2 = 70^2 - x^2 \Longrightarrow x^2 = 70^2 - 2y^2.$$

So $\Delta V \approx 2(70^2 - 2y^2) \cdot \Delta y$. We get the Riemann sum

$$V \approx \sum_i 2(70^2 - 2y^2) \cdot \Delta y.$$

We convert it to the integral. We need the height of the bent pyramid. From Figure 10 we get the height, when $x = 0$, i.e. $H = 70/\sqrt{2}$.

$$V = \int_0^{70/\sqrt{2}} 2(70^2 - 2y^2)\, dy = \left[2(70^2 y - \frac{2}{3}y^3)\right]_0^{70/\sqrt{2}}$$

$$= 2\left(70^2 \cdot \frac{70}{\sqrt{2}} - \frac{2}{3}\frac{70^3}{2\sqrt{2}}\right) = 2 \cdot \frac{70^3}{\sqrt{2}}\left(1 - \frac{1}{3}\right) = \frac{4 \cdot 70^3}{3\sqrt{2}}.$$

14. *First method:* The method of cylindrical shells. If the shell occurs at x, then the height of the shell is $2y$, where $y = \sqrt{1-(x-2)^2}$. So the height is $2\sqrt{1-(x-2)^2}$. The perimeter of the shell is $2\pi x$ and the thickness of the shell is Δx. The volume

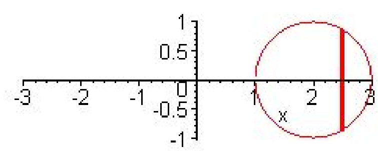


Figure 11: The donut as volume of revolution around the y-axis: Cylindrical shells

of the shell is approximately $2\pi x \cdot 2\sqrt{1-(x-2)^2}\,\Delta x$. The Riemann sum we get is

$$V \approx \sum_i 2\pi x \cdot 2\sqrt{1-(x-2)^2}\,\Delta x.$$

We convert the Riemann sum to an integral to get the exact volume of the donut:

$$V = \int_1^3 2\pi x \cdot 2\sqrt{1-(x-2)^2}\, dx = 4\pi \int_1^3 x\sqrt{1-(x-2)^2}\, dx.$$

We substitute $u = x - 2$, $du = dx$. We also change the limits of integration:

$$x = 1 \Longrightarrow u = -1, \quad x = 3 \Longrightarrow u = 1.$$

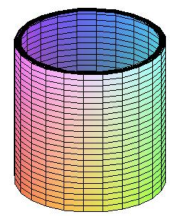


Figure 12: A typical cylindrical shell of thickness Δx

$$V = 4\pi \int_{-1}^{1} (u+2)\sqrt{1-u^2}\,du = 4\pi \left(\int_{-1}^{1} u\sqrt{1-u^2}\,du + 2\int_{-1}^{1} \sqrt{1-u^2}\,du \right)$$
$$= 4\pi \left(\left[-\frac{1}{3}(1-u^2)^{3/2} \right]_{-1}^{1} + 2\pi/2 \right) = 4\pi(0 - 0 + \pi) = 4\pi^2.$$

In the first integral we performed the substitution $w = 1-u^2$, $dw = -2u\,du$, which gives

$$\int u\sqrt{1-u^2}, du = -\frac{1}{2}\int \sqrt{w}\,dw = -\frac{1}{2}\int w^{1/2}\,dw = -\frac{1}{2}\frac{w^{3/2}}{3/2} = -\frac{1}{3}w^{3/2} = -\frac{1}{3}(1-u^2)^{3/2}.$$

The second integral represents the area under the circle $x^2+y^2=1$ and above the x-axis. The area of this half-disc is $\pi R^2/2 = \pi 1^2/2 = \pi/2$.

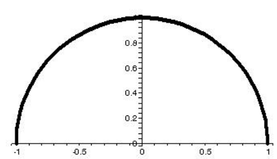


Figure 13: $\int_{-1}^{1} \sqrt{1-u^2}\,du$

Second method: The method of washers. $(x-2)^2+y^2=1$ gives

$$(x-2)^2 = 1-y^2 \Longrightarrow x-2 = \pm\sqrt{1-y^2} \Longrightarrow x = 2 \pm \sqrt{1-y^2}.$$

The inner radius of the washer is $2-\sqrt{1-y^2}$ and the outer radius of the washer is $2+\sqrt{1-y^2}$. The area of the washer is

$$A = \pi(2+\sqrt{1-y^2})^2 - \pi(2-\sqrt{1-y^2})^2 = \pi(4+4\sqrt{1-y^2}+1-y^2) - \pi(4+1-y^2-4\sqrt{1-y^2})$$
$$= 8\pi\sqrt{1-y^2}.$$

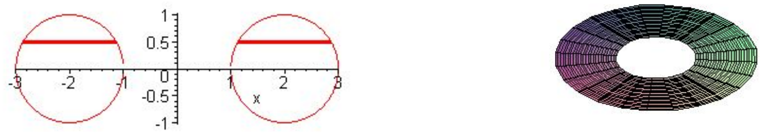


Figure 14: Washers for the donut

The volume of the donut is

$$V = \int_{-1}^{1} 8\pi\sqrt{1-y^2}\,dy = 8\pi\int_{-1}^{1}\sqrt{1-y^2}\,dy = 8\pi\pi/2 = 4\pi^2,$$

since the last integral represents the area of a half-disc of radius 1.

15. The volume of the fattened disc is $\pi y^2\Delta x = \pi(1-x^2/4)\Delta x$. This gives the Riemann sum $\sum_i \pi(1-x^2/4)\Delta x$ and then the integral

$$\int_{-2}^{2}\pi\left(1-\frac{x^2}{4}\right)dx = \pi\left[x-\frac{x^3}{3\cdot 4}\right]_{-2}^{2} = \pi\left(2-\frac{8}{12}\right)-\pi\left(-2-\frac{-8}{12}\right) = \pi\left(4-\frac{16}{12}\right) = \frac{8\pi}{3}.$$

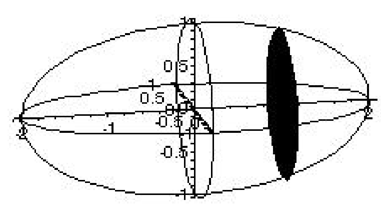


Figure 15: The ellipsoid and one disc in it

16. The annular region has area approximately the area of the strip. The length of the circle of radius r is $2\pi r$ and the area of the strip is $2\pi r\Delta r$. The mass of the oil in the

ring is the area of the ring times the density of the oil:

$$\rho(r) \cdot 2\pi r \Delta r = \frac{50}{1+r} 2\pi r \Delta r.$$

This gives the Riemann sum

Figure 16: A ring of oil of radius r and thickness Δr gives a strip with length $2\pi r$

$$\sum_i \frac{50}{1+r} 2\pi r \Delta r$$

and finally the integral

$$\int_0^{10000} \frac{50}{1+r} 2\pi r\, dr = 100\pi \int_0^{10000} \frac{r}{1+r}\, dr = 100\pi \int_0^{10000} \frac{1+r-1}{1+r}\, dr$$

$$= 100\pi \int_0^{10000} \left(1 - \frac{1}{1+r}\right) dr = 100\left[r - \ln(1+r)\right]_0^{10000} = 100\pi(10000 - \ln(10001))$$

17.

$$y' = \frac{e^x - e^{-x}}{2} \Longrightarrow (y')^2 = \left(\frac{e^x - e^{-x}}{2}\right)^2 = \frac{e^{2x} - 2e^x e^{-x} + e^{-2x}}{4} = \frac{e^{2x} - 2 + e^{-2x}}{4}$$

$$\Longrightarrow 1 + (y')^2 = \frac{4}{4} + \frac{e^{2x} - 2 + e^{-2x}}{4} = \frac{e^{2x} + 2 + e^{-2x}}{4} = \left(\frac{e^x + e^{-x}}{2}\right)^2.$$

The arclength is

$$\int_0^1 \frac{e^x + e^{-x}}{2}\, dx = \left[\frac{e^x}{2} - \frac{e^{-x}}{2}\right]_0^1 = \frac{e - e^{-1}}{2} - \frac{e^0 - e^0}{2} = \frac{e - e^{-1}}{2}.$$

18. (a) Integration by parts: $u = x \Longrightarrow u' = 1$, $v' = \sin(6x) \Longrightarrow v = -\cos(6x)/6$.

$$\int x \sin(6x)\, dx = -\frac{1}{6}\cos(6x) \cdot x + \int \frac{\cos(6x)}{6} dx = -\frac{x\cos(6x)}{6} + \frac{\sin(6x)}{36} + c.$$

(b) Integration by parts: $u = x^2 \Longrightarrow u' = 2x$, $v' = e^x \Longrightarrow v = e^x$.

$$\int x^2 e^x dx = x^2 e^x - \int 2x e^x dx = x^2 e^x - (2xe^x - \int e^x \cdot 2dx) = x^2 e^x - 2xe^x + 2\int e^x dx$$

$$= x^2e^x - 2xe^x + 2e^x + c,$$

where we have used another integration by parts with $u = 2x \implies u' = 2$ and $v' = e^x \implies v = e^x$.

(c) $\displaystyle\int \frac{x}{e^x}\,dx = \int xe^{-x}\,dx = x(-e^{-x}) - \int (-e^{-x})\cdot 1\,dx = -xe^{-x} + \int e^{-x}\,dx = -xe^{-x} - e^{-x} + c,$

where we have used integration by parts with $u = x \implies u' = 1$ and $v' = e^{-x} \implies v = -e^{-x}$.

(d) Integration by parts: $u = \ln(3x) = \ln(3) + \ln(x) \implies u' = 1/x$ and $v' = 1 \implies v = x$.

$$\int \ln(3x)\,dx = \int 1\cdot\ln(3x)\,dx = x\ln(3x) - \int x\frac{1}{x}\,dx = x\ln(3x) - \int 1\,dx = x\ln(3x) - x + c.$$

(e) Integration by parts: $u = x^2 \implies u' = 2x$, $v' = \cos(x) \implies v = \sin(x)$.

$$\int x^2\cos(x)\,dx = x^2\sin(x) - \int 2x\sin(x)\,dx = x^2\sin(x) - (2x(-\cos(x)) - \int 2(-\cos(x))\,dx$$

here we need another integration by parts with $u = 2x \implies u' = 2$ and $v' = \sin(x) \implies v = -\cos(x)$

$$= x^2\sin(x) + 2x\cos(x) - 2\int \cos(x)\,dx = x^2\sin(x) + 2x\cos(x) - 2\sin(x) + c.$$

(f) Integration by parts: $u = \theta \implies u' = 1$ and $v' = \sec\theta\tan\theta \implies v = \sec\theta$.

$$\int \theta\sec\theta\tan\theta\,d\theta = \theta\sec\theta - \int 1\cdot\sec\theta\,d\theta = \theta\sec\theta - \ln|\sec\theta + \tan\theta| + c.$$

(g) Integration by parts with $u = \arctan(x) \implies u' = 1/(1+x^2)$ and $v' = 1 \implies v = x$.

$$\int \arctan(x)\,dx = \int 1\cdot\arctan(x)\,dx = x\arctan(x) - \int \frac{x}{1+x^2}\,dx = x\arctan(x) - \frac{1}{2}\int \frac{1}{u}\,du$$

where we substitute $u = 1 + x^2 \implies du = 2x\,dx$

$$= x\arctan(x) - \frac{1}{2}\ln|u| + c = x\arctan(x) - \frac{1}{2}\ln(1+x^2) + c.$$

(h) Integration by parts: $u = \sin(x) \implies u' = \cos(x)$ and $v' = e^{2x} \implies v = e^{2x}/2$.

$$\int e^{2x}\sin(x)\,dx = \sin(x)\frac{e^{2x}}{2} - \int \cos(x)\frac{e^{2x}}{2}\,dx = \sin(x)\frac{e^{2x}}{2} - \frac{1}{2}\int e^{2x}\cos(x)\,dx$$

and now we do another integration by parts with $u = \cos(x) \implies u' = -\sin(x)$ and $v' = e^{2x} \implies v = e^{2x}/2$.

$$= \sin(x)\frac{e^{2x}}{2} - \frac{1}{2}\left(\cos(x)\frac{e^{2x}}{2} - \int -\sin(x)\frac{e^{2x}}{2}\,dx\right) = \sin(x)\frac{e^{2x}}{2} - \frac{1}{4}\cos(x)e^{2x} - \frac{1}{4}\int \sin(x)e^{2x}\,dx.$$

We end up with the integral we started with. If we call $I = \int e^{2x}\sin(x)\,dx$ we get the equation

$$I = \frac{e^{2x}}{2}\sin(x) - \frac{e^{2x}}{4}\cos(x) - \frac{1}{4}I$$

which we solve to I to get

$$(1+1/4)I = e^{2x}\left(\frac{\sin(x)}{2} - \frac{\cos(x)}{4}\right) \Longrightarrow \int e^{2x}\sin(x)\,dx = \frac{4}{5}e^{2x}\left(\frac{\sin(x)}{2} - \frac{\cos(x)}{4}\right)+c.$$

Second method. $u = e^{2x} \Longrightarrow u' = 2e^{2x}$ and $v' = \sin(x) \Longrightarrow v = -\cos(x)$.

$$\int e^{2x}\sin(x)\,dx = e^{2x}(-\cos(x)) - \int 2e^{2x}(-\cos(x))\,dx = -e^{2x}\cos(x) + 2\int e^{2x}\cos(x)\,dx$$

Now we do another integration by parts with $u = e^{2x} \Longrightarrow u' = 2e^{2x}$ and $v' = \cos(x) \Longrightarrow v = \sin(x)$.

$$\int e^{2x}\sin x dx = -e^{2x}\cos x + 2(e^{2x}\sin x - \int 2e^{2x}\sin x dx) = -e^{2x}\cos x + 2e^{2x}\sin x - 4\int e^{2x}\sin x dx.$$

This gives $I = -e^{2x}\cos(x) + 2e^{2x}\sin(x) - 4I \Longrightarrow 5I = -e^{2x}\cos(x) + 2e^{2x}\sin(x)$

$$\Longrightarrow I = -\frac{1}{5}e^{2x}\cos(x) + \frac{2}{5}e^{2x}\sin(x) + c.$$

(i) Integration by parts: $u = \ln(t+1) \Longrightarrow u' = 1/(t+1)$ and $v' = t \Longrightarrow v = t^2/2$.

$$\int t\ln(t+1)\,dt = \frac{t^2}{2}\ln(t+1) - \int \frac{t^2}{2}\frac{1}{t+1}\,dt = \frac{t^2}{2}\ln(t+1) - \frac{1}{2}\int \frac{t^2-1+1}{t+1}\,dt$$

$$= \frac{t^2}{2}\ln(t+1) - \frac{1}{2}\int \frac{t^2-1}{t+1}\,dt - \frac{1}{2}\int \frac{1}{t+1}\,dt = \frac{t^2}{2}\ln(t+1) - \frac{1}{2}\int (t-1)\,dt - \frac{1}{2}\ln(t+1)$$

$$= \frac{t^2}{2}\ln(t+1) - \frac{1}{2}\left(\frac{t^2}{2} - t\right) - \frac{1}{2}\ln(t+1) + c.$$

(j) Integration by parts: $u = \ln(1+x^2) \Longrightarrow u' = \frac{2x}{1+x^2}$, $v' = 1 \Longrightarrow v = x$.

$$\int_0^1 \ln(1+x^2)\,dx = \left[\ln(1+x^2)\cdot x\right]_0^1 - \int_0^1 x\frac{2x}{1+x^2}\,dx = \ln(1+1^2)\cdot 1 - \ln(1+0^2)\cdot 0 - 2\int_0^1 \frac{x^2}{1+x^2}\,dx$$

$$= \ln(2) - 2\int_0^1 \frac{x^2+1-1}{x^2+1}\,dx = \ln(2) - 2\left(\int_0^1 1 - \frac{1}{x^2+1}\,dx\right) = \ln(2) - 2[x - \arctan(x)]_0^1$$

$$= \ln(2) - 2(1 - \arctan(1)) + 2(0 - \arctan(0)) = \ln(2) - 2 + \frac{\pi}{2},$$

since $\arctan(1) = \pi/4$ and $\arctan(0) = 0$.

$$\text{(k)} \int \frac{x^3}{\sqrt{4+x^2}}\,dx = \int \frac{x}{\sqrt{4+x^2}}x^2\,dx$$

Integration by parts with $u = x^2 \implies u' = 2x$ and $v' = \dfrac{x}{\sqrt{4+x^2}} \implies v = \sqrt{4+x^2}$!!!

Check by differentiation: $((4+x^2)^{1/2})' = \frac{1}{2}(4+x^2)^{-1/2}(2x) = x(4+x^2)^{-1/2}$

$$\int \frac{x^3}{\sqrt{4+x^2}}\,dx = x^2\sqrt{4+x^2} - \int 2x\sqrt{4+x^2}\,dx = x^2\sqrt{4+x^2} - \int \sqrt{u}\,du$$

with the substitution $u = 4 + x^2 \implies du = 2x\,dx$

$$= x^2\sqrt{4+x^2} - \frac{2}{3}u^{3/2} + c = x^2\sqrt{4+x^2} - \frac{2}{3}(4+x^2)^{3/2} + c.$$

Second Method. Substitution $u = 4 + x^2 \implies du = 2x\,dx$

$$\int \frac{x^3}{\sqrt{4+x^2}}\,dx = \int \frac{x^2 \cdot 1/2 \cdot 2x}{\sqrt{4+x^2}}\,dx = \int \frac{x^2/2\,du}{\sqrt{u}} = \frac{1}{2}\int \frac{u-4}{u^{1/2}}\,du$$

$$= \frac{1}{2}\int (u^{1/2} - 4u^{-1/2})\,du = \frac{1}{2}\left(\frac{2}{3}u^{3/2} - 4\cdot 2u^{1/2}\right) + c = \frac{1}{3}\sqrt{4+x^2}^{3} - 4\sqrt{4+x^2} + c$$

$$= \left(\frac{1}{3}(4+x^2) - 4\right)\sqrt{4+x^2} + c.$$

The answers are the same!!! We check this:

$$x^2\sqrt{4+x^2} - \frac{2}{3}(4+x^2)^{3/2} = x^2\sqrt{4+x^2} - \frac{2}{3}(4+x^2)\sqrt{4+x^2} = \left(x^2 - \frac{8}{3} - \frac{2}{3}x^2\right)\sqrt{4+x^2}$$

$$= \left(\frac{1}{3}x^2 - \frac{8}{3}\right)\sqrt{4+x^2}.$$

19. (a) Complete the square: $y^2 + 4y + 5 = (y+2)^2 + 1$, then substitute $u = y + 1 \implies du = dy$.

$$\int \frac{1}{y^2+4y+5}\,dy = \int \frac{1}{(y+2)^2+1}\,dy = \int \frac{1}{u^2+1}\,du = \arctan(u) + c = \arctan(y+1) + c.$$

(b) Substitute $u = 2\theta \implies du = 2d\theta$.

$$\int \frac{1}{\sin^2(2\theta)}\,d\theta = \frac{1}{2}\int \frac{1}{\sin^2 u}\,du \overset{19}{=} \frac{1}{2}\left(-\frac{1}{1}\frac{\cos u}{\sin u} + \frac{0}{1}\int \frac{1}{\sin^0 u}\,du\right) = -\frac{1}{2}\cot u + c = -\frac{1}{2}\cot(2\theta) + c.$$

(c) $$\int (2x^3 + 3x + 4)\cos(2x)\,dx \overset{16}{=} \frac{1}{2}(2x^3 + 3x + 4)\sin(2x) + \frac{1}{2^2}(6x^2 + 3)\cos(2x)$$

$$-\frac{1}{8}12x\sin(2x) - \frac{1}{16}12\cos(2x) + c.$$

(d) $$\int \frac{1}{z^2+z}\,dz = \int \frac{1}{(z-0)(z-(-1))}\,dz \overset{26}{=} \frac{1}{0-(-1)}(\ln|z-0| - \ln|z-(-1)|) + c$$

$$= \ln|z| - \ln|z+1| + c.$$

(e) Substitute $u = 2z^2 \implies du = 4z\,dz$.

$$\int ze^{2z^2}\cos(2z^2)\,dz = \int \frac{1}{4}e^u\cos(u)\,du \overset{9}{=} \frac{1}{4}\frac{1}{1^2+1^2}e^u(1\cos(u)+1\sin(u))+c$$

$$= \frac{1}{8}e^{2z^2}(\cos(2z^2)+\sin(2z^2))+c.$$

(f) $$\int \tan^4(x)\,dx = \int \frac{\sin^4(x)}{\cos^4(x)}\,dx = \int \frac{(1-\cos^2 x)^2}{\cos^4(x)}\,dx = \int \frac{1-2\cos^2 x+\cos^4 x}{\cos^4 x}\,dx$$

$$= \int \frac{1}{\cos^4 x}\,dx - 2\int \frac{1}{\cos^2 x}\,dx + \int 1\,dx \overset{21}{=} \frac{1}{3}\frac{\sin x}{\cos^3 x} + \frac{2}{3}\int \frac{1}{\cos^2 x}\,dx - 2\int \frac{1}{\cos^2 x}\,dx + x$$

$$= \frac{\sin x}{3\cos^3 x} - \frac{4}{3}\int \frac{1}{\cos^2 x}\,dx + x = \frac{\sin x}{3\cos^3 x} - \frac{4}{3}\tan(x) + x + c.$$

(g) $$\int x^4 e^{3x}\,dx \overset{14}{=} \frac{1}{3}x^4e^{3x} - \frac{1}{3^2}4x^3e^{3x} + \frac{1}{3^3}12x^2e^{3x} - \frac{1}{3^4}24xe^{3x} + \frac{1}{3^5}24e^{3x} + c.$$

(h) $$\int \frac{x^2+1}{x^2-3x+2}\,dx = \int \frac{x^2-3x+2+3x-1}{x^2-3x+2}\,dx = \int \frac{x^2-3x+2}{x^2-3x+2} + \frac{3x-1}{x^2-3x+2}\,dx$$

$$= \int 1\,dx + \int \frac{3x-1}{x^2-3x+2}\,dx = x + \int \frac{3x-1}{(x-2)(x-1)}\,dx \overset{27}{=} x + \frac{1}{2-1}((2\cdot 3-1)\ln|x-2|$$

$$-(1\cdot 3-1)\ln|x-1|) + c = x + 5\ln|x-2| - 2\ln|x-1| + c$$

(i) $$\int \frac{5x+6}{x^2+4}\,dx \overset{25}{=} \frac{5}{2}\ln(x^2+4) + \frac{6}{2}\arctan(x/2) + c.$$

20. (a) Substitution $u = x^3 - 1 \implies du = 3x^2\,dx$. This gives

$$\int (x^3-1)^4 x^2\,dx = \int u^4\frac{1}{3}du = \frac{1}{3}\int u^4\,du = \frac{1}{3}\frac{u^5}{5} + c = \frac{(x^3-1)^5}{15} + c.$$

(b) Substitution $u = x^2 + 1 \implies du = 2x\,dx$. This gives

$$\int xe^{x^2+1}\,dx = \int e^u\frac{1}{2}\,du = \frac{1}{2}\int e^u\,du = \frac{1}{2}e^u + c = \frac{1}{2}e^{x^2+1} + c.$$

(c) Integration by parts: $u = x^2 \implies u' = 2x$, $v' = e^{2x} \implies v = e^{2x}/2$.

$$\int x^2e^{2x}\,dx = x^2\frac{e^{2x}}{2} - \int 2x\frac{e^{2x}}{2}\,dx = x^2\frac{e^{2x}}{2} - \int xe^{2x}\,dx.$$

Now we do another integration by parts with $u = x \implies u' = 1$ and $v' = e^{2x} \implies v = e^{2x}/2$.

$$\int x^2e^{2x}\,dx = x^2\frac{e^{2x}}{2} - \left(x\frac{e^{2x}}{2} - \int \frac{e^{2x}}{2}\,dx\right)$$

$$= \frac{x^2e^{2x}}{2} - \frac{xe^{2x}}{2} + \frac{1}{2}\int e^{2x}\,dx = \frac{x^2e^{2x}}{2} - \frac{xe^{2x}}{2} + \frac{1}{4}e^{2x} + c.$$

(d) Substitution $u = 1 - x \Longrightarrow du = -dx$. Since $x = 1 - u$, we get

$$\int x\sqrt{1-x}\,dx = \int (1-u)\sqrt{u}(-du) = -\int (1-u)u^{1/2}\,du = -\int u^{1/2} - u^{3/2}\,du$$

$$= -\frac{u^{3/2}}{3/2} + \frac{u^{5/2}}{5/2} + c = -\frac{2(1-x)^{3/2}}{3} + \frac{2(1-x)^{5/2}}{5} + c.$$

(e) Integration by parts: $u = (\ln x)^2 \Longrightarrow u' = 2\ln x(\ln x)' = 2\ln x \cdot 1/x$ by the chain rule. We choose $v' = 1 \Longrightarrow v = x$. We get

$$\int (\ln x)^2\,dx = (\ln x)^2 x - \int 2\ln x \cdot \frac{1}{x} x\,dx = x(\ln x)^2 - 2\int \ln x\,dx.$$

For the new integral we use another integration by parts with $u = \ln x \Longrightarrow u' = 1/x$, $v' = 1 \Longrightarrow v = x$ to get

$$\int (\ln x)^2\,dx = x(\ln x)^2 - 2(x\ln x - \int x \cdot 1/x\,dx) = x(\ln x)^2 - 2x\ln x + 2x + c.$$

Alternatively we find the integral $\int \ln x\,dx$ in the table to be $x\ln x - x + c$.

(f) First method: By properties of logarithms $\ln(x^2) = 2\ln x$. This gives

$$\int \ln(x^2)\,dx = \int 2\ln x\,dx = 2\int \ln x\,dx = 2(x\ln x - x) + c$$

by (e). There is an alternate method to use integration by parts with $u = \ln(x^2) \Longrightarrow u' = \frac{2x}{x^2} = \frac{2}{x}$, by the chain rule and $v' = 1 \Longrightarrow v = x$.

$$\int \ln(x^2)\,dx = \ln(x^2)x - \int \frac{2}{x}x\,dx = x\ln(x^2) - \int 2\,dx = x\ln(x^2) - 2x + c.$$

(g) Integration by parts with $u = \arcsin x \Longrightarrow u' = \frac{1}{\sqrt{1-x^2}}$, $v' = 1 \Longrightarrow v = x$. This gives

$$\int \arcsin x\,dx = x\arcsin x - \int x\frac{1}{\sqrt{1-x^2}}\,dx$$

Now we use the substitution $u = 1 - x^2 \Longrightarrow du = -2x\,dx$. This gives

$$\int \arcsin x\,dx = x\arcsin x - \int \frac{1}{u^{1/2}}\frac{-du}{2} = x\arcsin x + \frac{1}{2}\int u^{-1/2}\,du$$

$$= x\arcsin x + \frac{1}{2}\frac{u^{1/2}}{1/2} + c = x\arcsin x + (1-x^2)^{1/2} + c.$$

$$\text{(h)} \int \frac{(2x-1)e^{x^2}}{e^x}\,dx = \int (2x-1)e^{x^2-x}\,dx$$

which suggests the substitution $u = x^2 - x \Longrightarrow du = (2x-1)\,dx$.

$$\int \frac{(2x-1)e^{x^2}}{e^x}\,dx = \int e^u\,du = e^u + c = e^{x^2-x} + c.$$

(i) The substitution $u = \arcsin x \implies \sin u = x \implies \cos u\, du = dx$ gives

$$\int x \arcsin x\, dx = \int \sin u \cdot u \cos u\, du = \int u \sin u \cos u\, du = \int u \frac{1}{2} \sin(2u)\, du$$

$$= \frac{1}{2} \int u \sin(2u)\, du = \frac{1}{2} \left(u \frac{-\cos(2u)}{2} - \int 1 \cdot \frac{-\cos(2u)}{2}\, du \right)$$

using an integration by parts with $f = u \implies f' = 1$, $g' = \sin(2u) \implies g = -\cos(2u)/2$.

$$\int x \arcsin x\, dx = -\frac{1}{4} u \cos(2u) + \frac{1}{4} \int \cos(2u)\, du = -\frac{1}{4} u \cos(2u) + \frac{1}{8} \sin(2u) + c$$

$$= -\frac{1}{4} \arcsin x \cos(2 \arcsin x) + \frac{1}{8} \sin(2 \arcsin x) + c.$$

We can leave it like this, or we can simplify it by noticing that $\sin(2u) = 2 \sin u \cos u = 2x\sqrt{1 - x^2}$ and $\cos(2u) = 1 - 2\sin^2 u = 1 - 2x^2$. This gives:

$$\int x \arcsin x\, dx = -\frac{1}{4} (\arcsin x)(1 - 2x^2) + \frac{1}{8} 2x\sqrt{1 - x^2} + c.$$

21. (a) Substitute $u = 2x \implies du = 2dx$ first. This gives

$$\int \cos^3(2x) \sin(2x)\, dx = \int \cos^3 u \sin u \frac{du}{2} = \frac{1}{2} \int \cos^3 u \sin u\, du.$$

There are now two possibilities, since the exponents of sin and cos are both odd.

Method 1: Substitute $w = \sin u \implies dw = \cos u\, du$ and use $\cos^2 u = 1 - \sin^2 u$ to get

$$\int \cos^3(2x) \sin(2x)\, dx = \frac{1}{2} \int \cos^2 u \sin u \cos u\, du = \frac{1}{2} \int (1 - w^2) w\, dw$$

$$= \frac{1}{2} \int w - w^3\, dw = \frac{1}{2} \left(\frac{w^2}{2} - \frac{w^4}{4} \right) + c = \frac{1}{2} \left(\frac{\sin^2 u}{2} - \frac{\sin^4 u}{4} \right) + c$$

$$= \frac{1}{2} \left(\frac{\sin^2(2x)}{2} - \frac{\sin^4(2x)}{4} \right) + c.$$

Method 2: Substitute $w = \cos u \implies dw = -\sin u\, du$.

$$\int \cos^3(2x) \sin(2x)\, dx = \frac{1}{2} \int w^3 (-\, dw) = -\frac{1}{2} \int w^3\, dw = -\frac{1}{2} \frac{w^4}{4} + c$$

$$= -\frac{1}{8} \cos^4 u + c = -\frac{1}{8} \cos^4(2x) + c.$$

(b) Since the exponents of sin and cos are both odd there are two possibilities.

Method 1: Substitute $u = \sin t \implies du = \cos t\, dt$ and use $\cos^2 t = 1 - \sin^2 t$ to get

$$\int \cos^3 t \sin^3 t\, dt = \int \cos^2 t \sin^3 t \cos t\, dt = \int (1 - u^2) u^3\, du = \int u^3 - u^5\, du$$

$$= \frac{u^4}{4} - \frac{u^6}{6} + c = \frac{\sin^4 t}{4} - \frac{\sin^6 t}{6} + c.$$

Method 2: Substitute $u = \cos t \Longrightarrow du = -\sin t\, dt$ and use $\sin^2 t = 1 - \cos^2 t$.

$$\int \sin^3 t \cos^3 t\, dt = \int \sin^2 t \cos^3 t \sin t\, dt = \int (1-u^2)u^3(-du) = \int -u^3 + u^5\, du$$

$$= -\frac{u^4}{4} + \frac{u^6}{6} + c = -\frac{\cos^4 t}{4} + \frac{\cos^6 t}{6} + c.$$

(c) Since the exponent of cos is odd we substitute $u = \sin t \Longrightarrow du = \cos t\, dt$ and use $\cos^2 t = 1 - \sin^2 t$.

$$\int \sin^2 t \cos^3 t\, dt = \int \sin^2 t \cos^2 t \cos t\, dt = \int u^2(1-u^2)\, du$$

$$= \int u^2 - u^4\, du = \frac{u^3}{3} - \frac{u^5}{5} + c = \frac{\sin^3 t}{3} - \frac{\sin^5 t}{5} + c.$$

(d) Since the exponents of sin and cos are both even we convert to powers of cos using $\sin^2 t = 1 - \cos^2 t$.

$$\int \sin^2 t \cos^4 t\, dt = \int (1-\cos^2 t)\cos^4 t\, dt = \int \cos^4 t - \cos^6 t\, dt$$

$$= \frac{1}{4}\cos^3 t \sin t + \frac{3}{4}\int \cos^2\, dt - \left(\frac{1}{6}\cos^5 t \sin t + \frac{5}{6}\int \cos^4 t\, dt\right)$$

$$= \frac{1}{4}\cos^3 t \sin t + \frac{3}{4}\left(\frac{1}{2}\cos t \sin t + \frac{1}{2}\int 1\, dt\right) - \left(\frac{1}{6}\cos^5 t \sin t + \frac{5}{6}\int \cos^4 t\, dt\right)$$

$$= \frac{1}{4}\cos^3 t \sin t + \frac{3}{4}\left(\frac{1}{2}\cos t \sin t + \frac{1}{2}t\right) - \frac{1}{6}\cos^5 t \sin t - \frac{5}{6}\left(\frac{1}{4}\cos^3 t \sin t + \frac{3}{4}\int \cos^2 t\, dt\right)$$

$$= \frac{1}{4}\cos^3 t \sin t + \frac{3}{8}\cos t \sin t + \frac{3}{8}t - \frac{1}{6}\cos^5 t \sin t - \frac{5}{24}\cos^3 t \sin t - \frac{5}{8}\int \cos^2 t\, dt$$

$$= \frac{1}{4}\cos^3 t \sin t + \frac{3}{8}\cos t \sin t + \frac{3}{8}t - \frac{1}{6}\cos^5 t \sin t - \frac{5}{24}\cos^3 t \sin t - \frac{5}{8}\left(\frac{1}{2}\cos t \sin t + \frac{1}{2}t\right) + c$$

22. (a) The denominators are powers of 2, starting with $a_1 = 1 = 1/2^0$. The formula is $a_n = 1/2^{n+1}$. Since $\lim a^n = 0$ for $|a| < 1$ and $1/2 < 1$, we get

$$\lim_n \frac{1}{2^{n+1}} = \lim \frac{1}{2}\left(\frac{1}{2}\right)^n = \frac{1}{2}\cdot 0 = 0.$$

(b) We notice that $6 = 1\cdot 2\cdot 3$, $24 = 1\cdot 2\cdot 3\cdot 4$, $120 = 1\cdot 2\cdot 3\cdot 4\cdot 5$. The denominator of a_n is the product of numbers up to n. This is denoted by $n!$, with the convention that $1! = 1$. So $a_n = 1/n!$. To find the limit we notice that $n! > n$ as long as $n > 2$, since $n!$ contains the factor n. Consequently $1/n! < 1/n$. Using the comparison theorem with $b_n = 1/n$, i.e. $|a_n| < b_n$, we get that

$$\lim_n \frac{1}{n!} = 0.$$

(c) We notice that both numerators and denominators are squares: $a_1 = 1^2/2^2$, $a_2 = 2^2/3^2$, $a_3 = 3^2/4^2$, etc. The formula is $a_n = n^2/(n+1)^2$.

$$\lim_n \frac{n^2}{(n+1)^2} = \left(\lim_n \frac{n}{n+1}\right)^2 = \left(\lim_n \frac{n}{n(1+1/n)}\right)^2$$

$$= \left(\lim_n \frac{1}{1+1/n}\right)^2 = \left(\frac{1}{1+\lim 1/n}\right)^2 = \left(\frac{1}{1+0}\right)^2 = 1.$$

23. The first is a geometric series with ratio $r = 1.055 > 1$, Consequently it diverges. The second is a geometric series with ratio $r = 0.9$ and first term $a = 0.9^0 = 1$. Since $|r| < 1$ the geometric series converges and its sum is

$$\frac{a}{1-r} = \frac{1}{1-0.9} = \frac{1}{0.1} = 10.$$

The last series is not geometric. It reminds us of the series $\sum_n b_n = \frac{1}{n(n+1)}$. So we try to write the expresion $6/(n(n+3)$ as a sum of two fractions with denominators n and $n+3$.

$$\frac{6}{n(n+3)} = \frac{A}{n} + \frac{B}{n+3} = \frac{A(n+3)}{n(n+3)} + \frac{Bn}{n(n+3)} = \frac{A(n+3)+Bn}{n(n+3)}.$$

It will work out as long as for all n we have the numerators to agree

$$A(n+3) + Bn = 6$$

We plug $n = 0$ to get $3A = 6 \Longrightarrow A = 2$. Plugging now $n = -3$ we get $B(-3) = 6 \Longrightarrow B = -2$. Since

$$\frac{6}{n(n+3)} = \frac{2}{n} - \frac{2}{n+3}$$

we get

$$\begin{aligned}
\frac{6}{1(1+3)} &= \frac{2}{1} - \frac{2}{4} \\
\frac{6}{2(2+3)} &= \frac{2}{2} - \frac{2}{5} \\
\frac{6}{3(3+3)} &= \frac{2}{3} - \frac{2}{6} \\
\frac{6}{4(4+3)} &= \frac{2}{4} - \frac{2}{7} \\
\cdots & \quad \cdots \\
\frac{6}{(n-3)n} &= \frac{2}{n-3} - \frac{2}{n} \\
\frac{6}{(n-2)(n+1)} &= \frac{2}{n-2} - \frac{2}{n+1} \\
\frac{6}{(n-1)(n+2)} &= \frac{2}{n-1} - \frac{2}{n+2}
\end{aligned}$$

$$\frac{6}{n(n+3)} = \frac{2}{n} - \frac{2}{n+3}$$

We notice that lots of terms cancel when we sum up. This is the feature of telescoping series. We get

$$\frac{6}{1(1+3)} + \frac{6}{2(2+3)} + \cdots + \frac{6}{n(n+3)} = \frac{2}{1} + \frac{2}{2} + \frac{2}{3} - \frac{2}{n+1} - \frac{2}{n+2} - \frac{2}{n+3}.$$

The last three terms tend to 0 and we get for the sum of the series

$$2 + 1 + \frac{2}{3} = 3\frac{2}{3}.$$

24. The function $f(x) = \dfrac{1}{x \ln x}$ is decreasing and positive for $x \geq 2$. So we can apply the integral test. We look at the improper integral

$$\int_2^\infty \frac{1}{x \ln x}\, dx = \int_{\ln 2}^\infty \frac{1}{u}\, du$$

by using the substitution $u = \ln x \implies du = dx/x$. We also changed the limits of integration.

$$\int_{\ln 2}^\infty \frac{1}{u}\, du = \lim_{M\to\infty} \int_{\ln 2}^M \frac{1}{u} du = \lim_{M\to\infty} [\ln u]_{\ln 2}^M = \lim_{M\to\infty} (\ln M - \ln\ln 2) = \infty$$

The improper integral diverges, so the series diverges.

The function $g(x) = \dfrac{1}{x(\ln x)^2}$ is decreasing and positive for $x \geq 2$. So we can apply the integral test. We look at the improper integral

$$\int_2^\infty \frac{1}{x(\ln x)^2}\, dx = \int_{\ln 2}^\infty \frac{1}{u^2}\, du$$

by using the substitution $u = \ln x \implies du = dx/x$. We also changed the limits of integration.

$$\int_{\ln 2}^\infty \frac{1}{u^2}\, du = \lim_{M\to\infty} \int_{\ln 2}^M \frac{1}{u^2} du = \lim_{M\to\infty} [-1/u]_{\ln 2}^M = \lim_{M\to\infty} (-1/M + 1/\ln 2) = 1/\ln 2.$$

The improper integral converges, so the series converges.

25. We set $f(x) = \sec x$ and compute $f'(x) = \sec x \tan x$, $f''(x) = (\sec x)' \tan x + \sec x (\tan x)' = \sec x \tan x \tan x + \sec x \sec^2 x = \sec x \tan^2 x + \sec^3 x$. We calculate $f(\pi/4) = \sec \pi/4 = \sqrt{2}$, $f'(\pi/4) = \sqrt{2} \cdot 1 = \sqrt{2}$, $f''(x) = \sqrt{2} \cdot 1^2 + \sqrt{2}^3 = \sqrt{2} + 2\sqrt{2} = 3\sqrt{2}$.

$$T_2(x) = f(\pi/4) + f'(\pi/4)(x-\pi/4) + \frac{1}{2} f''(\pi/4)(x-\pi/4)^2 = \sqrt{2} + \sqrt{2}(x-\pi/4) + \frac{3}{2}\sqrt{2}(x-\pi/4)^2$$

26. Since the Taylor series of $\sin t$ around $t = 0$ is

$$\sin t = t - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!} + \cdots = \sum_{m=0}^{\infty} (-1)^m \frac{t^{2m+1}}{(2m+1)!}$$

$$\frac{\sin t}{t} = 1 - \frac{t^2}{3!} + \frac{t^4}{5!} - \frac{t^6}{7!} + \cdots = \sum_{m=0}^{\infty} (-1)^m \frac{t^{2m}}{(2m+1)!}$$

We integrate the series termwise to get

$$\int_0^x \frac{\sin t}{t}\,dt = \int_0^x 1 dt - \int_0^x \frac{t^2}{3!}\,dt + \int_0^x \frac{t^4}{5!}\,dt - \int_0^x \frac{t^6}{7!}\,dt + \cdots = \sum_{m=0}^{\infty} \int_0^x (-1)^m \frac{t^{2m}}{(2m+1)!}\,dt$$

$$= [t]_0^x - \left[\frac{t^3}{3\cdot 3!}\right]_0^x + \left[\frac{t^5}{5\cdot 5!}\right]_0^x - \left[\frac{t^7}{7\cdot 7!}\right]_0^x = \sum_{m=0}^{\infty} (-1)^m \left[\frac{t^{2m+1}}{(2m+1)(2m+1)!}\right]_0^x$$

$$= x - \frac{x^3}{3\cdot 3!} + \frac{x^5}{5\cdot 5!} - \frac{x^7}{7\cdot 7!} + \cdots = \sum_{m=0}^{\infty} (-1)^m \frac{x^{2m+1}}{(2m+1)(2m+1)!}$$

Quiz 1

1. The indefinite integral
$$\int (x^2-4)x+\cos x\,dx$$
is equal to

   A. $\dfrac{x^4}{4}-\sin x+c$ B. $\dfrac{x^3}{3}-2x^2+\sin x+c$

   C. $\dfrac{x^4}{4}-2x^2+\sin x+c$ D. $\dfrac{x^3}{3}-2x^2-\sin x+c$

2. The indefinite integral
$$\int \sec^2 x+\frac{6}{x}\,dx$$
is equal to

   A. $\dfrac{\sec^3 x}{3}+6+c$ B. $\tan x+6\ln|x|+c$

   C. $\dfrac{\sec^3 x}{3}+6\ln|x|+c$ D. $\tan x+6+c$

3. Your speed is given by the table

t sec	speed
0	5
1	7
2	10
3	20

   Then

   A. LHS(3) = 22 and RHS(3) = 37 B. LHS(3) = 42 and RHS(3) = 37

   B. LHS(3) = 37 and RHS(3) = 42 D. LHS(3) = 22 and RHS(3) = 42

4. Your speed is given by the function $v(t)=3t^2+5$. Then your acceleration $a(t)$ and the distance $s(t)$ you travel are given by

   A. $a(t)=t^3+5t+c$ and $s(t)=6t$ B. $a(t)=6t+c$ and $s(t)=t^3+3t$

   C. $a(t)=t^3+5t$ and $s(t)=6t+c$ D. $a(t)=6t$ and $s(t)=t^3+5t+c$

Answers:
1. C 2. B 3. A 4.D

Quiz 2

1. The definite integral
$$\int_1^4 \frac{1}{x}\,dx$$
is equal to

   A. $\frac{15}{2}$ B. 8

   C. $\ln 4$ D. $\ln 4 + c$

2. The definite integral
$$\int_0^{\pi/2} \cos x + x\,dx$$
is equal to

   A. $\frac{\pi^2}{8}$ B. $\frac{\pi^2}{8} + 1$

   C. 1 D. $\frac{\pi}{2} + 1$

3. The integral
$$\int_{-2}^{0} \sqrt{4 - x^2}\,dx$$
is equal to

   A. 4π B. $\pi/2$

   B. 2π D. π

   Hint: The integral represents a certain area. Which area?

4. If $F(x) = \int_3^x \sqrt{1+t^3}\,dt$, then $F'(x)$ equals

   A. $\sqrt{1+x^3}$ B. $\sqrt{1+t^3}$

   C. $\frac{(1+x^3)^{3/2}}{3/2} + c$ D. $x + \frac{2}{5}x^{5/2} + c$

5. The sum $\sum_{i=1}^{2000} i$ equals

   A. 2001000 B. 2000^2

   C. 1002000 D. $2000 \cdot 2001$

Answers:
1. C 2. B 3. D 4. A 5. A

Quiz 3

1. If the washer in the picture has thickness Δy, then its volume is

   A. $2\pi(R^2 - r^2)\Delta y$ B. $\pi(R^2 - r^2)\Delta y$

   C. $2\pi(R - r)\Delta y$ D. $2\pi(R - r)(\Delta y)^2$

2. The volume of the cylindrical shell of thickness Δx in the picture is

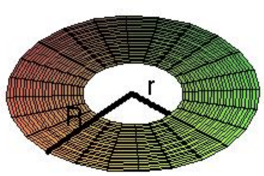
   A. $\pi x^2 H \Delta x$ B. $\pi H^2 x \Delta x$

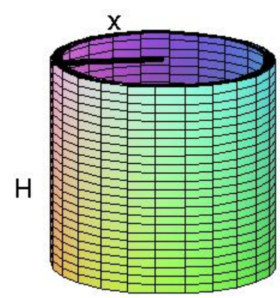
   C. $2\pi x^2 H \Delta x$ D. $2\pi x H \Delta x$

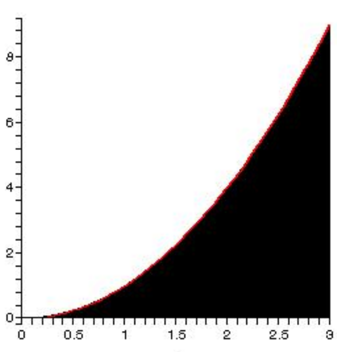
3. The following is the graph the function $y = x^2$ in the interval $0 \leq x \leq 3$. We rotate the curve around the x-axis. The volume of the solid we get is

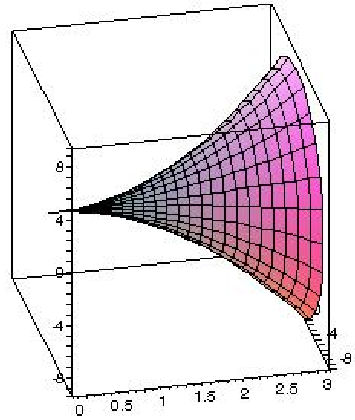
   A. $\pi \frac{3^5}{5}$ B. 9π

   C. $2\pi \frac{3^5}{5}$ D. 18π









Answers:
1.B 2. D 3. A

Exam 1

1. Evaluate the integrals. (5 points each)

$$\text{(a)} \int_0^1 x(x^{2004} + x^{998})\,dx$$

$$\text{(b)} \int e^\theta \sin(e^\theta + 1)\,d\theta$$

$$\text{(c)} \int \frac{x+5}{x^2+1}\,dx$$

$$\text{(d)} \int \frac{x^3+x}{x^4+2x^2+5}\,dx$$

2. Find the derivative of the function (4 points)

$$F(x) = 3 + \int_0^x \frac{1}{\sqrt{1-u^4}}\,du.$$

3. Compute the area bounded by the parabola $y = x^2$ and the parabola $y = -x^2 + 2$. *Hint:* Sketch and shade the region between the two curves. (5 points)

4. Compute the left-hand sum, the right-hand sum, and the midpoint rule with $n = 2$ subintervals for the integral: (6 points)

$$\int_0^4 x^2\,dx.$$

5. Graph the region of integration for the integral

$$\int_0^{3/2} (\sqrt{9-x^2} - x)\,dx$$

and evaluate the integral.

Hint: You cannot find an antiderivative. (5 points)

Exam 1: Solutions

1. Evaluate the integrals. (5 points each)

$$\text{(a)} \int_0^1 x(x^{2004} + x^{998})\,dx$$

$$\int_0^1 x(x^{2004}+x^{998})\,dx = \int_0^1 (x^{2005}+x^{999})\,dx = \left[\frac{x^{2006}}{2006} + \frac{x^{1000}}{1000}\right]_0^1 = \frac{1^{2006}}{2006}+\frac{1^{1000}}{1000} = \frac{1}{2006}+\frac{1}{1000}.$$

$$\text{(b)} \quad \int e^{\theta} \sin(e^{\theta}+1)\, d\theta$$

We substitute $u = e^{\theta} + 1$, which gives $\dfrac{du}{d\theta} = e^{\theta} \Longrightarrow du = e^{\theta}\, d\theta$.

$$\int e^{\theta} \sin(e^{\theta}+1)\, d\theta = \int \sin(u)\, du = -\cos(u) + c = -\cos(e^{\theta}+1) + c.$$

$$\text{(c)} \quad \int \frac{x+5}{x^2+1}\, dx$$

Since $(x^2+1)' = 2x$, we split the integrand as a sum and substitute $u = x^2 + 1$ in the first integral. This gives $du = 2x\, dx$.

$$\int \frac{x+5}{x^2+1}\, dx = \int \frac{x}{x^2+1}\, dx + \int \frac{5}{x^2+1}\, dx = \int \frac{x}{x^2+1} + 5\arctan(x)$$

$$= \frac{1}{2}\int \frac{1}{u}\, du + \arctan(x) = \frac{1}{2}\ln|u| + \arctan(x) = \frac{1}{2}\ln(x^2+1) + \arctan(x) + c.$$

$$\text{(d)} \quad \int \frac{x^3+x}{x^4+2x^2+5}\, dx$$

Since $(x^4+2x^2+5)' = 4x^3+4x$, which is four times the numerator, we substitute $u = x^4+2x^2+5$. This gives $du = 4(x^3+x)\, dx$.

$$\int \frac{x^3+x}{x^4+2x^2+5}\, dx = \frac{1}{4}\int \frac{1}{u}\, du = \frac{1}{4}\ln|u| + c = \frac{1}{4}\ln(x^4+2x^2+5) + c.$$

2. Find the derivative of the function (4 points)

$$F(x) = 3 + \int_0^x \frac{1}{\sqrt{1-u^4}}\, du.$$

Answer: Using the second fundamental theorem of calculus we get $F'(x) = 0 + 1/\sqrt{1-x^4} = 1/\sqrt{1-x^4}$.

3. Compute the area bounded by the parabola $y = x^2$ and the parabola $y = -x^2 + 2$. *Hint:* Sketch and shade the region between the two curves. (5 points)
We find the points of intersection of the two parabolas:

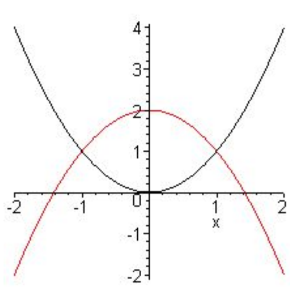


Figure 17: The two parabolas

$$x^2 = -x^2 + 2 \Longrightarrow 2x^2 = 2 \Longrightarrow x^2 = 1 \Longrightarrow x = \pm 1.$$

The area is

$$\int_{-1}^{1} (-x^2+2)-x^2\,dx = \int_{-1}^{1} -2x^2+2\,dx = \left[-\frac{2x^3}{3} + 2x\right]_{-1}^{1} = -\frac{2\cdot 1^3}{3}+2\cdot 1-\left(-\frac{2\cdot(-1)^3}{3} + 2(-1)\right)$$

$$= -\frac{2}{3} + 2 - \left(\frac{2}{3} - 2\right) == -\frac{2}{3} + 2 - \frac{2}{3} + 2 = 4 - \frac{4}{3} = \frac{12}{3} - \frac{4}{3} = \frac{8}{3}.$$

4. Compute the left-hand sum, the right-hand sum, and the midpoint rule with $n = 2$ subintervals for the integral: $\int_0^4 x^2\,dx$.

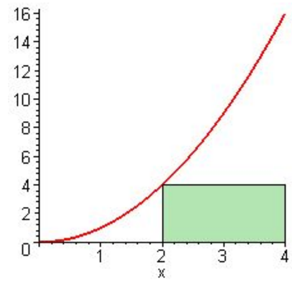


Figure 18: LHS(2)

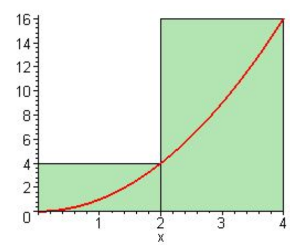


Figure 19: RHS(2)

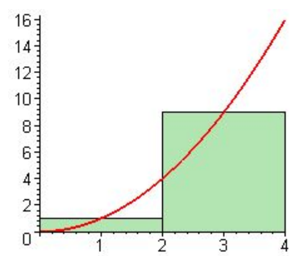


Figure 20: MID(2)

$$\Delta x = (4-0)/2 = 2$$

$$\text{LHS}(2) = 0^2 \cdot 2 + 2^2 \cdot 2 = 8, \quad \text{RHS}(2) = 2^2 \cdot 2 + 4^2 \cdot 2 = 8 + 32 = 40.$$

$$\text{MID}(2) = 1^2 \cdot 2 + 3^2 \cdot 2 = 2 + 18 = 20.$$

5. Graph the region of integration for the integral

$$\int_0^{3/2} (\sqrt{9-x^2} - x)\,dx$$

and evaluate the integral.

$$\int_0^{3/2} (\sqrt{9-x^2} - x)\,dx = \int_0^{3/2} \sqrt{9-x^2}\,dx - \int_0^{3/2} x\,dx = \int_0^{3/2} \sqrt{9-x^2}\,dx - \left[\frac{x^2}{2}\right]_0^{3/2}$$

$$= \int_0^{3/2} \sqrt{9-x^2}\,dx - \frac{9/4}{2} = \int_0^{3/2} \sqrt{9-x^2}\,dx - \frac{9}{8}.$$

So we need to compute the integral $\int_0^{3/2} \sqrt{9-x^2}\,dx$. This represents the total area of the regions which are shaded in the picture. We find the area of the sector (lightly-shaded) first. Since the hypotenuse of the triangle is 3 and the horizontal side is 3/2, the sector spans an angle

$$\pi/2 - \arccos\left(\frac{3/2}{3}\right) = \frac{\pi}{2} - \arccos(1/2) = \frac{\pi}{2} - \frac{\pi}{3} = \frac{\pi}{6}.$$

The area of the sector is proportional to the angle, so its area is

$$\pi R^2 \frac{\pi/6}{2\pi} = \pi 3^2 \frac{1}{12} = \pi \frac{9}{12} = \frac{3\pi}{4}.$$

For the triangle (darkly-shaded) we need the height h. Since the angle is $\pi/3$, trigonometry gives

$$\sin(\pi/3) = h/3 \Longrightarrow \frac{\sqrt{3}}{2} = \frac{h}{3} \Longrightarrow h = \frac{3\sqrt{3}}{2}.$$

The area of the triangle is $B \times h \times 1/2 = 3/2 \times 3\sqrt{3}/2 \times 1/2 = 9\sqrt{3}/8$. The given integral is equal to $3\pi/4 + 9\sqrt{3}/8 - 9/8$.

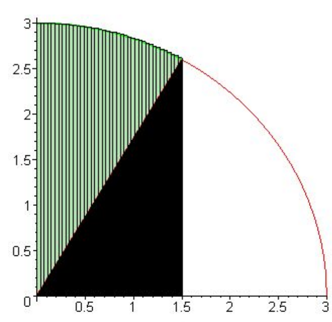


Figure 21: The circle and line of problem 5

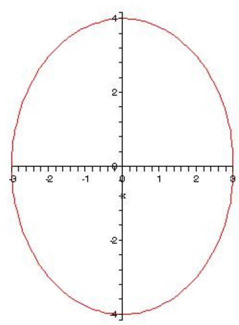
Exam 2

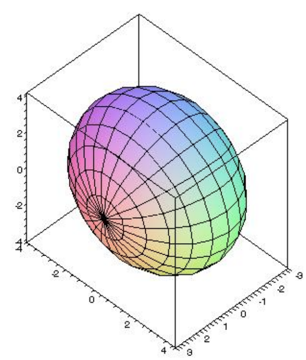
1. Evaluate the integrals. (6 points each)

$$\text{(a)} \int x\cos(x)\,dx, \qquad \text{(b)} \int (x^3-1)^4 x^2\,dx, \qquad \text{(c)} \int \arctan x\,dx,$$

$$\text{(d)} \int \sin^2 t\cos^3 t\,dt \quad (\text{use } u=\sin t), \qquad \text{(e)} \int \frac{e^x}{e^{2x}+1}dx.$$

2. When the ellipse $\dfrac{x^2}{9}+\dfrac{y^2}{16}=1$ is rotated around the x-axis we get a solid called ellipsoid. Compute its volume. (10 points)





3. The eruption of Mt. St. Helens in Washington State in 1980 deposited ash in a vast area. We can assume this area is a circular disc with center the crater of Mt. St. Helens and radius 30 km. The density of the ash found at a distance x km from the crater is given by the formula

$$\rho(x)=\frac{10{,}000}{x^2+1}\ \text{ton/km}^2.$$

(a) Write a Riemann sum that approximates the total mass of the ash deposited on the area.

(b) Find the exact total mass of ash deposited in the area above. (10 points)

Exam 2: Solutions

3. Evaluate the integrals.

$$\text{(a)} \int x\cos(x)\,dx = x\sin(x) - \int 1\cdot\sin(x)\,dx = x\sin(x)+\cos(x)+c$$

using integration by parts with $u=x \Longrightarrow u'=1$ and $v'=\cos(x) \Longrightarrow v=\sin(x)$.

$$\text{(b)} \quad \int (x^3-1)^4 x^2\,dx$$

Substitution $u = x^3 - 1 \Longrightarrow du = 3x^2\,dx$. This gives

$$\int (x^3-1)^4 x^2\,dx = \int u^4 \frac{1}{3} du = \frac{1}{3}\int u^4\,du = \frac{1}{3}\frac{u^5}{5} + c = \frac{(x^3-1)^5}{15} + c.$$

$$\text{(c)} \quad \int \arctan x\,dx,$$

Integration by parts with $u = \arctan(x) \Longrightarrow u' = 1/(1+x^2)$ and $v' = 1 \Longrightarrow v = x$.

$$\int \arctan(x)\,dx = \int 1{\cdot}\arctan(x)\,dx = x\arctan(x) - \int \frac{x}{1+x^2}\,dx = x\arctan(x) - \frac{1}{2}\int \frac{1}{u}\,du$$

where we substitute $u = 1 + x^2 \Longrightarrow du = 2x\,dx$

$$= x\arctan(x) - \frac{1}{2}\ln|u| + c = x\arctan(x) - \frac{1}{2}\ln(1+x^2) + c.$$

$$\text{(d)} \quad \int \sin^2 t\cos^3 t\,dt \quad (\text{use } u = \sin t)$$

Since the exponent of cos is odd we substitute $u = \sin t \Longrightarrow du = \cos t\,dt$ and use $\cos^2 t = 1 - \sin^2 t$.

$$\int \sin^2 t\cos^3 t\,dt = \int \sin^2 t\cos^2 t\cos t\,dt = \int u^2(1-u^2)\,du$$

$$= \int u^2 - u^4\,du = \frac{u^3}{3} - \frac{u^5}{5} + c = \frac{\sin^3 t}{3} - \frac{\sin^5 t}{5} + c.$$

$$\text{(e)} \quad \int \frac{e^x}{e^{2x}+1}dx = \int \frac{1}{u^2+1}\,du = \arctan(u) + c = \arctan(e^x) + c,$$

using the substitution $u = e^x \Longrightarrow du = e^x\,dx$.

3. When the ellipse $\dfrac{x^2}{9} + \dfrac{y^2}{16} = 1$ is rotated around the x-axis we get a solid called ellipsoid. Compute its volume. (10 points)

We solve for y^2 to get

$$y^2 = 16\left(1 - \frac{x^2}{9}\right).$$

We use the method of discs. The volume of the fattened disc is

$$\pi y^2 \Delta x = \pi \cdot 16(1 - x^2/9)\Delta x.$$

This gives the Riemann sum $\sum_i 16\pi(1 - x^2/9)\Delta x$ and then the integral

$$\int_{-3}^{3} 16\pi\left(1 - \frac{x^2}{9}\right)dx = 16\pi\left[x - \frac{x^3}{27}\right]_{-3}^{3} = 16\pi\left(3 - \frac{27}{27}\right) - 16\pi\left(-3 - \frac{-27}{27}\right)$$

$$= 16\pi\,(2 - (-2)) = 64\pi.$$

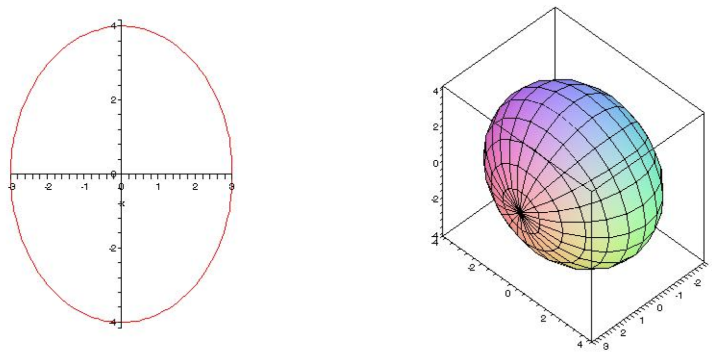


Figure 22: The ellipse and the ellipsoid

3. The eruption of Mt. St. Helens in Washington State in 1980 deposited ash in a vast area. We can assume this area is a circular disc with center the crater of Mt. St. Helens and radius 30 km. The density of the ash found at a distance x km from the crater is given by the formula

$$\rho(x) = \frac{10,000}{x^2+1}\,\text{ton/km}^2.$$

(a) Write a Riemann sum that approximates the total mass of the ash deposited on the area.

(b) Find the exact total mass of ash deposited in the area above.

The annular region has area approximately the area of the strip. The length of the circle of radius x is $2\pi x$ and the area of the strip is $2\pi x \Delta x$. The mass of the ass deposited in the ring is the area of the ring times the density of the ash:

$$\rho(x) \cdot 2\pi x \Delta x = \frac{10000}{x^2+1} 2\pi x \Delta x.$$

This gives the Riemann sum

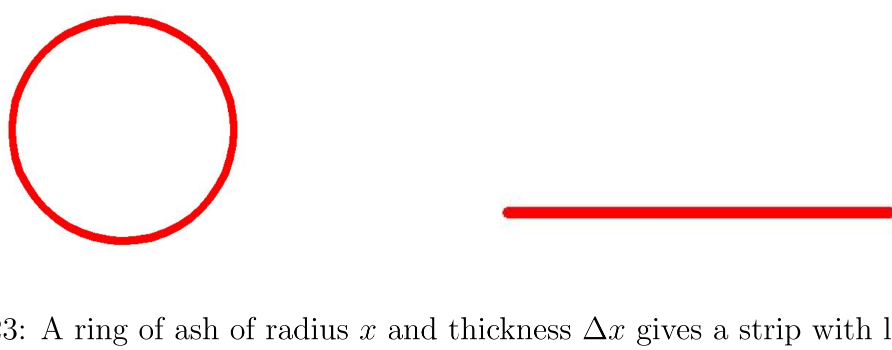


Figure 23: A ring of ash of radius x and thickness Δx gives a strip with length $2\pi x$

$$\sum_i \frac{10000}{x^2+1} 2\pi x \Delta x$$

and finally the integral

$$\int_0^{30} \frac{10000}{x^2+1} 2\pi x\, dx = 10000\pi \int_0^{30} \frac{2x}{x^2+1}\, dx$$

$$= 10000\pi[\ln(x^2+1)]_0^{30} = 10000\pi[\ln(30^2+1) - \ln(1)] = 10000\pi \ln 901.$$

Final Exam

1. Evaluate the integrals. (7 points each)

$$\text{(a)} \int \frac{x^2-1}{x^3} dx, \quad \text{(b)} \int xe^x\, dx, \quad \text{(c)} \int \arcsin(x)\, dx, \quad \text{(d)} \int \frac{1}{x^2+2x+2}\, dx$$

$$\text{(e)} \int x^2 \cos(x^3+2)\, dx, \quad \text{(f)} \int \sin^3 x \cos^3 x\, dx, \quad \text{(g)} \int \frac{1}{x\sqrt{\ln x}}\, dx.$$

2. (12 points) Compute the left-hand sum, the right-hand sum, the midpoint rule and the trapezoid rule with $n = 2$ subintervals for the integral

$$\int_0^4 f(x)\, dx,$$

where the function $f(x)$ has the following table of values:

x	$f(x)$
0	10
1	9
2	7
3	4
4	0

Hint: Sketch the areas representing the sums you need to compute.

3. Find the area bounded between the parabola $y = 4-x^2$ and the line $y = x+2$. Sketch the region of integration. (13 points)

4. (20 points) Do either of the following two problems.

(a) Humpty Dumpty looks like an egg. Humpty Dumpty can be considered as a volume of revolution. His profile can be described by the equations:

$$\frac{x^2}{1600} + \frac{y^2}{2500} = 1$$

for the upper part of his body above the x-axis (it is hard to say whether this is his waist or his neck) and

$$x^2 + y^2 = 1600$$

for the lower part of his body (below the x-axis). All numbers in this problem are in centimeters.

Find the volume of Humpty Dumpty.

(b) Humpty Dumpty decided to eat a donut. We get this donut by rotating around the y-axis the circle with equation

$$(x-2)^2 + y^2 = 1.$$

Compute the volume of the donut.

Figure 24: Alice meets Humpty Dumpty (Illustration to Chapter 6 of Through the Looking-Glass) by John Tenniel. Wood-engraving by the Dalziels.

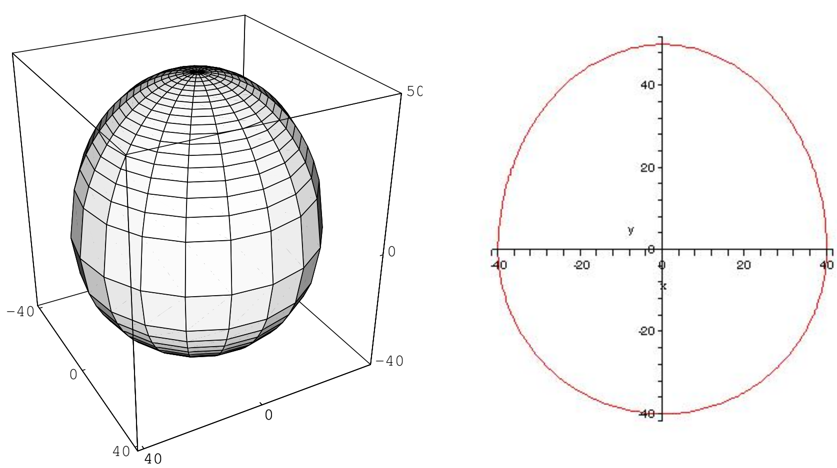


Figure 25: Humpty Dumpty

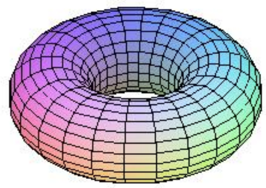


Figure 26: The donut

5. Decide whether the following series converge or diverge. If they converge find their sum. (6 points)

$$\sum_{n=0}^{\infty} 3 \cdot \left(\frac{1}{2}\right)^n, \qquad \sum_{n=1}^{\infty} \left(\frac{3}{2}\right)^n.$$

Final Exam: Solutions

May 26, 2004

1. Evaluate the integrals. (7 points each)

$$\text{(a)} \int \frac{x^2-1}{x^3} dx = \int \frac{x^2}{x^3} - \frac{1}{x^3}\, dx = \int (x^{-1} - x^{-3})\, dx = \ln|x| - \frac{x^{-2}}{-2} + c.$$

$$\text{(b)} \int xe^x\, dx = xe^x - \int 1 \cdot e^x\, dx = xe^x - e^x + c,$$

using integration by parts with $u = x \Longrightarrow u' = 1$, $v' = e^x \Longrightarrow v = e^x$.

$$\text{(c)} \int \arcsin(x)\, dx = x\arcsin(x) - \int x \frac{1}{\sqrt{1-x^2}}\, dx$$

using integration by parts with $u = \arcsin(x) \Longrightarrow u' = 1/\sqrt{1-x^2}$, $v' = 1 \Longrightarrow v = x$. Now we use substitution with $u = 1 - x^2 \Longrightarrow du = -2x\, dx \Longrightarrow x\, dx = -du/2$.

$$\int \arcsin(x)\, dx = x\arcsin(x) - \int \frac{-du/2}{\sqrt{u}} = x\arcsin(x) + \frac{1}{2}\int u^{-1/2}\, du$$

$$= x\arcsin(x) + \frac{1}{2}\frac{u^{1/2}}{1/2} + c = x\arcsin(x) + (1-x^2)^{1/2} + c$$

$$\text{(d)} \int \frac{1}{x^2+2x+2}\,dx = \int \frac{1}{(x+1)^2+1}\,dx$$

by completing the square. We substitute $u = x+1 \Longrightarrow du = dx$ to get

$$\int \frac{1}{x^2+2x+2}\,dx = \int \frac{1}{u^2+1}\,du = \arctan(u)+c = \arctan(x+1)+c.$$

$$\text{(e)} \int x^2\cos(x^3+2)\,dx = \int \cos(u)\frac{1}{3}\,du$$

using the substitution $u = x^3+2 \Longrightarrow du = 3x^2\,dx \Longrightarrow du/3 = x^2 dx$. This gives

$$\int x^2\cos(x^3+2)\,dx = \frac{1}{3}\sin(u)+c = \frac{1}{3}\sin(x^3+2)+c.$$

$$\text{(f)} \int \sin^3 x\cos^3 x\,dx = \int \sin^3 x\cos^2 x\cdot\cos x\,dx = \int \sin^3 x(1-\sin^2 x)\cdot\cos x\,dx$$

using the trigonometric identity $\cos^2 x = 1-\sin^2 x$. Now we substitute $u = \sin x \Longrightarrow du = \cos x\,dx$.

$$\int \sin^3 x\cos^3 x\,dx = \int u^3(1-u^2)du = \int u^3-u^5\,du = \frac{u^4}{4}-\frac{u^6}{6}+c = \frac{\sin^4 x}{4}-\frac{\sin^6 x}{6}+c$$

The substitution was motivated by the odd exponent of $\cos x$. However, here the exponent of $\sin x$ is also odd. This suggests an alternate substitution. We substitute $w = \cos x \Longrightarrow dw = -\sin x\,dx$.

$$\int \sin^3 x\cos^3 x\,dx = \int \sin^2 x\cos^3 x\cdot\sin x\,dx = \int (1-\cos^2 x)\cos^3 x\cdot\sin x\,dx$$

$$= \int (1-w^2)w^3(-dw) = \int -w^3+w^5\,dw = -\frac{w^4}{4}+\frac{w^6}{6}+c = -\frac{\cos^4 x}{4}+\frac{\cos^6 x}{6}+c$$

$$\text{(g)} \int \frac{1}{x\sqrt{\ln x}}\,dx = \int \frac{1}{\sqrt{u}}\,du = \int u^{-1/2}\,du = \frac{u^{1/2}}{1/2}+c$$
$$= 2u^{1/2}+c = 2(\ln(x))^{1/2}+c$$

using the substitution $u = \ln(x) \Longrightarrow du = dx/x$.

2. (12 points) Compute the left-hand sum, the right-hand sum, the midpoint rule and the trapezoid rule with $n=2$ subintervals for the integral

$$\int_0^4 f(x)\,dx,$$

where the function $f(x)$ has the following table of values:

x	$f(x)$
0	10
1	9
2	7
3	4
4	0

$$\Delta x = (4-0)/2 = 2$$

$$\text{LHS}(2) = f(0)\Delta x + f(2)\Delta x = 10 \cdot 2 + 7 \cdot 2 = 34.$$

$$\text{RHS}(2) = f(2)\Delta x + f(4)\Delta x = 7 \cdot 2 + 0 \cdot 2 = 14.$$

$$\text{MID}(2) = f(1)\Delta x + f(3)\Delta x = 9 \cdot 2 + 4 \cdot 2 = 26.$$

$$\text{TRAP}(2) = (\text{LHS}(2) + \text{RHS}(2))/2 = (34 + 14)/2 = 24.$$

Hint: Sketch the areas representing the sums you need to compute.

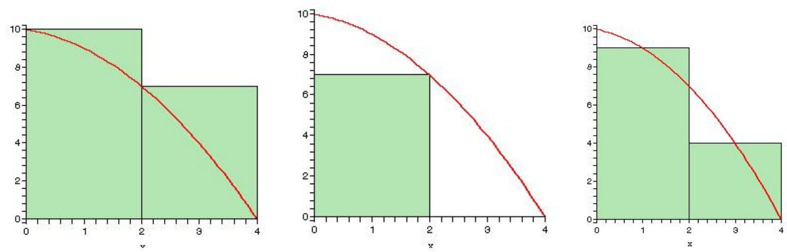


Figure 27: Left-hand sum, Right-hand sum, and Midpoint sum with $n = 2$

3. Find the area bounded between the parabola $y = 4 - x^2$ and the line $y = x + 2$. Sketch the region of integration. (13 points)

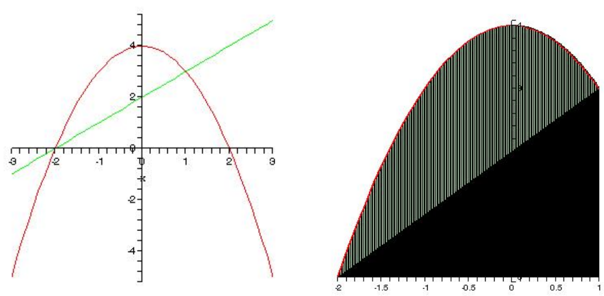


Figure 28: The parabola, the line and the region between them

First we find the points of intersection. We set

$$x + 2 = 4 - x^2 \Longrightarrow x^2 + x - 2 = 0 \Longrightarrow (x + 2)(x - 1) = 0 \Longrightarrow x = -2, x = 1.$$

From the graph we see that $y = 4 - x^2$ is larger than $y = x + 2$ in this range. Consequently, the area is equal to

$$\int_{-2}^{1} (4 - x^2) - (x + 2)\, dx = \int_{-2}^{1} 2 - x^2 - x\, dx = [2x - x^3/3 - x^2/2]_{-2}^{1} = (2 \cdot 1 - 1^3/3 - 1^2/2)$$

$$-(2(-2) - (-2)^3/3 - (-2)^2/2) = 2 - \frac{1}{3} - \frac{1}{2} + 4 - \frac{8}{3} + 2 = 8 - 3 - \frac{1}{2} = \frac{9}{2}.$$

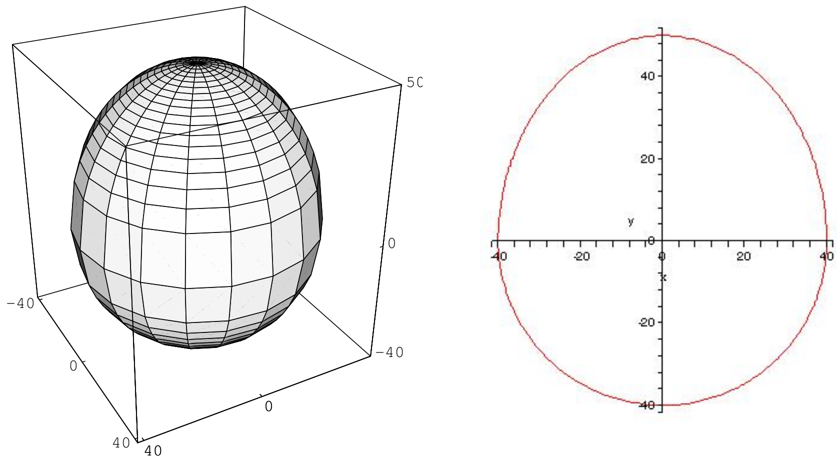


Figure 29: Humpty Dumpty

4. (20 points) Do either of the following two problems.

(a) Humpty Dumpty looks like an egg. Humpty Dumpty can be considered as a volume of revolution. His profile can be described by the equations:

$$\frac{x^2}{1600} + \frac{y^2}{2500} = 1$$

for the upper part of his body above the x-axis (it is hard to say whether this is his waist or his neck) and

$$x^2 + y^2 = 1600$$

for the lower part of his body (below the x-axis). All numbers in this problem are in centimeters.

Find the volume of Humpty Dumpty.

First method: The method of discs. The discs are horizontal. Each one has thickness Δy and radius x. The cross section has area πx^2 and the volume of the disc is $\pi x^2 \Delta y$. The relation between x and y is different for the upper part of Humpty Dumpty's body and for the lower part. In the upper part we get

$$\frac{x^2}{1600} + \frac{y^2}{2500} = 1 \Longrightarrow \frac{x^2}{1600} = 1 - \frac{y^2}{2500} \Longrightarrow x^2 = 1600(1 - y^2/2500).$$

For the lower part we get

$$x^2 + y^2 = 1600 \Longrightarrow x^2 = 1600 - y^2.$$

The volume of the upper part of Humpty Dumpty's body can be approximated by the Riemann sum

$$\sum \pi \cdot 1600 \left(1 - \frac{y^2}{2500}\right) \Delta y.$$

The horizontal discs extend from the waist of Humpty Dumpty $y = 0$ to the top of his head $y = 50$. We get this number by setting $x = 0$ into the equation $x^2/1600 + y^2/2500 = 1$. This gives the integral

$$\int_0^{50} \pi \cdot 1600 \left(1 - \frac{y^2}{2500}\right) dy = 1600\pi \int_0^{50} 1 - \frac{y^2}{2500}\, dy = 1600\pi \left[y - \frac{y^3}{3 \cdot 50^2}\right]_0^{50}$$

$$= 1600\pi \left[50 - \frac{50^3}{3 \cdot 50^2}\right] = 1600\pi(50 - 50/3) = 1600\pi \cdot 50(2/3) = 160000\pi/3.$$

The lower part of Humpty Dumpty gives us the Riemann sum

$$\sum \pi(1600 - y^2)\Delta y$$

and the discs extend from $y = -40$ to $y = 0$. The lowest point of Humpty Dumpty indeed occurs at $y = -40$ as we see by setting $x = 0$ in $x^2 + y^2 = 1600$. This gives as integral for the volume of the lower part of Humpty Dumpty

$$\int_{-40}^{0} \pi(1600-y^2)\, dy = \pi \left[1600y - \frac{y^3}{3}\right]_{-40}^{0} = -\pi(1600(-40)-(-40)^3/3) = 128000\pi/3.$$

The lower part of Humpty Dumpty is a hemisphere, so we do not really need the integral. As the volume of a sphere of radius R is $4\pi R^3/3$, we get $2\pi 40^3/3 = 128000\pi/3$ for the hemisphere. The total volume of Humpty Dumpty is $(160000+128000)\pi/3 = 96000\pi$.

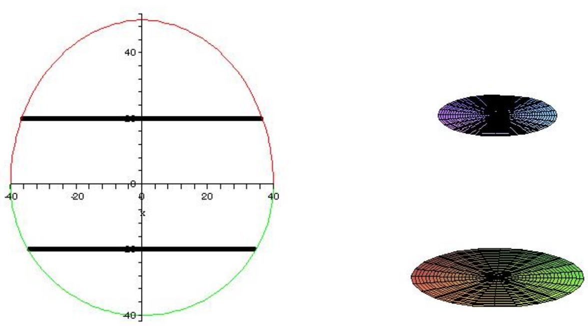


Figure 30: The volume of Humpty Dumpty using discs

Second method: The method of cylindrical shells. The radius of a typical shell is x, its perimeter $2\pi x$. The height depends on x. The upper edge of the shell goes as high as described by the equation $x^2/1600 + y^2/2500 = 1$, which gives, when we solve for y:

$$\frac{y^2}{2500} = 1-\frac{x^2}{1600} \Longrightarrow y^2 = 2500\left(1 - \frac{x^2}{1600}\right) \Longrightarrow y = 50\sqrt{1 - \frac{x^2}{1600}} = \frac{50}{40}\sqrt{1600 - x^2}.$$

The lower edge of the shell goes as low as described by the part of the equation $x^2 + y^2 = 1600$ below the x-axis. We solve for y to get

$$y^2 = 1600 - x^2 \Longrightarrow y = -\sqrt{1600 - x^2}.$$

The height of the shell is the difference of the heights of the two edges, i.e.

$$f_1(x)-f_2(x) = \frac{50}{40}\sqrt{1600 - x^2}-(-\sqrt{1600 - x^2}) = (50/40+1)\sqrt{1600 - x^2} = (9/4)\sqrt{1600 - x^2}.$$

The volume of a shell is $2\pi x(f_1(x) - f_2(x))\Delta x$. This gives the integral

$$V = \int_a^b 2\pi x(f_1(x) - f_2(x))\, dx = \int_0^{40} 2\pi x(9/4)\sqrt{1600 - x^2}\, dx$$

for the volume of Humpty Dumpty. The shells extend from $x = 0$ to $x = 40$ in the horizontal direction. We substitute $u = 1600 - x^2 \implies du = -2x\, dx$. We also change the limits of integration:

$$x = 0 \implies u = 1600 - 0^2 = 1600, \quad x = 40 \implies u = 1600 = 40^2 = 0.$$

$$V = \int_{1600}^{0} 2\pi(9/4)\sqrt{u}(-du/2) = \frac{9\pi}{4}\int_0^{1600} u^{1/2}\, du = \frac{9\pi}{4}\left[\frac{u^{3/2}}{3/2}\right]_0^{1600} = \frac{9\pi}{4}\frac{2}{3}1600^{3/2} = \frac{3\pi}{2}40^3$$
$$= 96000\pi.$$

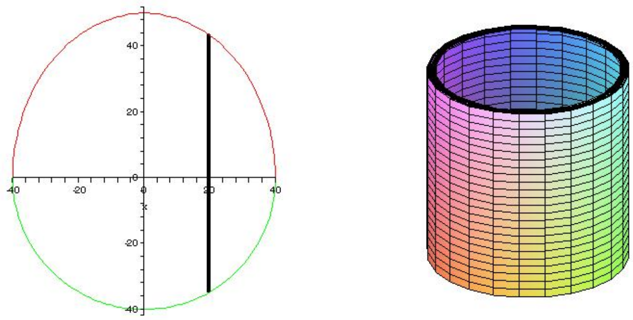


Figure 31: The volume of Humpty Dumpty using cylindrical shells

(b) Humpty Dumpty decided to eat a donut. We get this donut by rotating around the y-axis the circle with equation

$$(x-2)^2 + y^2 = 1.$$

Compute the volume of the donut.

5. *First method:* The method of cylindrical shells. If the shell occurs at x, then the height of the shell is $2y$, where $y = \sqrt{1-(x-2)^2}$. So the height is $2\sqrt{1-(x-2)^2}$. The perimeter of the shell is $2\pi x$ and the thickness of the shell is Δx. The volume of the shell is approximately $2\pi x \cdot 2\sqrt{1-(x-2)^2}\,\Delta x$. The Riemann sum we get is

$$V \approx \sum_i 2\pi x \cdot 2\sqrt{1-(x-2)^2}\,\Delta x.$$

We convert the Riemann sum to an integral to get the exact volume of the donut:

$$V = \int_1^3 2\pi x \cdot 2\sqrt{1-(x-2)^2}\, dx = 4\pi \int_1^3 x\sqrt{1-(x-2)^2}\, dx.$$

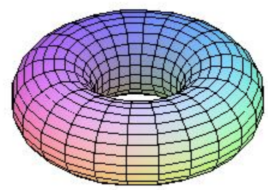


Figure 32: The donut

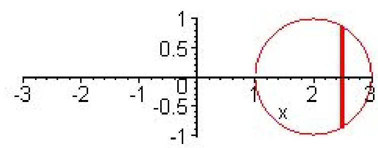


Figure 33: The donut as volume of revolution around the y-axis: Cylindrical shells

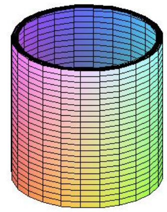


Figure 34: A typical cylindrical shell of thickness Δx

We substitute $u = x - 2$, $du = dx$. We also change the limits of integration:

$$x = 1 \Longrightarrow u = -1, \quad x = 3 \Longrightarrow u = 1.$$

$$V = 4\pi \int_{-1}^{1} (u+2)\sqrt{1-u^2}\,du = 4\pi \left(\int_{-1}^{1} u\sqrt{1-u^2}\,du + 2\int_{-1}^{1} \sqrt{1-u^2}\,du \right)$$

$$= 4\pi \left(\left[-\frac{1}{3}(1-u^2)^{3/2} \right]_{-1}^{1} + 2\pi/2 \right) = 4\pi(0 - 0 + \pi) = 4\pi^2.$$

In the first integral we performed the substitution $w = 1 - u^2$, $dw = -2u\,du$, which gives

$$\int u\sqrt{1-u^2}, du = -\frac{1}{2}\int \sqrt{w}\,dw = -\frac{1}{2}\int w^{1/2}\,dw = -\frac{1}{2}\frac{w^{3/2}}{3/2} = -\frac{1}{3}w^{3/2} = -\frac{1}{3}(1-u^2)^{3/2}.$$

The second integral represents the area under the circle $x^2 + y^2 = 1$ and above the x-axis. The area of this half-disc is $\pi R^2/2 = \pi 1^2/2 = \pi/2$.

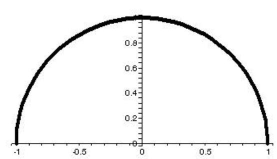


Figure 35: $\int_{-1}^{1} \sqrt{1-u^2}\,du$

Second method: The method of washers. $(x-2)^2 + y^2 = 1$ gives

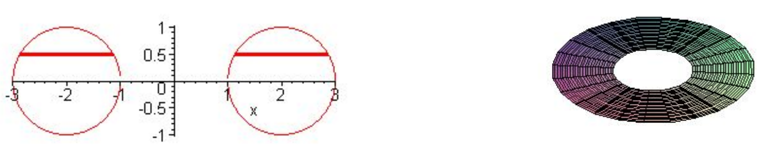


Figure 36: Washers for the donut

$$(x-2)^2 = 1 - y^2 \Longrightarrow x - 2 = \pm\sqrt{1-y^2} \Longrightarrow x = 2 \pm \sqrt{1-y^2}.$$

The inner radius of the washer is $2 - \sqrt{1-y^2}$ and the outer radius of the washer is $2 + \sqrt{1-y^2}$. The area of the washer is

$$A = \pi(2+\sqrt{1-y^2})^2 - \pi(2-\sqrt{1-y^2})^2 = \pi(4+4\sqrt{1-y^2}+1-y^2) - \pi(4+1-y^2-4\sqrt{1-y^2})$$

$$= 8\pi\sqrt{1-y^2}.$$

The volume of the donut is

$$V = \int_{-1}^{1} 8\pi\sqrt{1-y^2}\,dy = 8\pi\int_{-1}^{1}\sqrt{1-y^2}\,dy = 8\pi\pi/2 = 4\pi^2,$$

since the last integral represents the area of a half-disc of radius 1.

6. Decide whether the following series converge or diverge. If they converge find their sum. (6 points)

$$\sum_{n=0}^{\infty} 3\cdot\left(\frac{1}{2}\right)^n, \qquad \sum_{n=1}^{\infty}\left(\frac{3}{2}\right)^n.$$

Both series are geometric series. In the first problem the series has ratio $1/2$, which is less than 1 and positive. Consequently it converges. The sum of the series $\sum_{n=0} ar^n = a/(1-r)$, where a is the first term of the series. Therefore

$$\sum_{n=0}^{\infty} 3\cdot\left(\frac{1}{2}\right)^n = \frac{3}{1-1/2} = \frac{3}{1/2} = 6.$$

The second series has ratio $3/2 > 1$ and consequently diverges.

Tables of Integrals

Differentiation Formulas

1. $(f(x) \pm g(x))' = f'(x) \pm g'(x)$
2. $(kf(x))' = kf'(x)$
3. $(f(x)g(x))' = f(x)g'(x) + f'(x)g(x)$
4. $\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}$
5. $(f(g((x)))' = f'(g(x)) \cdot g'(x)$
6. $\frac{d}{dx}(x^n) = nx^{n-1}$
7. $\frac{d}{dx}(e^x) = e^x$
8. $\frac{d}{dx}(a^x) = a^x \ln a \quad (a > 0)$
9. $\frac{d}{dx}(\ln x) = \frac{1}{x}$
10. $\frac{d}{dx}(\sin x) = \cos x$
11. $\frac{d}{dx}(\cos x) = -\sin x$
12. $\frac{d}{dx}(\tan x) = \frac{1}{\cos^2 x}$

I. Basic Functions

1. $\int x^n \, dx = \frac{1}{n+1} x^{n+1} + c, \quad n \neq -1$
2. $\int \frac{1}{x} dx = \ln |x| + c$
3. $\int a^x \, dx = \frac{1}{\ln a} a^x + c, \quad a > 0$
4. $\int \ln x \, dx = x \ln x - x + c$
5. $\int \sin x \, dx = -\cos x + c$
6. $\int \cos x = \sin x + c$
7. $\int \tan x \, dx = -\ln |\cos x| + c$

II. Products of e^x, $\cos x$, and $\sin x$

8. $\int e^{ax} \sin(bx) \, dx = \frac{1}{a^2 + b^2} e^{ax} [a \sin(bx) - b \cos(bx)] + c$
9. $\int e^{ax} \cos(bx) \, dx = \frac{1}{a^2 + b^2} e^{ax} [a \cos(bx) + b \sin(bx)] + c$
10. $\int \sin(ax) \sin(bx) \, dx = \frac{1}{b^2 - a^2} [a \cos(ax) \sin(bx) - b \sin(ax) \cos(bx)] + c, \quad a \neq b$
11. $\int \cos(ax) \cos(bx) \, dx = \frac{1}{b^2 - a^2} [b \cos(ax) \sin(bx) - a \sin(ax) \cos(bx)] + c, \quad a \neq b$
12. $\int \sin(ax) \cos(bx) \, dx = \frac{1}{b^2 - a^2} [b \sin(ax) \sin(bx) + a \cos(ax) \cos(bx)] + c, \quad a \neq b$

III. Product of Polynomial $p(x)$ with $\ln x$, e^x, $\cos x$, $\sin x$

13. $\displaystyle\int x^n \ln x\, dx = \frac{1}{n+1}x^{n+1}\ln x - \frac{1}{(n+1)^2}x^{n+1} + c, \quad n \neq -1$

14. $\displaystyle\int p(x)e^{ax}\, dx = \frac{1}{a}p(x)e^{ax} - \frac{1}{a}\int p'(x)e^{ax}\, dx = \frac{1}{a}p(x)e^{ax} - \frac{1}{a^2}p'(x)e^{ax} + \frac{1}{a^3}p''(x)e^{ax} - \cdots$

(signs alternate)

15. $$\begin{aligned}\int p(x)\sin ax\, dx &= -\frac{1}{a}p(x)\cos ax + \frac{1}{a}\int p'(x)\cos ax\, dx \\ &= -\frac{1}{a}p(x)\cos ax + \frac{1}{a^2}p'(x)\sin ax + \frac{1}{a^3}p''(x)\cos ax - \cdots\end{aligned}$$

$(-++--++\ldots)$ (signs alternate in pairs after first term)

16. $$\begin{aligned}\int p(x)\cos ax\, dx &= \frac{1}{a}p(x)\sin ax - \frac{1}{a}\int p'(x)\sin ax\, dx \\ &= \frac{1}{a}p(x)\sin ax + \frac{1}{a^2}p'(x)\cos ax - \frac{1}{a^3}p''(x)\sin ax - \cdots\end{aligned}$$

$(++--++--\ldots)$(signs alternate in pairs)

IV. Integer Powers of $\sin x$ and $\cos x$

17. $\displaystyle\int \sin^n x\, dx = -\frac{1}{n}\sin^{n-1}x\cos x + \frac{n-1}{n}\int \sin^{n-2}x\, dx, \quad n \text{ positive}$

18. $\displaystyle\int \cos^n x\, dx = \frac{1}{n}\cos^{n-1}x\sin x + \frac{n-1}{n}\int \cos^{n-2}x\, dx, \quad n \text{ positive}$

19. $\displaystyle\int \frac{1}{\sin^m x}\, dx = -\frac{1}{m-1}\frac{\cos x}{\sin^{m-1}x} + \frac{m-2}{m-1}\int \frac{1}{\sin^{m-2}x}\, dx, \quad m \neq 1,\ m \text{ positive}$

20. $\displaystyle\int \frac{1}{\sin x}\, dx = \frac{1}{2}\ln\left|\frac{(\cos x)-1}{(\cos x)+1}\right| + c$

21. $\displaystyle\int \frac{1}{\cos^m x}\, dx = \frac{1}{m-1}\frac{\sin x}{\cos^{m-1}x} + \frac{m-2}{m-1}\int \frac{1}{\cos^{m-2}x}\, dx, \quad m \neq 1,\ m \text{ positive}$

22. $\displaystyle\int \frac{1}{\cos x}\, dx = \frac{1}{2}\ln\left|\frac{(\sin x)+1}{(\sin x)-1}\right| + c$

23. $\displaystyle\int \sin^m x\cos^n x\, dx$: If m is odd, let $w = \cos x$. If n is odd, let $w = \sin x$.

If both m and n are even and non-negative, convert all to $\sin x$ or all to $\cos x$ (using $\sin^2 x + \cos^2 x = 1$), and use IV-17 or IV-18. If m and n are even and one of them is negative, convert to whichever function is in the denominator and use IV-19 or IV-21. If both m and n are even and negative, the substitution $w = \cos x$ converts the integral into a rational function which can be integrated by the method of partial fractions.

V. Quadratic in the Denominator

24. $$\int \frac{1}{x^2 + a^2}\, dx = \frac{1}{a} \arctan \frac{x}{a} + c, \quad a \neq 0$$

25. $$\int \frac{bx + c}{x^2 + a^2}\, dx = \frac{b}{2} \ln |x^2 + a^2| + \frac{c}{a} \arctan \frac{x}{a} + C, \quad a \neq 0$$

26. $$\int \frac{1}{(x-a)(x-b)}\, dx = \frac{1}{a-b}(\ln|x-a| - \ln|x-b|) + c, \quad a \neq b$$

27. $$\int \frac{cx + d}{(x-a)(x-b)}\, dx = \frac{1}{a-b}\left[(ac+d)\ln|x-a| - (bc+d)\ln|x-b|\right] + C, \quad a \neq b$$

VI. Integrands Involving $\sqrt{a^2+x^2}$, $\sqrt{a^2-x^2}$, $\sqrt{x^2-a^2}$, $a > 0$

28. $$\int \frac{1}{\sqrt{a^2 - x^2}}\, dx = \arcsin \frac{x}{a} + c$$

29. $$\int \frac{1}{\sqrt{x^2 \pm a^2}}\, dx = \ln|x + \sqrt{x^2 \pm a^2}| + c$$

30. $$\int \sqrt{a^2 \pm x^2}\, dx = \frac{1}{2}\left(x\sqrt{a^2 \pm x^2} + a^2 \int \frac{1}{\sqrt{a^2 \pm x^2}}\, dx\right) + c$$

31. $$\int \sqrt{x^2 - a^2}\, dx = \frac{1}{2}\left(x\sqrt{x^2 - a^2} - a^2 \int \frac{1}{\sqrt{x^2 - a^2}}\, dx\right) + c$$